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Some identities of symmetry for q-Bernoulli polynomials under symmetric group S_3

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Abstract

In this paper, we consider the q-Bernoulli polynomials which are slightly different from Carlitz q-Bernoulli polynomials and give new identities of symmetry in three variables.

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1. Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$ and let q be an indeterminate such that $|1-q|_p < p^{-\frac{1}{p-1}}$. By $UD(\mathbb{Z}_p)$ we denote the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [3,4,6])}. \tag{1}$$

From (1), we note that

$$I_0(f_1) = I_0(f) + f'(0), \quad \text{where } f_1(x) = f(x+1). \tag{2}$$

In general, by (2), we get

$$I_0(f_n) - I_0(f) = \sum_{l=0}^{n-1} f'(l), \quad (n \geq 1), \tag{3}$$

where $f_n(x) = f(x+n)$, (see [4,6]).

As is well knwon, the q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. The ordinary Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{4}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [1-12]). When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. From (2), we can derive the following equation [1,3,9]:

$$\int_{\mathbb{Z}_p} e^{xt}d\mu_0(x) = \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \tag{5}$$

and

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{6}$$

With the viewpoint of (6), the modified Carlitz's q-Bernoulli polynomials are defined by Kim to be

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_0(y) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \text{ (cf. [4,6]).} \tag{7}$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the modified Carlitz's q-Bernoulli numbers (or the q-Bernoulli numbers of the second kind).

From (2), we note that

$$B_{0,q} = 1, \quad (qB_q + 1)^n - B_{n,q} = \begin{cases} \dfrac{\log q}{q-1}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \tag{8}$$

with the usual convention about replacing B_q^n by $B_{n,q}$ (see [4,6]).

The purpose of this paper is to give some new identities of symmetry in three variables for the modified Carllitz's q-Bernoulli polynomials which are derived from the p-adic invariant integrals on $\mathbb{Z}_p$.

2. Some identities of the modified q-Bernoulli polynomials

Let w_1, w_2, w_3 be integers with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$. Then by (1), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t} d\mu_0(y) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t} \\ &= \lim_{N\to\infty} \frac{1}{w_1 p^N} \sum_{k=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_2 w_3 (k + w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t}. \end{aligned} \tag{9}$$

Thus, from (9), we have

$$\begin{aligned} &\frac{1}{w_2 w_3} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t} d\mu_0(y) \\ &= \lim_{N\to\infty} \frac{1}{w_1 w_2 w_3 p^N} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_1-1} \sum_{y=0}^{p^N-1} e^{[w_2 w_3 (k + w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t}. \end{aligned} \tag{10}$$

By the same method as (10), we see that

$$\begin{aligned}&\frac{1}{w_3w_1}\sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1}\int_{\mathbb{Z}_p} e^{[w_3w_1y+w_2w_3w_1x+w_2w_1i+w_2w_3j]_q t}d\mu_0(y)\\&=\lim_{N\to\infty}\frac{1}{w_1w_2w_3p^N}\sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1}\sum_{k=0}^{w_2-1}\sum_{y=0}^{p^N-1} e^{[w_3w_1(k+w_2y)+w_2w_3w_1x+w_2w_1i+w_2w_3j]_q t}.\end{aligned} \tag{11}$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2.1. Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$. Then the following expressions

$$\frac{1}{w_{\sigma(2)}w_{\sigma(3)}}\sum_{i=0}^{w_{\sigma(2)}-1}\sum_{j=0}^{w_{\sigma(3)}-1}\int_{\mathbb{Z}_p} e^{[w_{\sigma(2)}w_{\sigma(3)}y+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}x+w_{\sigma(1)}w_{\sigma(3)}i+w_{\sigma(1)}w_{\sigma(2)}j]_q t}d\mu_0(y)$$

are the same for any $\sigma \in S_3$.

From (7), we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p} e^{[w_2w_3y+w_1w_2w_3x+w_1w_3i+w_1w_2j]_q t}d\mu_0(y)\\&=\sum_{n=0}^{\infty}[w_2w_3]_q^n\int_{\mathbb{Z}_p}\left[y+w_1x+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right]_{q^{w_2w_3}}^n d\mu_0(y)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}[w_2w_3]_q^n B_{n,q^{w_2w_3}}\left(w_1x+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\frac{t^n}{n!}.\end{aligned} \tag{12}$$

Therefore by Theorem 2.1 and (12), we obtain the following theorem.

Theorem 2.2. Let $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$. For any nonnegative integer n, the following expressions

$$\frac{[w_{\sigma(2)}w_{\sigma(3)}]_q^n}{w_{\sigma(2)}w_{\sigma(3)}}\sum_{i=0}^{w_{\sigma(2)}-1}\sum_{j=0}^{w_{\sigma(3)}-1}B_{n,q^{w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(1)}x+\frac{w_{\sigma(1)}}{w_{\sigma(2)}}i+\frac{w_{\sigma(1)}}{w_{\sigma(3)}}j\right)$$

are the same for any $\sigma \in S_3$.

Now, we observe that

$$\begin{aligned} &\left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right]_{q^{w_2 w_3}} \\ &= \frac{1-q^{w_1}}{1-q^{w_2 w_3}} \frac{1-q^{w_1(w_3 i + w_2 j)}}{1-q^{w_1}} + q^{w_1 w_3 i + w_1 w_2 j}[y + w_1 x]_{q^{w_2 w_3}} \\ &= \frac{[w_1]_q}{[w_2 w_3]_q}[w_3 i + w_2 j]_{q^{w_1}} + q^{w_1 w_3 i + w_1 w_2 j}[y + w_1 x]_{q^{w_2 w_3}}. \end{aligned} \tag{13}$$

Thus, by (13), we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} \left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right]^n_{q^{w_2 w_3}} d\mu_0(y) \\ &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{[w_1]_q}{[w_2 w_3]_q}\right)^{n-k} [w_3 i + w_2 j]^{n-k}_{q^{w_1}} q^{k(w_1 w_3 i + w_1 w_2 j)} B_{k, q^{w_2 w_3}}(w_1 x). \end{aligned} \tag{14}$$

From (10) and (14), we have

$$\begin{aligned} &\frac{[w_2 w_3]^n_q}{w_2 w_3} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \int_{\mathbb{Z}_p} \left[y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right]^n_{q^{w_2 w_3}} d\mu_0(y) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{[w_2 w_3]^k_q}{w_2 w_3} [w_1]^{n-k}_q B_{k, q^{w_2 w_3}}(w_1 x) \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{k(w_1 w_3 i + w_1 w_2 j)} [w_3 i + w_2 j]^{n-k}_{q^{w_1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{[w_2 w_3]^k_q}{w_2 w_3} [w_1]^{n-k}_q B_{k, q^{w_2 w_3}}(w_1 x) K_{n, q^{w_1}}(w_2, w_3 | k), \end{aligned} \tag{15}$$

where

$$K_{n,q}(w_1, w_2 | k) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{(w_2 i + w_1 j)k} [w_2 i + w_1 j]^{n-k}_q. \tag{16}$$

As this expression is invariant under any permutations in w_1, w_2, w_3, we have the following theorem.

Theorem 2.3. Let w_1, w_2, w_3 be any positive integers. Then for any nonnegative integer n, the following expressions

$$\sum_{k=0}^{n} \binom{n}{k} \frac{[w_{\sigma(2)} w_{\sigma(3)}]^k_q}{w_{\sigma(2)} w_{\sigma(3)}} [w_{\sigma(1)}]^{n-k}_q B_{k, q^{w_{\sigma(2)} w_{\sigma(3)}}}(w_{\sigma(1)} x) K_{n, q^{w_{\sigma(1)}}}(w_{\sigma(2)}, w_{\sigma(3)} | k)$$

are all the same for any $\sigma \in S_3$.

Remark 2.4. Using (16) and by specializing $w_3 = 1$ or $w_2 = w_3 = 1$, we can obtain many interesting identities. However, as this requires much space, we will omit those.

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