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Common Properties, Implicit Function, Rational Contractive Map and Related Common Fixed Point Theorems in Multiplicative Metric Space

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Abstract

The purpose of this paper is to prove common fixed point theorems using rational contraction for weekly compatible mappings along with property (E.A), common limit range properties satisfying implicit functions in the setup of multiplicative metric space.

Keyword: Rational contractive maps, property (E.A), common limit range properties, contractive map, common fixed point, multiplicative metric space.
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1. INTRODUCTION AND PRELIMINARIES

Common Fixed point theorems satisfying certain contraction conditions has many applications and has been at the Centre of various research activities also it is the key concepts to solve many problems of calculus. It is well known that set of all positive real numbers is not complete with respect to usual metric but it is observed that set of all real numbers is a complete multiplicative metric space with respect to the multiplicative absolute value function. This problem was solved in 2008, by Bashirov [1] by introducing the new metric space named Multiplicative Metric Space. The theory of convergence in multiplicative metric space and related fixed point theorems in multiplicative metric space was introduced by Özavsar and Cevikel initiated [6].
Let us start with topological definitions of multiplicative metric space. The notations $\mathbb{R}$ and $\mathbb{R}_+$ are used to represent the set of real numbers and the set of all positive real numbers respectively.

Definition 1.1. [6] Let X be a nonempty set. A multiplicative metric is a mapping $d: X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(1.1) $d(x,y) \geq 1\ \forall\ x,y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(1.2) $d(x,y) = d(y,x)\ \forall\ x, y \in X$;

(1.3) $d(x,y) \leq d(x,z)+d(z,y)\ x, y \in X$ (multiplicative triangle inequality).

Definition 1.2. [6] Let (X, d) be a multiplicative metric space. Then a sequence $\{x_n\}$ in X is said to be

(1.4) a *multiplicative convergent* to x if for every multiplicative open ball

$$B_\varepsilon(x)=\{\ y \mid d(\ x\ ,y\ ) \leq \varepsilon, \varepsilon > 1\ ,$$

there exists a natural number N such that $n \geq N$, *then* $x_n \in B_\varepsilon(x)$, *i.e.* $d(x_n, x) \to 1$ as $n \to \infty$.

(1.5) *a multiplicative Cauchy sequence* if $\forall \varepsilon > 1$ for all m, n >N, that is, $d(x_n, x_m) \to 1$ as $n \to \infty$.

(1.6) we call a multiplicative metric space *complete* if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

The concept of multiplicative contraction was given by Özavsar and Cevikel [6] in 2012, some fixed point theorem were also proved.

Definition 1.3. [6] Let f be a mapping of a multiplicative metric space (X, d) into itself. Then f is said to be a *multiplicative* contraction if there exist a real constant $\omega \in [0,1)$ such that

$d(fx,fy) \leq d^{\,\omega}\,(x, y)$ for all $x,y \in X$.

Definition 1.4. [2] Let f and g be two mappings of a multiplicative metric space (X, d) into itself, then f and g are said to be

(1.7) *commutative mapping* if $fgx=gfx\ \forall x \in X$.

(1.8) *Weak commutative mapping* if $d(fgx, gfx) \leq d(fx, gx)\ \forall x \in X$.

(1.9) *Weakly compatible* if f and g commute at coincidence points, that is, $ft=gt$ for some $t \in X$. Implies that $fgt = gft$.

(1.10) *E.A property* if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

(1.11) CLR_g *property* (common limit range of g property)if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$ for some $t \in X$.

(1.12) CLR_f *property* (common limit range of f property)if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = ft$ for some $t \in X$.

2. MAIN RESULTS

Implicit functions was used by Popa [9], it was a good contractive condition in multiplicative metric space. Many authors have been using the concept of multiplicative metric space e.g. [see 5, 7, 10]. To prove the main result we define a suitable class of the implicit function involving three real non-negative arguments as follows:

Let Ψ is family of functions such that $\phi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ is continuous and increasing in each coordinate variable and

i. $\phi\,(t.t_1, 1, t) \leq t.t_1$
ii. $\phi\,(1, t.t_1, t_1) \leq t.t_1$
iii. $\phi\,(t, 1, 1) \leq t$
iv. $\phi\,(1, t, 1) \leq t$
v. $\phi\,(t, t, 1) \leq t$

for every $t, t_1 \in \mathbb{R}_+ (t, t_1 \geq 1)$. It is obvious that $\phi\,(1, 1, 1) = 1$. There exist many functions$\phi \in \Psi$.

Now we prove the following theorems for weakly compatible mappings satisfying implicit function in a multiplicative metric space.

Theorem 2.1. *Let A, B, γ, θ be self mappings of a multiplicative metric space (X, d) such that*

(E1) $\gamma X \subset BX$ *and* $\theta X \subset AX$

(E2) $$d(\gamma x, \theta\, y) \leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(Ax,By)[d(\theta\, y,\gamma x)+d(Ax,\theta\, y)]}{d(By,Ax)+d(\gamma x,By)}, \\ \dfrac{d(\theta\, y,\gamma x)[d(By,Ax)+d(\gamma x,By)]}{d(\theta\, y,\gamma x)+d(Ax,\theta\, y)}, \dfrac{d(Ax,\gamma x)[d(By,\theta\, y)+d(By,Ax)]}{d(Ax,\gamma x)+d(\theta\, y,\gamma x)} \end{matrix} \right\} \right\}^{\omega}$$

for all x, y $\in$ *X, where* $\omega \in (0, \frac{1}{2})$ *and* $\phi \in \Psi$;

(E3) let us suppose that the pairs (A, γ) and (B, θ) are weakly compatible;

(E4) One of the subspaces AX or BX or γ X or θX is complete

Then A, B, and have a unique common fixed point.

Proof. Let x_0 be any arbitrary point of metric space X. since γ X $\subset BX$, hence $\exists\, x_1 \in X$ such that $\gamma x_0 = Bx_1 = y_0$, for this $x_1\ \exists\, x_2 \in X$ such that $Ax_2 = \theta\, x_1 = y_1$. Similarly, an inductive sequence $\{ y_n \}$ can be defined in such a way that,

$\gamma x_{2n} = Bx_{2n+1} = y_{2n}$, $Ax_{2n+2} = \theta\, x_{2n+1} = y_{2n+1}$.

Next, we prove that $\{ y_n \}$ is a multiplicative sequence in *X*. in fact, $\forall\, n \in \mathbb{N}$, we have, From *(E2),* we haves

$d(y_{2n}, y_{2n+1}) \leq d(\gamma x_{2n}, \theta x_{2n+1})$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Ax_{2n}, Bx_{2n+1})[d(\theta x_{2n+1}, \gamma x_{2n}) + d(Ax_{2n}, \theta x_{2n+1})]}{d(Bx_{2n+1}, Ax_{2n}) + d(\gamma x_{2n}, Bx_{2n+1})}, \\ \dfrac{d(\theta x_{2n+1}, \gamma x_{2n})[d(Bx_{2n+1}, Ax_{2n}) + d(\gamma x_{2n}, Bx_{2n+1})]}{d(\theta x_{2n+1}, \gamma x_{2n}) + d(Ax_{2n}, \theta x_{2n+1})} \\ \dfrac{d(Ax_{2n}, \gamma x_{2n})[d(Bx_{2n+1}, \theta x_{2n+1}) + d(Bx_{2n+1}, Ax_{2n})]}{d(Ax_{2n}, \gamma x_{2n}) + d(\theta x_{2n+1}, \gamma x_{2n})} \end{array} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})]}{d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n})}, \\ \dfrac{d(y_{2n+1}, y_{2n})[d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n})]}{d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})}, \\ \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})]}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})} \end{array} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1})]}{d(y_{2n}, y_{2n-1}) + 1}, \\ \dfrac{d(y_{2n+1}, y_{2n})[d(y_{2n}, y_{2n-1}) + 1]}{d(y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1})}, \\ \dfrac{d(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1})]}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})} \end{array} \right\} \right\}^{\omega}$$

$$\leq \{\phi\{d(y_{2n-1}, y_{2n}).d(y_{2n}, y_{2n+1}), 1, d(y_{2n-1}, y_{2n})\}\}^{\omega}$$

$d(y_{2n}, y_{2n+1}) \leq d^{\omega}(y_{2n-1}, y_{2n}).d^{\omega}(y_{2n}, y_{2n+1})$

This implies that,

$d(y_{2n}, y_{2n+1}) \leq d^{\frac{\omega}{1-\omega}}(y_{2n-1}, y_{2n})$

On substituting, h = $\frac{\omega}{1-\omega} \in (0, \frac{1}{2})$

$d(y_{2n}, y_{2n+1}) \leq d^{h}(y_{2n-1}, y_{2n})$,

In a similar way we have,

$d(y_{2n+1}, y_{2n+2}) = d(Tx_{2n+1}, \gamma x_{2n+2}) = d(\gamma x_{2n+2}, Tx_{2n+1})$

$$\leq \left\{\phi\left\{\begin{array}{c}\frac{d(Ax_{2n+2},Bx_{2n+1})[d(Tx_{2n+1},\gamma x_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n+2})+d(\gamma x_{2n+2},Bx_{2n+1})},\\ \frac{d(Tx_{2n+1},\gamma x_{2n+2})[d(Bx_{2n+1},Ax_{2n+2})+d(\gamma x_{2n+2},Bx_{2n+1})]}{d(Tx_{2n+1},\gamma x_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})},\\ \frac{d(Ax_{2n+2},\gamma x_{2n+2})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})]}{d(Ax_{2n+2},\gamma x_{2n+2})+d(Tx_{2n+1},\gamma x_{2n+2})}\end{array}\right\}\right\}^{\omega}$$

$$\leq \left\{\phi\left\{\begin{array}{c}\frac{d(y_{2n+1},y_{2n})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})]}{d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})},\\ \frac{d(y_{2n+1},y_{2n+2})[d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})},\\ \frac{d(y_{2n+1},y_{2n+2})[d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+2})}\end{array}\right\}\right\}^{\omega}$$

$$\leq \left\{\phi\{1, d(y_{2n+1},y_{2n+2}).d(y_{2n+1},y_{2n}), d(y_{2n},y_{2n+1})\}\right\}^{\omega}$$

$d(y_{2n+1},y_{2n+2}) \leq d^{\omega}(y_{2n},y_{2n+1}).d^{\omega}(y_{2n+1},y_{2n+2})$

This implies that,

$d(y_{2n+1},y_{2n+2}) \leq d^{\frac{\omega}{1-\omega}}(y_{2n},y_{2n+1})$

On substituting, h = $\frac{\omega}{1-\omega} \in (0,\frac{1}{2})$

$d(y_{2n+1},y_{2n+2}) \leq d^{h}(y_{2n},y_{2n+1})$,

Hence

$$d(y_n,y_{n+1}) \leq d^{h^1}(y_{n-1},y_n) \leq d^{h^2}(y_{n-2},y_{n-1}) \leq \cdots \leq d^{h^n}(y_0,y_1)$$

For all n≥2, let m,n∈ℕ such that m≥ n. Using the triangular multiplicative inequality, we obtain

$$\begin{aligned} d(y_m,y_n) &\leq d(y_m,y_{m-1}).d(y_{m-1},y_{m-2}) \ldots d(y_{n+1},y_n) \\ &\leq d^{h^{m-1}}(y_1,y_0).d^{h^{m-2}}(y_1,y_0) \ldots d^{h^n}(y_1,y_0) \\ &\leq d^{\frac{h^n}{1-h}}(y_1,y_0) \end{aligned}$$

This implies that $d(y_m,y_n) \to 1$ as n and m → ∞, therefore $\{y_n\}$ is a multiplicative Cauchy sequence in X.

Now, suppose that AX is complete, there exist u ∈ AX such that

$$y_{n+1} = \theta\ x_{2n+1} = A\,x_{2n+2} \to u \quad (n \to \infty).$$

Consequently, we can find $v \in X$ such that *Av=u.* Further a multiplicative Cauchy sequence$\{ y_n \}$ has a convergent subsequence $\{ y_{2n+1} \}$, therefore the sequence $\{ y_n \}$ converges and hence a subsequence $\{ y_{2n} \}$ also converges. Thus we have,
$y_{2n} = \gamma\, x_{2n} = B\, x_{2n+1} \to u (n \to \infty)$.

We claim that γ *v=u.* if possible γ *v≠ u*, substituting *x=v* and *y=*x_{2n+1} in *(E2),* we have

$$d(\gamma v, \theta\, x_{2n+1}) \le \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Bx_{2n+1})[d(\theta\, x_{2n+1}, \gamma v) + d(Av, \theta\, x_{2n+1})]}{d(Bx_{2n+1}, Av) + d(\gamma v, Bx_{2n+1})}, \\ \frac{d(\theta\, x_{2n+1}, \gamma v)[d(Bx_{2n+1}, Av) + d(\gamma v, Bx_{2n+1})]}{d(\theta\, x_{2n+1}, \gamma v) + d(Av, \theta\, x_{2n+1})}, \\ \frac{d(Av, \gamma v)[d(Bx_{2n+1}, \theta\, x_{2n+1}) + d(Bx_{2n+1}, Av)]}{d(Av, \gamma v) + d(\theta\, x_{2n+1}, \gamma v)} \end{array} \right\} \right\}^{\omega}$$

Taking $n \to \infty$, On the two sides of the above inequality,

$$d(\gamma v, u) \le \left\{ \phi \left\{ \begin{array}{c} \frac{d(u,u)[d(u,\gamma v) + d(u,u)]}{d(u,u) + d(\gamma v, u)}, \frac{d(u,\gamma v)[d(u,u) + d(\gamma v, u)]}{d(u,\gamma v) + d(u,u)}, \\ \frac{d(u,\gamma v)[d(u,u) + d(u,u)]}{d(u,\gamma v) + d(u,\gamma u)} \end{array} \right\} \right\}^{\omega}$$

$$\le \{\phi\{1, d(u,\gamma v), 1\}\}^{\omega}$$

$d(u,\gamma v) \le d^{\omega}(u,\gamma v)$, a contradiction, since $\omega \in (0, \frac{1}{2})$
This implies that, *d(γ v,u) = 1* implies γ *v = u.*
Since *u =* γ *v* ∈ $\gamma X \subset BX$, there exist *w* ∈ *X* such that *u=Bw.*
Claim that θ*w=u*, if possible *Tw≠ u.* substituting *x=v* and *y=w* in *(E2),* we have
d(u, θ*w) = d(γ v,* θ*w)*

$$\le \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Bw)[d(\theta\, w, \gamma v) + d(Av, \theta\, w)]}{d(Bw, Av) + d(\gamma v, Bw)}, \\ \frac{d(\theta\, w, \gamma v)[d(Bw, Av) + d(\gamma v, Bw)]}{d(\theta\, w, \gamma v) + d(Av, \theta\, w)}, \frac{d(Av, \gamma v)[d(Bw, \theta\, w) + d(Bw, Av)]}{d(Av, \gamma v) + d(\theta\, w, \gamma v)} \end{array} \right\} \right\}^{\omega}$$

As *Av = u =* γ *v = Bw,* we have

$$\leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(u,u)[d(\theta\, w,u)+d(u,\theta\, w)]}{d(u,u)+d(u,u)} \\ \dfrac{d(\theta\, w,u)[d(u,u)+d(u,u)]}{d(\theta\, w,u)+d(u,\theta\, w)}, \dfrac{d(u,u)[d(u,\theta\, w)+d(u,u)]}{d(u,u)+d(\theta\, w,u)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \{\phi\{d(\theta\, w,u),1,1\}\}^{\omega}$$

$$d(u,\theta\, w) \leq d^{\omega}(u,Tw)$$

A contradiction, since $\omega \in (0, \frac{1}{2})$ implies $u = \theta w$.

Hence we get $u=Av=\gamma\, v$, that is, v is a coincidence point of A, γ, also $u=Bw=\theta w$, that is w is coincidence point of B and θ. Therefore

$$Av=\gamma\, v=Bw=\theta w=u.$$

Since the pairs A, γ and B, are weakly compatible, we have

$$\gamma\, u=\gamma\,(Av)=A(\gamma\, v)=Au=w_1 \text{ (say)}$$

And

$$\theta u=\theta(Bw)=B(\theta w)=Bu=w_2 \text{ (say)}$$

From *(E2)*, we have

$$d(w_1,w_2) = d(\gamma\, u,vu)$$

$$\leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(Au,Bu)[d(\theta\, u,\gamma u)+d(Au,\theta\, u)]}{d(Bu,Au)+d(\gamma u,Bu)}, \\ \dfrac{d(\theta\, u,\gamma u)[d(Bu,Au)+d(\gamma u,Bu)]}{d(\theta\, u,\gamma u)+d(Au,\theta\, u)}, \dfrac{d(Au,\gamma u)[d(Bu,\theta\, u)+d(Bu,Au)]}{d(Au,\gamma u)+d(\theta\, u,\gamma u)} \end{matrix} \right\} \right\}^{\omega}$$

Using symmetry and above conditions of w_1 *and* w_2 we have

$$\leq \left\{ \phi \left\{ \begin{matrix} \dfrac{d(w_1,w_2)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)} \\ \dfrac{d(w_2,w_1)[d(w_2,w_1)+d(w_1,w_2)]}{d(w_2,w_1)+d(w_1,w_2)}, \dfrac{d(w_1,w_1)[d(w_2,w_2)+d(w_2,w_1)]}{d(w_1,w_1)+d(w_2,w_1)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \{\phi\{d(w_1,w_2),d(w_1,w_2),1\}\}^{\omega}$$

$$d(w_1,w_2) \leq d^{\omega}(w_1,w_2)$$

on the other hand, since $\omega \in (0, \frac{1}{2})$ implies, $d(w_1, w_2) = 1$, which implies that $w_1 = w_2$ and hence we have

$$\gamma\, u = Au = \theta u = Bu.$$

Again using (E2) and symmetry of multiplicative metric space we have,

$$d(\gamma\, v,\, \theta u) \leq \left\{ \phi \left\{ \begin{matrix} \frac{d(Av,Bu)[d(\theta\, u,\gamma v)+d(Av,\theta\, u)]}{d(Bu,Av)+d(\gamma v,Bu)}, \\ \frac{d(\theta\, u,\gamma v)[d(Bu,Av)+d(\gamma v,Bu)]}{d(\theta\, u,\gamma v)+d(Av,\theta\, u)}, \frac{d(Av,\gamma v)[d(Bu,\theta\, u)+d(Bu,Av)]}{d(Av,\gamma v)+d(\theta\, u,\gamma v)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{matrix} \frac{d(\gamma v,\theta\, u)[d(\gamma v,\gamma v)+d(\theta\, u,\gamma v)]}{d(\theta\, u,\theta\, u)+d(\theta\, u,\gamma v)}, \frac{d(\gamma v,\theta\, u)[d(\theta\, u,\gamma v)+d(\gamma v,\theta\, u)]}{d(\theta\, u,\gamma v)+d(\gamma v,\theta\, u)} \\ \frac{d(Tu,Sv)[d(Tu,Sv)+d(Sv,Tu)]}{d(Tu,Sv)+d(Sv,Tu)}, \frac{d(\gamma v,\gamma v)[d(\theta\, u,\theta\, u)+d(\theta\, u,\gamma v)]}{d(\gamma v,\gamma v)+d(\theta\, u,\gamma v)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \{\phi\{d(\gamma v,\theta\, u), d(\gamma v,\theta\, u), 1\}\}^{\omega}$$

$d(\gamma v,\theta\, u) \leq d^{\omega}(\gamma v,\theta\, u)$, on the other hand, since $\omega \in (0, \frac{1}{2})$ implies $d(\gamma v,\theta\, u) = 1$ $i.e. \gamma v = \theta\, u$ *(u= θ u)* and hence we have *u=γ u=Au= θ u=Bu.* Therefore u is a common fixed point of A, B, γ and , Similarly, we can complete the proof for the different case in which BX or θX or γ X is comsplete.

UNIQUENESS

Let p and q are two different common fixed points of *A., B, γ, θ* then using symmetry of multiplicative metric space and using equation *(E2)*

$$d(p,q) = d(\gamma p,\theta\, q)$$

$$\leq \left\{ \phi \left\{ \begin{matrix} \frac{d(Ap,Bq)[d(\theta\, q,\gamma p)+d(Ap,\theta\, q)]}{d(Bq,Ap)+d(\gamma p,Bq)}, \\ \frac{d(\theta\, q,\gamma p)[d(Bq,Ap)+d(\gamma p,Bq)]}{d(\theta\, q,\gamma p)+d(Ap,\theta\, q)}, \frac{d(Ap,\gamma p)[d(Bq,\theta\, q)+d(Bq,Ap)]}{d(Ap,\gamma p)+d(\theta\, q,\gamma p)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{matrix} \frac{d(p,q)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)}, \\ \frac{d(q,p)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)}, \frac{d(p,p)[d(q,q)+d(q,p)]}{d(p,p)+d(q,p)} \end{matrix} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{matrix} d(p,q) \\ d(p,q), 1 \end{matrix} \right\} \right\}^{\omega}$$

$d(p,q) \leq d^{\omega}(p,q)$,

on the other hand, since $\omega \in (0,\frac{1}{2})$ implies $d(p,q) = 1$ $i.e.$ $p = q$, which proves the uniqueness.

Corollary 2.2. *Let A, B, γ be mappings of a multiplicative metric space (X, d) into itself satisfying*

(E5) $\gamma X \subset BX$ *and* $\gamma X \subset AX$

(E6) $$d(\gamma x, \gamma y) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax,By)[d(\gamma y,\gamma x)+d(Ax,\gamma y)]}{d(By,Ax)+d(\gamma x,By)}, \\ \frac{d(\gamma y,\gamma x)[d(By,Ax)+d(\gamma x,By)]}{d(\gamma y,\gamma x)+d(Ax,\gamma y)}, \frac{d(Ax,\gamma x)[d(By,\gamma y)+d(By,Ax)]}{d(Ax,\gamma x)+d(\gamma y,\gamma x)} \end{array} \right\} \right\}^{\omega}$$

For all x, y $\in$ *X, where* $\omega \in (0,\frac{1}{2})$ *and* $\phi \in \Psi$;

(E3) let us suppose that the pairs (A, γ) and (B, γ) are weakly compatible;

(E4) One of the subspaces AX or BX or γ X is complete

Then A, B and γ have a unique common fixed point.

In theorem 2.1., if we put $\theta = \gamma$, then we obtain the corollary 2.2

Corollary 2.3.*Let γ and be mappings of a multiplicative metric space (X, d) into itself satisfying*

(E7) $$d(\gamma x, \theta y) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(x,y)[d(\theta y,\gamma x)+d(x,\theta y)]}{d(y,x)+d(\gamma x,y)}, \\ \frac{d(\theta y,\gamma x)[d(y,x)+d(\gamma x,y)]}{d(\theta y,\gamma x)+d(x,\theta y)}, \frac{d(x,\gamma x)[d(y,\theta y)+d(y,x)]}{d(x,\gamma x)+d(\theta y,\gamma x)} \end{array} \right\} \right\}^{\omega}$$

For all x,y $\in$ *X, where* $\omega \in (0,\frac{1}{2})$ *and* $\phi \in \Psi$,

(E8) One of the subspaces γ X or θX is complete

Then γ and have a unique common fixed point.

In theorem 2.1., if we put A=B=1, then we obtain the corollary 2.3.

Theorem 2.4. *Let A, B, γ, θ be mappings of a multiplicative metric space (X, d) into itself satisfying the conditions (E1), (E2), (E3) and the following conditions:*

(E9) one of the subspaces AX or BX or γ X or θX is closed subset of X

(E10) the pairs (A, γ) and (B, θ) satisfy the E.A. property.

Then A, B, γ, θ have a unique common fixed point.

Proof. Suppose that the pairs *A*, γ satisfies the E.A property. Then $\exists$ a sequence $\{x_n\}$ In X such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} \gamma x_n = z$ for some $z \in X$. Since $SX \subset BX$, $\exists$ a sequence $\{y_n\}$ in X such that γx_n =B y_n

Hence

$$\lim_{n\to\infty} B\, y_n = \text{z}$$

Now suppose that *BX* is closed subset of X, ∃ a point u∈ *X* such that Bu=z.

We will show that $\lim_{n\to\infty} \theta\ y_n$ = z, from inequality *(E2)*, we have

$$d(\gamma x_n\ ,\ \theta\ y_n\ ) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n\ ,By_n\ )[d(\ \theta\ y_n\ ,\gamma x_n\ )+d(Ax_n\ ,\ \theta\ y_n\ )]}{d(By_n\ ,Ax_n\ )+d(\gamma x_n\ ,By_n\ )}, \\ \frac{d(\ \theta\ y_n\ ,\gamma x_n\ )[d(By_n\ ,Ax_n\ )+d(\gamma x_n\ ,By_n\ )]}{d(\ \theta\ y_n\ ,\gamma x_n\ )+d(Ax_n\ ,\ \theta\ y_n\ )}, \\ \frac{d(Ax_n\ ,\gamma x_n\ )[d(By_n\ ,\ \theta\ y_n\ )+d(By_n\ ,Ax_n\ )]}{d(Ax_n ,\gamma x_n\ )+d(\ \theta\ y_n\ ,\gamma x_n\ )} \end{array} \right\} \right\}^{\omega}$$

Taking n approaches to infinity and using the symmetry property of multiplicative metric space, we have

$$d\left(z, \lim_{n\to\infty} \theta\ y_n\ \right) \leq \left\{\phi\left\{d\left(\lim_{n\to\infty} \theta\ y_n\ \ ,z\right), 1,1\right\}\right\}^{\omega}$$

$$d\left(z, \lim_{n\to\infty} \theta\ y_n\ \right) \leq\ d^{\,\omega}\left(z, \lim_{n\to\infty} \theta\ y_n\ \right),$$

[using (iii)]

since $\omega\ \in (0,\frac{1}{2})$ implies $d\left(z, \lim_{n\to\infty} \theta\ y_n\ \right) = 1\ i.e. z = \lim_{n\to\infty} \theta\ y_n$.

Thus we have

$$\lim_{n\to\infty} A\, x_n\ = \lim_{n\to\infty} B\, x_n\ = \lim_{n\to\infty} \gamma\, x_n\ = \lim_{n\to\infty} \theta\ x_n\ = z = Bu$$

(say)

for some *u* in *X.*

substituting *x=* x_n and *y=u* in *(E2),* we have

$$d(\gamma x_n\ ,\theta\ u) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Ax_n\ ,Bu)[d(\theta\ u,\gamma x_n\ )+d(Ax_n\ ,\theta\ u)]}{d(Bu,Ax_n\ )+d(\gamma x_n\ ,Bu)}, \\ \frac{d(\theta\ u,\gamma x_n\ )[d(Bu,Ax_n\ )+d(\gamma x_n\ ,Bu)]}{d(\theta\ u,\gamma x_n\ )+d(Ax_n\ ,\theta\ u)}, \\ \frac{d(Ax_n\ ,\gamma x_n\ )[d(Bu,\theta\ u)+d(Bu,Ax_n\ )]}{d(Ax_n\ ,\gamma x_n\ )+d(\theta\ u,\gamma x_n\ )} \end{array} \right\} \right\}^{\omega}$$

Taking n→ ∞, we have

$d(Bu, \theta\ u)$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Bu, Bu)[d(\theta\ u, Bu) + d(Bu, \theta\ u)]}{d(Bu, Bu) + d(Bu, Bu)}, \\ \frac{d(\theta\ u, Bu)[d(Bu, Bu) + d(Bu, Bu)]}{d(\theta\ u, Bu) + d(Bu, \theta\ u)}, \frac{d(Bu, Bu)[d(Bu, \theta\ u) + d(Bu, Bu)]}{d(Bu, Bu) + d(\theta\ u, Bu)} \end{array} \right\} \right\}^{\omega}$$

$$d(Bu, \theta\ u) \leq \left\{\phi\{d(Bu, \theta\ u), 1,1\}\right\}^{\omega}$$

$d(Bu, \theta\, u) \leq\ d^{\,\omega}\,(Bu, \theta\, u)$

On the other hand, since $\omega\ \in (0, \frac{1}{2})$ it implies that $d(Bu, \theta\, u) = 1\ i.e. Bu = \theta\, u.$

It is given that B and θ are weakly compatible, we have $B\ \theta u = \theta Bu$ and then $BBu = \theta\theta u = B\ \theta u = \theta But$ also

$\theta X \subset AX$, $\exists\ v \in X$ such that $\theta u = Av$.

Next we claim that $Av = \gamma\ v$, if possible $Av \neq \gamma v$ putting $x=v$ and $y=u$, we have

$$d(\gamma v, \theta\, u) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Bu)[d(\theta\, u, \gamma v) + d(Av, \theta\, u)]}{d(Bu, Av) + d(\gamma v, Bu)}, \\ \frac{d(\theta\, u, \gamma v)[d(Bu, Av) + d(\gamma v, Bu)]}{d(\theta\, u, \gamma v) + d(Av, \theta\, u)}, \\ \frac{d(Av, \gamma v)[d(Bu, \theta\, u) + d(Bu, Av)]}{d(Av, \gamma v) + d(\theta\, u, \gamma v)} \end{array} \right\} \right\}^{\omega}$$

$$d(\gamma v, Av) \leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av,Av)[d(Av,\gamma v)+d(Av,Av)]}{d(Av,Av)+d(\gamma v,Av)}, \\ \frac{d(Av,\gamma v)[d(Av,Av)+d(\gamma v,Av)]}{d(Av,\gamma v)+d(Av,Av)}, \\ \frac{d(Av,\gamma v)[d(Av,Av)+d(Av,Av)]}{d(Av,\gamma v)+d(Av,\gamma v)} \end{array} \right\} \right\}^{\omega}$$

[since, $\theta u = Av$ and $Bu = Av$]

$d(\gamma v, Av) \leq \left\{\phi\{1, d(Av, \gamma v), 1\}\right\}^{\omega}$

$d\ (\gamma v, Av) \leq\ d^{\,\omega}\,(\gamma v, Av)$

Which is a contradiction because the range of ω, this implies that $d(\gamma v, Av) = 1\ i.e. \gamma v = Av.$

Hence we have, $Bu = \theta u = Av = \gamma\ v$. Since the pair A, γ are weakly compatible, we have $A\ \gamma\ v = \gamma\ Av$ and then $\gamma\ \gamma\ v = \gamma\ Av = A\ \gamma\ v = AAv$.

Next we claim that $\gamma\ Av = Av$, if possible $\gamma\ Av \neq Av$ on substituting $x = Av$ and $y = u$, we have

$$d(\gamma\ Av, Av) = d(\gamma Av, \theta\, u)$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(AAv, Bu)[d(\theta\, u, \gamma Av) + d(AAv, \theta\, u)]}{d(Bu, AAv) + d(\gamma Av, Bu)}, \\ \frac{d(\theta\, u, \gamma Av)[d(Bu, AAv) + d(\gamma Av, Bu)]}{d(\theta\, u, \gamma Av) + d(AAv, \theta\, u)}, \\ \frac{d(AAv, \gamma Av)[d(Bu, \theta\, u) + d(Bu, AAv)]}{d(AAv, \gamma Av) + d(\theta\, u, \gamma Av)} \end{array} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(SAv,Av)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(Av,SAv)[d(Av,SAv)+d(SAv,Av)]}{d(Av,SAv)+d(SAv,Av)}, \\ \frac{d(SAv,SAv)[d(Av,Av)+d(Av,SAv)]}{d(SAv,SAv)+d(Av,SAv)} \end{array} \right\} \right\}^{\omega}$$

$$[since\ AAv = \gamma Av, \theta\ u = Av = Bu = Av]$$

$$\leq \{\phi\{d(\gamma Av, Av), d(Av, \gamma Av), 1\}\}^{\omega}$$

$$d(\gamma Av, Av) \leq\ d^{\omega}(\gamma Av, Av)$$

This is a contradiction, since $\omega\ \in \left(0, \frac{1}{2}\right)$, hence

$Av = \gamma Av$, hence $\gamma\ Av = Av = AAv$. Hence Av is a common fixed point of A and γ. Similarly, $BBu=Bu= \theta Bu$, i.e., Bu is a common fixed point of B and θ as $Av=Bu$, Av is a common fixed point of A, B, γ and θ.

Similarly we can complete the proof for cases in which AX *or* θX *or* $\gamma\ X$ is closed subset of X.

Uniqueness

Let Av and Pu are two distinct common fixed points of A, B, γ and θ. Using symmetry of multiplicative metric space and *(E2)*, we have

$$d(Av, Pu) = d(\gamma Av, \theta\ Pu)$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(AAv, BPu)[d(\theta\ Pu, \gamma Av) + d(AAv, \theta Pu)]}{d(BPu, AAv) + d(\gamma Av, BPu)}, \\ \frac{d(\theta\ Pu, \gamma Av)[d(BPu, AAv) + d(\gamma Av, BPu)]}{d(\theta\ Pu, \gamma Av) + d(AAv, \theta\ Pu)}, \\ \frac{d(AAv, \gamma Av)[d(BPu, \theta\ Pu) + d(BPu, AAv)]}{d(AAv, \gamma Av) + d(\theta\ Pu, \gamma Av)} \end{array} \right\} \right\}^{\omega}$$

$$\leq \left\{ \phi \left\{ \begin{array}{c} \frac{d(Av, Pu)[d(Pu, Av) + d(Av, Pu)]}{d(Pu, Av) + d(Av, Pu)}, \\ \frac{d(Pu, Av)[d(Pu, Av) + d(Av, Pu)]}{d(Pu, Av) + d(Av, Pu)}, \\ \frac{d(Av, Av)[d(Pu, Pu) + d(Pu, Av)]}{d(Av, Av) + d(Pu, Av)} \end{array} \right\} \right\}^{\omega}$$

$$\leq \{\phi\{d(Av, Pu), d(Av, Pu), 1\}\}^{\omega}$$

$$d(Av, Pu) \leq\ d^{\omega}(Av, Pu)$$

A contradiction, since $\omega\ \in (0, \frac{1}{2})$, hence $d(Av, Pu) = 1\ i.e.\ Av = Pu.$

Now we prove the following theorems for weakly compatible mappings with common limit range property satisfying the implicit function in a multiplicative metric space.

Lemma 2.5.*Let A, B, γ, θ be mappings of a multiplicative metric space (X, d) satisfying the conditions (E1) and (E2) and the following condition:*
(E11) the pairs (A, γ) satisfies CLR$_A$ property or the pair (B,) satisfies CLR$_B$ property,
Then the pairs (A, γ) and (B, θ) share the common limit in the range of A property or B property.

Proof. Let us consider that the pair (A, γ) satisfies the common limit range of A property. Then $\exists$ a sequence $\{x_n\}$ in X such that

$$\operatorname*{Lim}_{n\to\infty} A\,x_n = \lim_{n\to\infty} \gamma\, x_n = Az \text{ for some } z \in X.$$

Since $\gamma X \subset BX$, so for each x_n there exists y_n in X such thst $\gamma\, x_n = B\, y_n$. Then $\lim_{n\to\infty} B\, y_n$ =Az. hence, we have

$$\lim_{n\to\infty} \gamma\, x_n = \lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, y_n = Az.$$

Now we claim that $\lim_{n\to\infty} \theta\, y_n$ =Az. On substituting $x = x_n$ *and* $y = y_n$ in *(E2)*, we have

$$d(\gamma x_n, \theta\, y_n) \le \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Ax_n, By_n)[d(\theta\, y_n, \gamma x_n) + d(Ax_n, \theta\, y_n)]}{d(By_n, Ax_n) + d(\gamma x_n, By_n)}, \\ \dfrac{d(\theta\, y_n, \gamma x_n)[d(By_n, Ax_n) + d(\gamma x_n, By_n)]}{d(\theta\, y_n, \gamma x_n) + d(Ax_n, \theta\, y_n)}, \\ \dfrac{d(Ax_n, \gamma x_n)[d(By_n, \theta\, y_n) + d(By_n, Ax_n)]}{d(Ax_n, \gamma x_n) + d(\theta\, y_n, \gamma x_n)} \end{array} \right\} \right\}^{\omega}$$

Taking $n \to \infty$, we have

$$d\left(Az, \lim_{n\to\infty} \theta\, y_n\right) \le \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Az, Az)\left[d\left(\lim_{n\to\infty} \theta\, y_n, Az\right) + d\left(Az, \lim_{n\to\infty} \theta\, y_n\right)\right]}{d(Az, Az) + d(Az, Az)}, \\ \dfrac{d\left(\lim_{n\to\infty} \theta\, y_n, Az\right)[d(Az, Az) + d(Az, Az)]}{d\left(\lim_{n\to\infty} \theta\, y_n, Az\right) + d\left(Az, \lim_{n\to\infty} \theta\, y_n\right)}, \\ \dfrac{d(Az, Az)[d\left(Az, \lim_{n\to\infty} \theta\, y_n\right) + d(Az, Az)]}{d(Az, Az) + d(\lim_{n\to\infty} \theta\, y_n, Az)} \end{array} \right\} \right\}^{\omega}$$

$$d\left(Az, \lim_{n\to\infty} \theta\, y_n\right) \le \left\{ \phi \left\{ d\left(\lim_{n\to\infty} \theta\, y_n\right), 1, 1 \right\} \right\}^{\omega}$$

$$d\left(Az, \lim_{n\to\infty} \theta\, y_n\right) \le d^{\,\omega}\left(Az, \lim_{n\to\infty} \theta\, y_n\right)$$

This is a contradiction, since $\omega \in \left(0,\frac{1}{2}\right)$, hence

$d\left(Az, \lim_{n\to\infty} \theta\, y_n\right) = 1.$ Therefore, $Az = \lim_{n\to\infty} \theta\, y_n$ and hence $\lim_{n\to\infty} \theta\, y_n = Az$.

$$\text{i.e. } \lim_{n\to\infty} \theta\, y_n = \lim_{n\to\infty} B\, y_n = \lim_{n\to\infty} \gamma\, x_n = \lim_{n\to\infty} A\, x_n = Az$$

Then the pairs (A, γ) and (B, θ) share the common limit range of A property.

for the other pair (B, θ) which shares common limit range of B property, Since the pair (B, θ) satisfies common limit range of B property. Then $\exists$ a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} B\, y_n = \lim_{n\to\infty} \theta\, y_n = Bz$$

for some $z \in X$. Since $\theta X \subset AX$, so for each $y_n \ \exists\ x_n$ in X such that $\theta\, y_n = A\, x_n$. Then $\lim_{n\to\infty} A\, x_n = Bz$. hence, we have

$$\lim_{n\to\infty} A\, x_n = \lim_{n\to\infty} B\, y_n = \lim_{n\to\infty} \theta\, y_n = Bz.$$

Now we claim that $\lim_{n\to\infty} \gamma\, x_n = Bz$. If possible $\lim_{n\to\infty} \gamma\, x_n \neq Bz$, On substituting $x = x_n$ and $y = y_n$ in *(E2)*, we have

$$d(\gamma x_n, \theta\, y_n) \le \left\{\phi\left\{\begin{array}{l}\dfrac{d(Ax_n, By_n)[d(\theta\, y_n, \gamma x_n) + d(Ax_n, \theta\, y_n)]}{d(By_n, Ax_n) + d(\gamma x_n, By_n)},\\ \dfrac{d(\theta\, y_n, \gamma x_n)[d(By_n, Ax_n) + d(\gamma x_n, By_n)]}{d(\theta\, y_n, \gamma x_n) + d(Ax_n, \theta\, y_n)},\\ \dfrac{d(Ax_n, \gamma x_n)[d(By_n, \theta\, y_n) + d(By_n, Ax_n)]}{d(Ax_n, \gamma x_n) + d(\theta\, y_n, \gamma x_n)}\end{array}\right\}\right\}^{\omega}$$

Taking $n \to \infty$, we have

$$d\left(\lim_{n\to\infty} \gamma x_n, Bz\right) \le \left\{\phi\left\{\begin{array}{l}\dfrac{d(Bz,Bz)\left[d\left(Bz, \lim_{n\to\infty}\gamma x_n\right) + d(Bz,Bz)\right]}{d(Bz,Bz) + d\left(\lim_{n\to\infty}\gamma x_n, Bz\right)},\\ \dfrac{d\left(Bz, \lim_{n\to\infty}\gamma x_n\right)\left[d(Bz,Bz) + d\left(\lim_{n\to\infty}\gamma x_n, Bz\right)\right]}{d\left(Bz, \lim_{n\to\infty}\gamma x_n\right) + d(Bz,Bz)},\\ \dfrac{d\left(Bz, \lim_{n\to\infty}\gamma x_n\right)[d(Bz,Bz) + d(Bz,Bz)]}{d\left(Bz, \lim_{n\to\infty}\gamma x_n\right) + d\left(Bz, \lim_{n\to\infty}\gamma x_n\right)}\end{array}\right\}\right\}^{\omega}$$

$$d\left(\lim_{n\to\infty} \gamma x_n, Bz\right) \le \left\{\phi\{1, d(Bz, \gamma x_{2n}), 1\}\right\}^{\omega}$$

$$d\left(\lim_{n\to\infty} \gamma x_n, Bz\right) \le d^{\omega}\left(\lim_{n\to\infty} \gamma x_n, Bz\right)$$

This is a contradiction, since $\omega \in \left(0,\frac{1}{2}\right)$, hence

Therefore, $\lim_{n\to\infty} \gamma\, x_n = Bz$.

Then the pairs (A, γ) and (B, θ) share the common limit range of B property.

Theorem 2.6. *let A, B, γ, θ be mappings of a multiplicative metric space (X, d) satisfying the conditions (E1) and (E2) and (E11)*

Then the pairs A, γ and B, have a coincidence point.

Moreover, assume that the pairs (A, γ) and (B, θ) are weakly compatible. Then A, B, γ and θ have a unique common fixed point.

Proof. Using the lemma (2.5), the pairs (A, γ) and (B, θ) share the common limit range of A property, that is there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} \theta\, y_n = \lim_{n\to\infty} B\, y_n = \lim_{n\to\infty} \gamma\, x_n = \lim_{n\to\infty} A\, y_n = Av$$

for some $v \in X$. First, we claim that $Av = \gamma\, v$, substituting $x=v$ and $y=y_n$ in *(E2)*, we have

$$d(\gamma v, \theta\, y_n) \le \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Av, By_n)[d(\theta\, y_n, \gamma v) + d(Av, \theta\, y_n)]}{d(By_n, Av) + d(\gamma v, By_n)}, \\ \dfrac{d(\theta\, y_n, \gamma v)[d(By_n, Av) + d(\gamma v, By_n)]}{d(\theta\, y_n, \gamma v) + d(Av, \theta\, y_n)}, \\ \dfrac{d(Av, \gamma v)[d(By_n, \theta\, y_n) + d(By_n, Av)]}{d(Av, \gamma v) + d(\theta\, y_n, \gamma v)} \end{array} \right\} \right\}^{\omega}$$

Taking n approaches to infinity, we have

$$d(\gamma v, Av) \le \left\{ \phi \left\{ \begin{array}{c} \dfrac{d(Av, Av)[d(Av, \gamma v) + d(Av, Av)]}{d(Av, Av) + d(\gamma v, Av)}, \\ \dfrac{d(Av, \gamma v)[d(Av, Av) + d(\gamma v, Av)]}{d(Av, \gamma v) + d(Av, Av)}, \\ \dfrac{d(Av, \gamma v)[d(Av, Av) + d(Av, Av)]}{d(Av, \gamma v) + d(Av, \gamma v)} \end{array} \right\} \right\}^{\omega}$$

$$d(\gamma v, Av) \le \{\phi\{1, d(Av, \gamma v), 1\}\}^{\omega}$$

$$d(\gamma v, Av) \le d^{\omega}(\gamma v, Av)$$

On the other hand $\omega \in \left(0,\frac{1}{2}\right)$, implies that $d(\gamma\, v, Av) = 1$ *i.e.,* $\gamma\, v = Av$.

Since $\gamma\, X \subset BX$, $\exists\, w \in X$ such that $Bw = \gamma\, v$. Now we claim that $Bw = \theta\, w$, if possible $Bw \neq \theta\, w$. *Putting* $x=v$ and $y=w$, in (E2), we have

$$d(Bw, \theta w) = d(\gamma v, \theta\, w)$$

$$\leq \left\{ \phi \begin{Bmatrix} \dfrac{d(Av,Bw)[d(\theta\, w,\gamma v)+d(Av,\theta\, w)]}{d(Bw,Av)+d(\gamma v,Bw)}, \\ \dfrac{d(\theta\, w,\gamma v)[d(Bw,Av)+d(\gamma v,Bw)]}{d(\theta\, w,\gamma v)+d(Av,\theta\, w)}, \\ \dfrac{d(Av,\gamma v)[d(Bw,\theta\, w)+d(Bw,Av)]}{d(Av,\gamma v)+d(\theta\, w,\gamma v)} \end{Bmatrix} \right\}^{\omega}$$

$$\leq \left\{ \phi \begin{Bmatrix} \dfrac{d(Bw,Bw)[d(\theta\, w,Bw)+d(Bw,\theta\, w)]}{d(Bw,Bw)+d(Bw,Bw)} \\ \dfrac{d(\theta\, w,Bw)[d(Bw,Bw)+d(Bw,Bw)]}{d(\theta\, w,Bw)+d(Bw,\theta\, w)}, \\ \dfrac{d(Bw,Bw)[d(Bw,\theta\, w)+d(Bw,Bw)]}{d(Bw,Bw)+d(\theta\, w,Bw)} \end{Bmatrix} \right\}^{\omega}$$

[since, $Av=Bw$, $\gamma\, v=Bw$]

$$d(Bw,\theta\, w) \leq \{\phi\{1, d(\theta\, w, Bw), 1\}\}^{\omega}$$

$$d(Bw, \theta w) = d^{\omega}(Bw, \theta w),$$

Which is a contradiction, hence since $\in \left(0,\frac{1}{2}\right)$, $Bw=\theta\, w$ and hence $\theta\, w=Av=\gamma\, v=Bw$.

since (A, γ) and (B, θ) are weakly compatible and $\gamma\, v=Av$ and $\theta w=Bw$. Hence $A\gamma\, v = \gamma\, Av = AAv = \gamma\,\gamma\, v$, $\theta Bw = B\,\theta w = BBw = \theta\theta w$.

Finally we claim that $\gamma\, Av=Av$. Putting $x=Av$ and $y=w$ in *(E2)* we have

$$d(\gamma\, Av,Av) = d(\gamma\, Av, \theta w)$$

$$\leq \left\{ \phi \begin{Bmatrix} \dfrac{d(AAv,Bw)[d(\theta\, w,\gamma Av)+d(AAv,\theta\, w)]}{d(Bw,AAv)+d(\gamma Av,Bw)}, \\ \dfrac{d(\theta\, w,\gamma Av)[d(Bw,AAv)+d(\gamma Av,Bw)]}{d(\theta\, w,\gamma Av)+d(AAv,\theta\, w)}, \\ \dfrac{d(AAv,\gamma Av)[d(Bw,\theta\, w)+d(Bw,AAv)]}{d(AAv,\gamma Av)+d(\theta\, w,\gamma Av)} \end{Bmatrix} \right\}^{\omega}$$

$$[\text{since}, AAv = \gamma Av,\ \theta\, w = Av, Bw = Av]$$

$$d(\gamma\, Av,Av) \leq \left\{ \phi \begin{Bmatrix} \frac{d(AAv,Av)[d(Av,\gamma Av)+d(\gamma Av,Av)]}{d(Av,\gamma Av)+d(\gamma Av,Av)}, \\ \frac{d(Av,\gamma Av)[d(Av,\gamma Av)+d(\gamma Av,Av)]}{d(Av,\gamma Av)+d(\gamma Av,Av)}, \\ \frac{d(\gamma Av,\gamma Av)[d(Av,Av)+d(Av,\gamma Av)]}{d(\gamma Av,\gamma Av)+d(Av,\gamma Av)} \end{Bmatrix} \right\}^{\omega}$$

$$d(\gamma Av, Av) \leq \{\phi\{d(\gamma Av, A), d(\gamma Av, Av), 1\}\}^{\omega}$$

$$d(\gamma\, Av,Av) = d^{\omega}(\gamma Av, Av),$$

which is a contradiction hence $\gamma\ Av=Av$ and hence $\gamma\ Av=Av=A\gamma\ v=AAv=\gamma\ \gamma\ v$, which implies that Av is a common fixed point of A and γ. also, one can easily prove that $BBw=\theta\ \theta\ w=\theta\ Bw=B\ \theta\ w$, *i.e.* Bw is common fixed pointof B and θ. As $Av=Bw$, Av is a common fixed point of A, B, γ and θ. The uniqueness follows easily from *(E2)*. This completes the proof.

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