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# A Characterization of Uniform Extending Modules as a Finite Direct Sum of Modules

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**Abstract**

In this paper, we investigate a generalization of CS-modules called uniform extending modules, studying the direct sums of uniform modules over the rings with finite uniform dimension. We study the interaction between uniform extending modules, relatively injective modules, quasi-continuous modules, quasi-injective modules and direct sum of indecomposable modules. We try to find out what are the conditions under which a module which is a finite direct sum of relatively injective modules $M_i$ is uniform-extending.

**Keywords:** CS-modules, uniform-extending modules, uniform modules.

## INTRODUCTION

It is known that a module over a right noetherian ring is extending if and only it is a direct sum of uniform modules, is uniform extending and every local summand is a direct summand [6], whereas a quasi-continuous module, which is a direct sum of uniform modules, is uniform extending [7]. In addition, a module with finite uniform dimension is extending if and only if it is uniform extending [2]. In this paper, we further investigate uniform-extending modules, employing the restriction ($1$-$C_1$), which is concerned with uniform submodules.

Throughout this paper, R will denote a ring with identity and M a unitary right R-module. A module M is *uniform* if any two nonzero submodules of M have nonzero intersection. A module M is CS (or extending) if every submodule of M is essential in a direct summand of M, whereas a module M is *uniform-extending* if every uniform submodule of M is essential in a direct summand of M.

Clearly, every CS module is uniform extending. Also every quasi-continuous module is uniform extending. A module M is called a (countably)∑uniform-extending module if $M^{(A)}$ (respectively $M^{(N)}$) is uniform-extending for every set A.

**Preliminary Lemmas:**

We start with a well known result by Dung *et al.*[2], followed by a keen observation by Huynh *et al.*

**Lemma 1.1:** Let $M=\oplus_{i\in I}M_i$, where each $M_i$ is an indecomposable module. Then M is a quasi-injective module if and only if M is uniform-extending and for any $i, j\in I$, $M_i$ is $M_j$-injective.

**Lemma 1.2:** [4, Lemma 2.1] Let $M=\oplus_{i\in I}M_i$ be a continuous module where each $M_i$ is uniform. Then the following conditions are equivalent:

(i) $M^{(N)}$ is uniform-extending;

(ii) $M^{(N)}$ is quasi-injective.

*We now come to our main results.*

**Theorem 1.3:** *Let* R *be a ring of finite right uniform dimension and* M *be a right* R-*module such that* $M=\oplus_{i=1}^{n}M_i$ *is a finite direct sum of uniform submodules*($M_i$)*. Then following conditions are equivalent:*

*(i)* M *is uniform-extending;*

*(ii)* M *is CS;*

*(iii)* M *is quasi-continuous and a direct sum of indecomposable modules;*

*(iv)* $M_i\oplus M_j$ *is quasi-continuous, for every* $1\le i<j\le n$*;*

*(v)* $M_i$ *is* $M_j$-*injective* $(i\neq j)$*;*

*(vi)* M *is quasi-injective;*

*(vii)* M *is uniform-extending and* $M_i$ *is* $M_j$-*injective* $(i\neq j)$*.*

**Proof :** Let $M = \oplus_{i=1}^{n} M_i$, where each $M_i$ is a uniform module.

(i) $\Rightarrow$ (ii) It is known that any finite direct sum of uniform modules is uniform-extending if and only if it is CS, by [2, 8.5]. Therefore, M is a CS-module.

(ii) $\Rightarrow$ (iii) If M is a CS-module, then it follows from the necessary condition of ([7], Theorem 13) that it is also a quasi-continuous module and is a direct sum of indecomposable submodules.

(iii) $\Rightarrow$ (iv) $M = \oplus_{i=1}^{n} M_i$ is a quasi-continuous module. It is known that a finite direct sum $(M_1 \oplus \dots \oplus M_n)$ is quasi-continuous, whenever $M_i \oplus M_j$ is quasi-continuous for all $1 \le i < j \le n$, by ([3], Corollary 11).

(iv) $\Rightarrow$ (v) Let $M_i \oplus M_j$, be quasi-continuous, then it clearly follows that $M_i$ is $M_j$-injective by ([8], Corollary 2.14).

(v) $\Rightarrow$ (vi) Let $M_i$ be $M_j$-injective then it follows from the sufficient condition of ([8], Corollary 1.19) that M ($M = \oplus_{i=1}^{n} M_i$) is a quasi-injective module.

(vi) $\Rightarrow$ (vii) M is quasi-injective as defined above, we then have from Lemma 1.1, that M is uniform-extending and $M_i$ is $M_j$-injective, for any $i, j \in I$ $(i \neq j)$.

(vii) $\Rightarrow$ (i) is trivial.

**Theorem 1.4 :** *Let* R *be a ring and* M *an R-module such that* $M = \oplus_{i=1}^{n} M_i$ *is a finite direct sum of relatively injective modules* $M_i$ *($1 \le i < j \le n$). Then* M *is a uniform-extending module.*

**Proof :** Let $M = \oplus_{i=1}^{n} M_i$ be a finite direct sum of relatively injective modules over R. Suppose $M_i$ is $M_j$-injective, then as seen in the previous result, M will be quasi-injective by ([8], Corollary 1.19) and hence continuous (as every quasi-injective module is a continuous module).

Now according to Harmanci *et al*. ([3], Theorem 12), if M is continuous then it is a finite direct sum of relatively injective continuous modules ($M_i$).

Therefore, each $M_i$ is a continuous module and hence quasi-continuous (as every continuous module is also quasi-continuous) and consequently, CS (being a quasi-continuous module).

Thus each $M_i$ is CS and $M_j$-injective module, $(1 \le i < j \le n)$. Then by ([3], Theorem 8), one can easily conclude that M being a finite direct sum of relatively injective CS modules ($M_i$) is a CS module. That is, M is a CS-module.

And it is well known that every CS-module is uniform-extending, hence, we conclude that M is a uniform extending module.

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