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αIg-Homeomorphism and α^*Ig-Homeomorphism in Ideal Topological Spaces

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Abstract

In this paper, we introduce and study two new homeomorphisms namely αIg-homeomorphism and α^*Ig-homeomorphism in ideal topological spaces.

Keywords: αIg-closed sets, αIg-continuous function, αIg-irresolute function, αIg-closed functions, αIg- open functions, αIg-homeomorphism, α^* Ig-homeomorphism.

I. INTRODUCTION

An Ideal I on a topological space (X, τ, I) is defined as a non-empty collection I of subsets of X satisfying the following two conditions (i) if $A \in I$ and $B \subset A$, then $B \in I$ (ii) If $A \in I$ and $B \in I$,then $A \cup B \in I$. We will make use of the basic facts about the local functions without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\tau, I)$ called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\tau, I)$. When there is no chance for confusion, we simply write A^* instead of $A^*(\tau, I)$ and τ^* for $\tau^*(\tau, I)$. For a subset $A \subset X$, $A^*(\tau, I) = \{x \in X / U \cap A \notin I$ for each neighborhood U of x$\}$. For every ideal topological space (X, τ, I), there exists a topology generated by $\beta(I, \tau) = \{U - J / U \in \tau$ and $J \in I\}$. In general β(I, τ) is not always a topology[1]. If I is an ideal on X, then it is called ideal space. By an ideal space, we always mean an ideal topological space with no separation properties assumed.

The notion homeomorphism plays a very important role in Topology. Many topologists have generalized the concept of homeomorphisms. Devi et al.[2],[3] introduced αI-homeomorphism, g-homeomorphisms and gc-homeomorphisms in topological spaces. S. Jafari[4] introduced $\breve{g}$-homeomorphisms in topological spaces. In this chapter, we introduce the concept of αIg-closed functions, αIg-open functions, αIg-homeomorphism in ideal topological spaces and study its relationship with existing homeomorphisms. A new class of functions α^*Ig–homeomorphism is introduced which form a subclass of αIg-homeomorphisms. We establish that the set of all α^* Ig-homeomorphism from (X, τ, I) onto itself is a group under the composition of functions.

II. PRELIMINARIES

Definition 2.1[5]: Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be an α-Ideal generalized closed set (αIg-closed set) if $A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open.

Definition 2.2[6]: A function f : (X, τ, I) → (Y, σ) is called αIg-continuous, if the inverse image of every closed set in Y is αIg-closed set in X.

Definition 2.3[6]: A function f : (X, τ, I) → (Y, σ, J) is said to be αIg-irresolute, if f^{-1} (A) is αIg-closed set in (X, τ, I), for every αIg-closed set A in (Y, σ, J) .

Definition 2.4[5]: A Subset A of an ideal space (X, τ, I) is said to be *-closed , if A^* **⊆ A.**

Definition 2.5: A function f : (X, τ, I) → (Y, σ, J) is said to be *-closed function, if for every closed subset A of (X, τ, I), f(A) is *-closed in (Y, σ, J).

Definition 2.6: A function f : (X, τ, I) → (Y, σ, J) is said to be closed function, if for every closed subset A of (X, τ, I), f(A) is closed in (Y, σ, J).

III. αIG-CLOSED AND αIG-OPEN FUNCTIONS

In this section, we introduce the concept of αIg-closed functions and αIg-open functions.

Definition 3.1: A function f : (X, τ, I) → (Y, σ, J) is called an αIg-closed (resp. αIg-open) function if for every closed (resp. open) subset A of X, f(A) is αIg-closed (resp. αIg-open) in Y.

Theorem 3.2: A function f : (X, τ, I) → (Y, σ, J) is αIg-closed if and only if for each open set U containing f^{-1} (S), there is an αIg-open set V of Y such that $S \subseteq V$ and f^{-1} (V) ⊆ U.

Proof: Necessity: Let U be open in X. Then U^c is closed in X. Since f is an αIg-closed function, f(U^c)is αIg-closed in Y. Thus Y – f (U^c) is αIg-open, say V containing S such that f^{-1} (V) ⊆ f^{-1} (Y − f(U^c)) = U.

Sufficiency: Let F be a closed set in X. Then F^c is open in X. By hypothesis, there exits an αIg-open set V of Y such that B ⊆ V and f^{-1} (V) ⊆ F^c and so F ⊆ (f^{-1} (V))^c = f^{-1} (V^c). Therefore, f (F) = V^c. Since V^c is αIg-closed, f (F) is αIg-closed in Y. Hence f is αIg-closed.

Theorem 3.3: If f : (X, τ, I) → (Y, σ, J) is a closed map, then it is αIg-closed map.

Proof: Let A be a closed subset of X. Since f is a closed map, f (A) is closed in Y. Every closed is αIg- closed, f (A) is αIg- closed in Y. Hence f is αIg-closed.

Remark 3.4: The converse of the above theorem need not be true as seen from the following example.

Example3.5: Let X = Y = {a, b, c, d}**,** τ = {ϕ, X,{a}, {b}, {a, b}, {b, c}, {a, b, c}}, I = {ϕ, {a}}, σ = {ϕ, Y, {a}, {b, d},{a, b, d}}, J = {ϕ, {a}}. Define a function f : (X, τ, I) → (Y, σ, J) by f(a) = b, f(b) = d , f(c) = a, f(d) = c. Then f is αIg-closed map but not closed map because the subset {a, c, d} is closed in X, f ({a, c, d}) = {a, b, c} is not closed in Y.

Theorem 3.6: If f : (X, τ, I) → (Y, σ, J) is a *-closed map , then it is αIg-closed map.

Proof: Let A be a closed subset of X. Since f is a *-closed map, f(A) is *-closed in Y. Every *-closed is αIg-closed, f (A) is αIg-closed in Y. Hence f is αIg-closed.

Remark 3.7: The converse of the above theorem need not be true as seen from the following example.

Example 3.8: Let X = Y = {a, b, c, d}**,** τ = { ϕ, X, {a}, {b}, {a, b}, {b, c}, {a, b, c}} I = {ϕ, {a}}, σ = {ϕ, Y, {a}, {b, d}, {a, b, d}}, J = {ϕ, {a}}. Define a function f: (X, τ, I) → (Y, σ, J) by f (a) = b, f (b) = d, f (c) = a, f (d) = c. Then f is αIg-closed map but not *-closed map because the subset {b, c, d} is closed in X, f({b, c, d}) = {a, c, d} is not *-closed in Y.

Remark 3.9: The composition of two αIg-closed functions need not be an αIg-closed function.

Example 3.10: Consider X = Y = Z = {a, b, c, d}, τ = {ϕ, X, {a}, {b, d}, {a, b, d}}, I = {ϕ, {a}}, σ = {ϕ, Y, {c}, {a, b, d}}, J = {ϕ, {a}} and η = {ϕ, Z, {a}, {b, c, d}}, K = {ϕ, {b}}. Define a function f : (X, τ, I) → (Y, σ, J) by f (a) = b, f (b) = a, f (c) = c, f

(d) = d and g : (Y, σ, J) $\rightarrow$ (Z, η, K) by g (a) = a, g (b) = b, g (c) = c, g (d) = d. Then both f and g are αIg-closed but g $\circ$ f is not αIg-closed because the subset {c} is closed in X, (g $\circ$ f)({c}) = {c} is not αIg-closed in Z.

IV. αIG-HOMEOMORPHISMS

Definition4.1: A bijection f : (X, τ, I) $\rightarrow$ (Y, σ, J) is called an αIg-homeomorphism, if both f and f^{-1} are αIg- continuous.

We say that the space (X, τ, I) and (Y, σ, J) are αIg-homeomorphic if there exists an αIg-homeomorphism from (X, τ, I) onto (Y, σ, J).

Theroem 4.2: Every homeomorphism is an αIg-homeomorphism.

Proof: Let f : (X, τ, I) $\rightarrow$ (Y, σ, J) be a homeomorphism. Then f and f^{-1} are continuous and f is a bijection. Since every continuous function is αIg-continuous, it follows that f is αIg-homeomorphism.

Remark 4.3: The converse of the above theorem need not be true as seen from the following example.

Example 4.4: Let X = Y = {a, b, c, d}, τ = {ϕ, X, {a}, {b, d}, {a, b, d}}, I = {ϕ, {a}}, σ = {ϕ, Y, {b, c, d}} and J = {ϕ, {b, c, d}, {c}}. Define a function f : (X, τ, I) $\rightarrow$ (Y, σ, J) by f (a) = a, f (b) = b, f (c) = c, f (d) = d. Then f is αIg-homeomorphism but not homeomorphism because the subset f^{-1}({a}) = {a} is closed in Y but not closed in X.

Remark 4.5: The composition of two αIg-homeomorphisms need not be αIg-homeomorphism.

Example 4.6: Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, I = {ϕ, {a}}, σ = {ϕ, {a, b}, Y}, J = {{a}, {a, b}, ϕ}, η = {ϕ, {a}, {a, b}, Z}, K = {ϕ} respectively. Let f : (X, τ, I) $\rightarrow$ (Y, σ, J) and g : (Y, σ, J) $\rightarrow$ (Z, η, K) be identity maps respectively. Then both f and g are αIg-homeomorphisms but their composition g $\circ$ f : (X, τ, I) $\rightarrow$ (Z, η, K), is not an αIg-homeomorphism, because for the open set {a, b} of (X, τ, I), g $\circ$ f({a, b}) = g (f ({a, b})) = g({a, b}) = {a, b} which is not αIg-open in (Z, η, K). Therefore, g $\circ$ f is not αIg-open and not an αIg-homeomorphism.

The following theorem gives a characterization of αIg-homeomorphism.

Theorem 4.7: For any bijection f : (X, τ, I) $\rightarrow$ (Y, σ, J), the following statements are equivalent.

(i) f^{-1} : (Y, σ, J) → (X, τ, I) is αIg-continuous.

(ii) f is an αIg-open map.

(iii) f is an αIg-closed map.

Proof: (i) ⇒(ii): Let U be an open set of (X, τ, I). By assumption $(f^{-1})^{-1}(U)$ = f (U) is αIg-open in (Y, σ , J). So f is αIg-open.

(ii)⇒(iii): Let F be closed set of (X, τ, I). Then F^c is open in (X, τ, I). Since f is αIg-open, f (F^c) is αIg-open in Y. This implies $[f(F)]^c$ is αIg-open in Y. This implies f (F) is αIg-closed in Y. Therefore, f is an αIg-closed map.

(iii)⇒(i): Suppose F is a closed set in (X, τ, I). Then by assumption the inverse image of F under f^{-1} namely if f (F) is αIg-closed in (Y, σ, J). Hence f^{-1} is αIg-continuous.

Theorem 4.8: Let f: (X, τ, I) → (Y, σ, J) be a bijective and αIg-continuous map. Then, the following statement are equivalent.

(i) f is an αIg-open map.

(ii) f is an αIg-homeomorphism.

(iii) f is an αIg-closed map.

Proof follows from Definition 3.1, Definition 4.1, Definition 2.2 and the Theorem 4.7.

IV. α*Ig-Homeomorphisms

We introduce a new class of maps called α* Ig-homeomorphisms which forms a subclass of αIg-homeomorphism. This class of maps is closed under composition of maps.

Definition 5.1: A map f : (X, τ, I) → (Y, σ, J) is called α*Ig-open if f(U) is αIg-open set U of (X, τ,I).

Theorem 5.2: For any bijection f:(X, τ, I) → (Y, σ, J), the following statements are equivalent.

(i) The inverse map f^{-1} : (Y, σ, J) → (X, τ, I) is αIg-irresolute.

(ii) f is α^*Ig-open map.

(iii) f is α^*Ig-closed map.

Proof: (i) $\Rightarrow$(ii): Let U be αIg-open in (X, τ, I). By (i), $(f^{-1})^{-1}(U)$ = f(U) is αIg-open in (Y, σ, J). Hence (ii) holds.

(ii) $\Rightarrow$(iii): Let V be αIg-closed in (X, τ, I). Then X$-$V is αIg-open and by (ii) f (X$-$V) = Y$-$ f(V) is αIg-open in (Y, σ, J). That is f (V) is αIg-closed in Y and so f is α^*Ig-closed map.

(iii) $\Rightarrow$(i): Let W be αIg-closed in (X, τ, I). By (iii), f(W) is αIg-closed in (Y, σ, J). But f (W) = $(f^{-1})^{-1}$ (W) . Thus (i) holds.

Next we introduce a new class of maps as follows.

Definition 5.3: A bijection f: (X, τ, I) $\rightarrow$ (Y, σ, J) is called α^*Ig-homeomorphism if both f and f^{-1} are αIg-irresolute.

We say that the spaces (X, τ, I) and (Y, σ, J) are α^*Ig-homeomorphic, if there exists an α^*Ig- homeomorphism form (X, τ, I) onto (Y, σ, J). The family of all αIg-homeomorphism (resp. α^* Ig- homeomorphism) from (X, τ, I) onto (Y, σ, J) is denoted by αIg-h(X, τ, I) (resp. α^*Ig-h(X, τ, I)).

Theorem 5.4: Every α^*Ig-homeomorphism is an αIg-homeomorphism.

Proof: Let f: (X, τ, I) $\rightarrow$ (Y, σ, J) be a α^*Ig-homeomorphism. Then f and f^{-1} are αIg-irresolute and f is bijection . Thus, f and f^{-1} are αIg-continuous. Hence f is αIg-homeomorphism.

Remark 5.5: The converse of the above theorem need not be true as seen from the following example.

Example 5.6: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, X}, I = {ϕ, {a}}, σ = {ϕ, {a, b}, Y}, J = {{a},{a, b}, ϕ} respectively. Let f: (X, τ, I) $\rightarrow$ (Y, σ, J) be identity map. Then f is αIg-homeomorphism but f is not α^*Ig-homeomorphism because the subset {a, b} is αIg-closed in Y, but f^{-1}({a, b}) = {a, b} is not αIg-closed in X.

Theorem 5.7: Let f : (X, τ, I) $\rightarrow$ (Y, σ, J) and g : (Y, σ, J) $\rightarrow$ (Z, η, K) be α^*Ig-homeomorphism then their composition g $\circ$ f : (X, τ, I) $\rightarrow$(Z, η, K) is α^* Ig-homeomorphism.

Proof: Let U be an αIg-open set in (Z, η, K). Since g is αIg-irresolute, g^{-1}(U) is αIg-open in (Y, σ, J). Since f is αIg-irresolute, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}$ (U) is an αIg-open set in (X, τ, I). Therefore (X, τ, I) is αIg-irresolute. Also, for a αIg-open set G in (X, τ, I), we have (g ∘ f) (G) = g (f (G)) = g (W), where W = f (G). By hypothesis f (G) is αIg-open in (Y, σ, J) and also by hypothesis g (f (G)) is a αIg-open set in (Z, η, K). That is, (g ∘ f) (G) is a αIg-open set in (Z, η, K) and therefore, $(g \circ f)^{-1}$ is αIg-irresolute. Also, (g ∘ f) is a bijection. Hence (g ∘ f) is α*Ig-homeomorphism.

Theorem 5.8: The set α*Ig-h(X, τ, I) is a group under the composition of maps.

Proof: Define a binary operation *: α*Ig-h(X, τ, I) *X* α*Ig-h(X, τ, I) by f * g = (g ∘ f) for all f, g ∈ α*Ig-h(X, τ, I) and ∘ is the usual operation of composition of maps. Then, by Theorem 5.7, (g ∘ f) ∈ α*Ig-h(X, τ, I). We know that the compostion of maps is associative and the identity map I : (X, τ, I) → (X, τ, I) belonging to α*Ig-h(X, τ, I) serves as the identity element. For any f ∈ α*Ig-h(X, τ, I), $f \circ f^{-1} = f^{-1} \circ f = I$. Hence inverse exists for each element of α*Ig-h(X, τ, I). α*Ig-h(X, τ, I) forms a group under the operation of composition of maps.

Theorem 5.9: Let f: (X, τ, I) → (Y, σ, J) be an α*Ig-homeomorphism. Then, f induces an isomorphism from the group α*Ig-h(X, τ, I) onto the group α*Ig-h(Y, σ, J).

Proof: Let f ∈ α*Ig-h(X, τ, I). We define a function φ_f: α*Ig-h(X, τ, I)→α*Ig-h(Y, σ, J) by $\varphi_f = f \circ h \circ f^{-1}$ for every h ∈ α*Ig-h(X, τ, I). Then f is a bijection. Now for all g, h ∈ α*Ig-h(X, τ, I), $\varphi_f(g \circ h) = f \circ (g \circ h) \circ f^{-1} = (f \circ g \circ f^{-1}) \circ (f \circ h \circ f^{-1}) = \varphi_f(g) \circ \varphi_f(h)$.

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