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The relation between Wiener index and Nandu sequence of SM family of Graphs

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Abstract

SM family of Graphs consists of SM sum graphs, SM balancing graphs, its complement graphs and its subgraphs. SM sum graph is associated with the relationship between the powers of 2 and positive integers. SM balancing graphs are formed by using the fact that all positive integers can be written as a linear combination of powers of 3. Nandu sequence is obtained from the number of edges of SM graphs. The relationships between Wiener indices and the Nandu sequence of these graphs are obtained in this work. Also the summation of Nandu series, generating function of Nandu sequence and girth of these family of graphs are discussed and some results are formed.

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Keywords: n^{th}SM Balancing graphs, n^{th}SM Sum graphs, Nandu sequence, Wiener index of SM graphs, Girth of SM graphs.

1. Introduction

For a fixed positive integer n, consider the set $P = \{2^m,\ 0 \leq m \leq n-1\}$. Any positive integer less than 2^n and not in P can be expressed as the sum of two or more distinct elements of P. If $p \notin P$ and $p = \sum x_i$, with distinct $x_i \in P$, then each x_i is called an additive component of p. The simple graph, $SM\left(\sum_n\right)$ is a graph with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by two distinct vertices v_i and v_j are adjacent if either i is an additive component of j or j is an additive component of i. Also consider the set $T = \{3^m, 0 \leq m \leq n-1\}$ for a fixed positive integer $n \geq 2$. Any positive integer $x \leq \frac{1}{2}(3^n - 1)$, which is not a power of 3 can be expressed as a linear combination of two or more distinct elements of the set T with coefficients $-1, 0$ or 1. The relation between x and the elements of T are used to form a new class of graphs called n^{th} SM Balancing graphs, $SM(B_n)$ and $SMD(B_n)$. Some preliminaries are given below.

2. n^{th} SM Balancing Graphs and n^{th} SM Sum Graphs

Definition 2.1. [8] Consider the set $T = \{3^m, 0 \leq m \leq n-1\}$ for a fixed positive integer $n \geq 2$. Let $I = \{-1, 0, 1\}$. Any positive integer $x \leq \frac{1}{2}(3^n - 1)$ which is not a power of 3 can be expressed as

$$x = \sum_{j=1}^{n} \alpha_j y_j \tag{1}$$

for some $\alpha_i \in I$ and $y_j \in T$. If $\alpha_i \neq 0$, then each y_j is called a balancing component of x.

Definition 2.2. [8] Let T be the set $T = \{3^m, 0 \leq m \leq n-1\}$ for a fixed positive integer $n \geq 2$. Consider the simple directed graph $G = (V, E)$, where the vertex set $V = \{v_1, v_2, \ldots, v_{\frac{1}{2}(3^n-1)}\}$ and adjacency of vertices defined by, two distinct vertices v_x is adjacent to v_{y_j} if (1) holds and $\alpha = -1$ and two distinct vertices v_{y_j} is adjacent to v_x if (1) holds and $\alpha = 1$. This directed graph G is called the $n^{th} SMD$ Balancing Graph, $SMD(B_n)$. The underlying undirected graph is called n^{th} SM Balancing Graphs, $SM(B_n)$.

Definition 2.3. If $p < 2^n$, is a positive integer which is not a power of 2, then $p = \sum_{1}^{n} x_i$, with $x_i = 0$ or 2^m, for some $0 \leq m \leq n-1$ and x_is are distinct. Here we call each $x_i \neq 0$ as an additive component of p.

Definition 2.4. For a fixed integer $n \geq 2$, define a simple graph $SM\left(\sum_n\right)$, called n^{th} SM sum graph, with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by, v_i

and v_j are adjacent if either i is an additive component of j or j is an additive component of i.

Definition 2.5. [3] If $G(V, E)$ is a graph, then Wiener index, $w(G)$ is defined as the sum of distances between all unordered pairs of vertices of G. $ie.$, $w(G) = \sum_{\{u,v\}\subseteq V} d(u, v)$, where $d(u, v)$ is the distance between u and v.

Definition 2.6. [3] If $G(V, E)$ is a graph, then the Hyper Wiener index, $ww(G)$ is defined as

$$ww(G) = \frac{1}{2} \sum_{\{u,v\}\subseteq V} [d(u, v)^2 + d(u, v)].$$

Definition 2.7. [8] Let $G = SM\left(\sum_n\right)$ or $SM(B_n)$ with vertex set V be an nth SM graph. The Adi-R-set of degrees, denoted by A_n^R is defined as $A_n^R = \{degv_{i(x)}, v_i \in V\}$, where x is the number of times each $degv_i$ repeats.

Definition 2.8. Consider the graph $SM\left(\sum_{n+1}\right)$ with vertex set $V = \{v_i, 1 \leq i \leq 2^{n+1} - 1\}$. Define the sequence $\{Nt_n\}$ as $Nt_n = \frac{1}{2}\sum_{v\in V} deg\ v$ as Nandu sequence and the sequence $\{DNt_n\}$ as $DNt_n = \sum_{v\in V} deg\ v$ as Double Nandu sequence. So we get $\{Nt_n\} = 2, 9, 28, 75, 186, \ldots$ and $\{DNt_n\} = 4, 18, 56, 150, 372, \ldots$

Definition 2.9. Consider the graph $SM(B_{n+1})$ with vertex set $V = \{v_i, 1 \leq i \leq \frac{1}{2}(3^{n+1} - 1)\}$. Define the sequence $\{Nt_n\}$ as $Nt_n = \frac{1}{2}\sum_{v\in V} deg\ v$ as Nandu sequence and the sequence $\{DNt_n\}$ as $DNt_n = \sum_{v\in V} deg\ v$ as Double Nandu sequence. So we get $\{Nt_n\} = 4, 24, 104, 400, 1452, \ldots$ and $\{DNt_n\} = 8, 48, 208, 800, 2904, \ldots$
Note: For a fixed integer $n \geq 2$, let $T = \{3^m : 0 \leq m \leq n - 1\}$, $N = \{1, 2, 3, \ldots, t\}$, where $t = \frac{1}{2}(3^n - 1)$. Also let $P = \{2^m : 0 \leq m \leq n - 1\}$, $M = \{1, 2, 3, \ldots, 2^n - 1\}$. Then consider $P^c = M - P$, $T^c = N - T$ throughout this paper unless otherwise specified.

3. Relationship between Wiener index and Nandu sequence of $SM\left(\sum_n\right)$ and $SM(B_n)$

Graph parameters are useful to study the structure of graphs and then to apply in real life problems. Nandu sequence is a sequence of number of edges of the SM family of graphs. We are finding out some graph parameters like Wiener index, Hyper Wiener index etc in connection with the terms of the Nandu sequence. The Wiener index, a topological

index of a graph, is used to explain the variation in boiling points, molar volumes, heat isomerization etc. Wiener index also used in networks.

Lemma 3.1. If $G = SM\left(\sum_n\right)$, $P = \{2^m : 0 \leq m \leq n-1\}$, then

$$d(v_i, v_j) = \begin{cases} 1 & \text{, if } i \text{ is an additive component of } j \text{ or } j \text{ is an additive component of } i \\ 2 & \text{, if } i, j \in P \text{ or } i, j \notin P, i \text{ and } j \text{ have atleast one common additive component} \\ 3 & \text{neither } i \text{ nor } j \text{ is an additive component but exactly one of them belongs to } P \\ 4 & \text{, } i, j \notin P, i \text{ and } j \text{ have no common additive component.} \end{cases}$$

Proposition 3.2. Let $G = SM\left(\sum_n\right)$ be an nth SM sum graph. Let $d_r(v_i, v_j)$ denote the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Let the $(n-1)^{th}$ term of the Nandu sequence of $G = SM\left(\sum_n\right)$ be Nt_{n-1}. Then:

$$d_r(v_i, v_j) = \begin{cases} Nt_{n-1} & \text{, if } r = 1 \\ \dfrac{n(n-1)}{2} + \left[\dfrac{\left(\frac{Nt_{n-1}}{n} - n\right)\left(\frac{Nt_{n-1}}{n} - n + 1\right)}{2} - \delta\right], & \text{if } r = 2 \\ Nt_{n-1} + n - n^2 & \text{, if } r = 3 \\ \delta & \text{, if } r = 4. \end{cases}$$

where $\delta = \dfrac{1}{2}\sum_{r=2}^{n-2}[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}]$.

Proof. We have $Nt_{n-1} = n.(2^{n-1} - 1)$. The proof is obvious when $r = 1$. When $r = 4$ and $n \geq 4$, then $d_4(v_i, v_j)$ is the total number of unordered pairs of disjoint subsets containing atleast 2 elements of a set. Hence the result follows.

Now suppose, $r = 2$. Let $P = \{2^m : 0 \leq m \leq n-1\}$ implies $|P^c| = 2\dfrac{Nt_{n-1}}{n} - n + 1$. So the total number of unordered pairs of elements of P^c is $\dfrac{(2\frac{Nt_{n-1}}{n} - n)(2\frac{Nt_{n-1}}{n} - n + 1)}{2}$, which follows the result.

Suppose $r = 3$. We have, $|V| = 2\dfrac{Nt_{n-1}}{n} + 1$. Let $d_r(v_i, v_j) = d_r$. In this case the number of unordered pairs is given by

$$d_3(v_i, v_j) = \frac{(2\frac{Nt_{n-1}}{n} + 1)(2\frac{Nt_{n-1}}{n})}{2} - (d_1 + d_2 + d_4) = Nt_{n-1} + n - n^2.$$

Hence proved. ■

Remark 3.3. $\delta = 0$ for $n = 2$ or 3.

Theorem 3.4. Let $G = SM\left(\sum_n\right)$, $n \geq 2$. Then $w(G) = 4.\left(\frac{Nt_{n-1}}{n}\right)^2 + 2.\frac{Nt_{n-1}}{n} - n^2 + n + 2\delta$.

Proof. By the definition of $w(G)$,

$$\begin{aligned} w(G) &= \sum_{\{u,v\}\subseteq V} d(u,v) \\ &= 1.Nt_{n-1} + 2[\frac{n(n-1)}{2} + \left[\frac{(2\frac{Nt_{n-1}}{n} - n)(2\frac{Nt_{n-1}}{n} - n + 1)}{2} - \delta\right] \\ &\quad + 3[Nt_{n-1} + n - n^2] + 4\delta \\ &= 4.(\frac{Nt_{n-1}}{n})^2 + 2.\frac{Nt_{n-1}}{n} - n^2 + n + 2\delta. \end{aligned}$$

■

Theorem 3.5. Let $G = SM\left(\sum_n\right)$, $n \geq 2$. Then

$$ww(G) = 6.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3.\frac{Nt_{n-1}}{n} + 2.Nt_{n-1} - 3n^2 + 3n + 7\delta$$

.

Proof. By using proposition 3.2, we get

$$\sum_{\{u,v\}\subseteq V} (d(u,v))^2 = 1^2.Nt_{n-1} + 4\left[\frac{n(n-1)}{2} + \frac{(2\frac{Nt_{n-1}}{n} - n)(2\frac{Nt_{n-1}}{n} - n + 1)}{2} - \delta\right]$$

$$\begin{aligned} &+ 9[Nt_{n-1} + n - n^2] + 16\delta \\ &= 8.\left(\frac{Nt_{n-1}}{n}\right)^2 + 4.\frac{Nt_{n-1}}{n} + 2.Nt_{n-1} - 5n^2 + 5n + 12\delta. \end{aligned}$$

Therefore,

$$\begin{aligned} ww(G) &= \frac{1}{2}[w(G) + \sum_{\{u,v\}\subseteq V} (d(u,v))^2] \\ &= \frac{1}{2}[8.(\frac{Nt_{n-1}}{n})^2 + 4.\frac{Nt_{n-1}}{n} + 2.Nt_{n-1} - 5n^2 + 5n + 12\delta + 4.(\frac{Nt_{n-1}}{n})^2 \\ &\quad + 2.\frac{Nt_{n-1}}{n} - n^2 + n + 2\delta] \\ &= 6.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3.\frac{Nt_{n-1}}{n} + 2.Nt_{n-1} - 3n^2 + 3n + 7\delta \end{aligned}$$

Hence the theorem. ■

Lemma 3.6. If $G = SM(B_n)$, $T = \{3^m : 0 \leq m \leq n-1\}$, $v_i, v_j \in V(G)$, then

$$d(v_i, v_j) = \begin{cases} 1 & \text{, if } i \text{ is a balancing component of } j \text{ or } j \text{ is a balancing component of } i. \\ 2 & \text{, if } i, j \in T \text{ or } i, j \notin T, i \text{ and } j \text{ have atleast one common balancing component.} \\ 3 & \text{, neither } i \text{ nor } j \text{ is a balancing component but exactly one of them belongs to } T. \\ 4 & \text{, } i, j \notin T, i \text{ and } j \text{ have no common balancing component.} \end{cases}$$

Proposition 3.7. Let $G = SM(B_n)$ be an nth SM Balancing graph. Let $d_r(v_i, v_j)$ denote the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Let $t = \frac{1}{2}(3^n - 1)$. Let the $(n-1)^{th}$ term of the Nandu sequence of $G = SM(B_n)$ be Nt_{n-1}. Then:

$$d_r(v_i, v_j) = \begin{cases} Nt_{n-1} & \text{, if } r = 1 \\ \frac{n(n-1)}{2} + \left[\frac{\left(\frac{3.Nt_{n-1}}{2n} - n\right)\left(\frac{3.Nt_{n-1}}{2n} - n + 1\right)}{2} - \sigma\right] & \text{, if } r = 2 \\ \frac{Nt_{n-1}}{2} + n - 2n^2 & \text{, if } r = 3 \\ \sigma & \text{, if } r = 4. \end{cases}$$

where $\sigma = \frac{1}{2}\sum_{r=2}^{n-2}[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}2^{r+k-2}]$.

Proof. Since the $(n-1)^{th}$ term of the Nandu sequence of $G = SM(B_n)$ is Nt_{n-1}, the number of cases when $r = 1$ is Nt_{n-1}. When $r = 4$ and $n \geq 4$, the $d_4(v_i, v_j)$ is the total number of unordered pairs of disjoint subsets containing atleast 2 elements of a set consisting of $n \geq 4$ elements. Hence the result follows.

Now consider the case when $r = 2$. Let $T = \{3^m : 0 \leq m \leq n-1\}$ implies $|T^c| = \dfrac{3.Nt_{n-1}}{2n} - n + 1$. So the total number of unordered pairs of elements of T^c is $\dfrac{\left(\frac{3.Nt_{n-1}}{2n} - n\right)\left(\frac{3.Nt_{n-1}}{2n} - n + 1\right)}{2}$, which follows the result.

Suppose $r = 3$. We have, $|V| = \dfrac{3.Nt_{n-1}}{2n} + 1$. Therefore the number of unordered pairs is $\dfrac{\left(\frac{3.Nt_{n-1}}{2n} + 1\right)\left(\frac{3.Nt_{n-1}}{2n}\right)}{2} - (d_1 + d_2 + d_4) = \dfrac{Nt_{n-1}}{2} + n - n^2$. Hence the result.

■

Remark 3.8. $\sigma = 0$ for $n = 2$ or 3.

Theorem 3.9. Let $G = SM(B_n)$, $n \geq 2$. Then $w(G) = \dfrac{9}{4}.\left(\dfrac{Nt_{n-1}}{n}\right)^2 + 3.\dfrac{Nt_{n-1}}{2n} + \dfrac{Nt_{n-1}}{2} + n - n^2 + 2\sigma$.

Proof. By the definition of $w(G)$,

$$\begin{aligned} w(G) &= \sum_{\{u,v\} \subseteq V} d(u,v) \\ &= 1.Nt_{n-1} + 2\Big[\frac{n(n-1)}{2} + \frac{\left(\frac{3.Nt_{n-1}}{2n} - n\right)\left(\frac{3.Nt_{n-1}}{2n} - n + 1\right)}{2} - \sigma\Big] \\ &\quad + 3.\left(\frac{Nt_{n-1}}{2} + n - n^2\right)\Big] + 4\sigma \\ &= \frac{9}{4}.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3.\frac{Nt_{n-1}}{2n} + \frac{Nt_{n-1}}{2} + n - n^2 + 2\sigma. \end{aligned}$$

■

Theorem 3.10. Let $G = SM(B_n)$,$n \geq 2$. Then

$$ww(G) = \frac{27}{4}.\left(\frac{Nt_{n-1}}{n}\right)^2 + \frac{1}{2}Nt_{n-1} + \frac{9Nt_{n-1}}{4n} + 3n - 3n^2 + 7\sigma.$$

Proof. By using the above proposition 3.7, we get

$$\sum_{\{u,v\}\subseteq V}(d(u,v))^2 = 1.Nt_{n-1} + 4\left[\frac{n(n-1)}{2} + \frac{\left(\frac{3.Nt_{n-1}}{2n} - n\right)\left(\frac{3.Nt_{n-1}}{2n} - n + 1\right)}{2} - \sigma\right]$$
$$+ 9.\left(\frac{Nt_{n-1}}{2} + n - n^2\right)\Big] + 16\sigma$$
$$= \frac{9}{2}.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3\frac{Nt_{n-1}}{n} + \frac{Nt_{n-1}}{2} + 5n - 5n^2 + 12\sigma$$

Therefore,

$$ww(G) = \frac{1}{2}\left[w(G) + \sum_{\{u,v\}\subseteq V}(d(u,v))^2\right]$$
$$= \frac{1}{2}\Big[\frac{9}{2}.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3\frac{Nt_{n-1}}{n} + \frac{Nt_{n-1}}{2} + 5n - 5n^2 + 12\sigma$$
$$+ \frac{9}{4}.\left(\frac{Nt_{n-1}}{n}\right)^2 + 3.\frac{Nt_{n-1}}{2n} + \frac{Nt_{n-1}}{2} + n - n^2 + 2\sigma\Big]$$
$$= \frac{27}{4}.\left(\frac{Nt_{n-1}}{n}\right)^2 + \frac{1}{2}Nt_{n-1} + \frac{9Nt_{n-1}}{4n} + 3n - 3n^2 + 7\sigma.$$

■

4. Summation of Nandu series and the generating function

Now let us have a look into the summation of the Nandu series of the SM balancing graphs and SM sum graphs. Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Also generating functions can be used to solve recurrence relations and to solve many types of counting problems such as the number of ways to select or distribute objects of different kinds, subject to variety of constraints. Here we provide generating functions for the Nandu sequences.

Lemma 4.1. Let $\{Nt_n\}$, where $Nt_n = \frac{1}{2}\sum_{v\in V} deg\ v$, be the Nandu sequence for the SM sum graphs. Then its summation is given by $\sum_{r=1}^{n} Nt_r = 2n.2^n - \frac{n(n+1)}{2} - n.$

Proof. We have,

$$\begin{aligned}\sum_{r=1}^{n} Nt_r &= \sum_{r=1}^{n}(r+1)(2^r-1) = \sum_{r=1}^{n}(r.2^r+2^r-r-1)\\ &= 2\Big[\frac{1-(n+1)2^n+n.2^{n+1}}{(1-2)^2}\Big]+2(2^n-1)-\frac{n(n+1)}{2}-n\\ &= 2n.2^n-\frac{n(n+1)}{2}-n.\end{aligned}$$

■

Lemma 4.2. Let $\{Nt_n\}$, where $Nt_n = \frac{1}{2}\sum_{v\in V} deg\ v$, be the Nandu sequence for the SM balancing graphs. Then $\sum_{r=1}^{n} Nt_r = \frac{3}{4}[2n.3^n+3^n-1]-n-\frac{n(n+1)}{2}$.

Proof.

$$\begin{aligned}\sum_{r=1}^{n} Nt_r &= \sum_{r=1}^{n}(r+1)(3^r-1) = \sum_{r=1}^{n}(r.3^r+3^r-r-1)\\ &= 2\left[\frac{1-(n+1)3^n+n.3^{n+1}}{(1-3)^2}\right]+\frac{3(3^n-1)}{2}-\frac{n(n+1)}{2}-n\\ &= \frac{3}{4}[2n.3^n+3^n-1]-n-\frac{n(n+1)}{2}.\end{aligned}$$

■

Definition 4.3. A closed form of the generating function of the Nandu sequence of SM sum graph is given by $G(x) = \sum_{r=1}^{n}(r+1)(2^r-1)x^r$.

Definition 4.4. A closed form of the generating function of the Nandu sequence of SM balancing graph is given by $G(x) = \sum_{r=1}^{n}(r+1)(3^r-1)x^r$.

5. Girth of SM graphs

The girth of a graph G is the length of the smallest cycle of G. A cycle of smallest length is called a girdle of G.

Theorem 5.1. The girth of a SM sum graph is given by $g\left(SM\left(\sum_n\right)\right) = 4$.

Proof. Consider the graph $G = SM\left(\sum_n\right), n > 2$. Since the SM sum graphs are triangle -free graphs, the girth of these graphs are more than 3. We can partition the vertex set of

G as $X \cup Y$, where $X = \{v_i : where\ i = 2^r\ for\ 0 \leq r \leq n-1\}$ and $Y = V - X$. Now let $v_i \in X$ and $v_j \in Y$, then there is a vertex v_t adjacent to v_j and v_{2^n-1}. So $v_i v_j v_t v_{2^n-1}$ is a cycle of length 4. Therefore a girdle of G is having a length 4. Hence the girth is $g\left(SM\left(\sum_n\right)\right) = 4$. ■

Theorem 5.2. The girth of a SM balancing graph is given by $g(SM(B_n)) = 4$.

Proof. Proof is same as above. ■

6. Conclusion

We derived a relationship between the Nandu sequence and Wiener index of of SM family of Graphs. This may be helpful to study the nature and different graph theoretic parameters related to n^{th} SM Balancing graphs and n^{th} SM sum graphs.The same way we can find some other parameters also in connection with Nandu sequence. There are many graph parameters related with the number of edges of a graph. Since these SM graphs are systematically arranged graphs, this type of relationships may help in further studies of nature and structure of Graphs.

References

[1] Arockiaraj S, "A study on distance in Graphs", Ph.D Thesis submitted to Madurai Kamraj University, September 2010.
[2] Harary F, "Graph theory", Addison-Wesly Publishing Company Inc. (1994).
[3] I Gutman, "Relation Between Hyper wiener and wiener index", Chem. Phys. Letter 364(2002), 352–356.
[4] Kinkal ch Das and Ivan Gutman, "Estimating the wiener index by means of number of vertices, Number of edges and diameters", Math Commun. Comp. Chem, 64(2010) 647–660.
[5] Liyan, Ying angLi, Wei Gao and Junsheng Li, "On the external Hyper-wiener index of graphs", Journal of Chemical and Pharmaceutical Research, 2014, 6(3), 477–481.
[6] Parthasarathy K R, "Basic Graph theory", Tata Mc Graw Hill New York, 1994.
[7] Sreekumar K.G. and Dr. Manilal K, "n^{th} SM balancing graphs and some of its parameters", International Journal of Computer and Mathematical Science, Vol 5, Issue 10, October 2016, P. 56–65.
[8] Sreekumar K.G. and Dr. Manilal K, "Some parameters of SM family of Graphs", International Organisation of Scientific Research-Mathematics journal, Vol 12, issue 6, Version 2(Nov-Dec 2016), P. 07–13.