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A note on the identities of symmetry for the generalized twisted q-Euler numbers and polynomials with weighted measure[1]

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Abstract

In this paper, by using Dirichlet's character χ, we construct a generalized twisted q-Euler numbers $E^{(\alpha,\beta)}_{n,\chi,q,w}$ and polynomials $E^{(\alpha,\beta)}_{n,\chi,q,w}(x)$ with weight $(\alpha,\ \beta)$. And we give some interesting relations between the generalized twisted q-Euler polynomials $E^{(\alpha,\beta)}_{n,\chi,q,w}(x)$ with weight $(\alpha,\ \beta)$ and the generalized twisted q-Euler zeta functions with weight (α,β). Finally, we investigate some symmetric properties by using zeta function.

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1. Introduction

In [1–2], there are some symmetric properties for the q-Euler polynomials and the Carlitz's type of generalized q-Bernoulli polynomials, respectively. And in [3], q-Volkenborn integration is defined by Kim. It is used in functional equation of q-zeta function. Moreover, the q-zeta function of Carlitz's q-Bernoulli numbers and polynomials is used for investigating symmetric identities in [12].

The aim of this paper is to define a generalized twisted q- Euler numbers $E_{n,\chi,q,w}^{(\alpha,\beta)}$ and polynomials $E_{n,\chi,q,w}^{(\alpha,\beta)}(x)$ with weight (α, β). We also construct the zeta function for the generalized twisted q-Euler numbers with weight (α, β) and Hurwitz zeta function which interpolate the generalized twisted q-Euler numbers with weight (α, β). And using Hurwitz zeta function, we find some symmetric property of the generalized twisted q-Euler functions with weight (α, β).

Throughout this paper, p is fixed on odd prime number. Using the odd prime number p, we use the following notations. $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$ (see[1-11]). If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

Also in this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{(see [1 − 12]).} \tag{1.1}$$

Hence, $\lim_{q\to 1}[x]_q = x$ for all x with $|x|_p \le 1$.

Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a (\mod dp^N)\}, \end{aligned} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 < a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q} \tag{1.3}$$

is known to be a distribution on X.

For $g \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x \text{ (see [3]).} \tag{1.4}$$

Let $g_n(x)$ be the translation with $g_n(x) = g(x+n)$. Then we have the following integral equation:

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (l). \tag{1.5}$$

Let

$$T_p = \cup_{m\geq 1} C_{p^m} = \lim_{m\to\infty} C_{p^m},$$

where $C_{p^m} = \{w | w^{p^m} = 1\}$ is the cyclic group of order p^m. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$ (see [9 , 10]).

The generalized Euler number and polynomials are introduced in [7]. Let χ be a primitive Dirichlet character of conductor $d \in \mathbb{N}$. We assume that d is odd. Then the generalized Euler numbers associated with χ, $E_{n,\chi}$, are defined by

$$F_\chi(t) = \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a e^{at}}{e^{dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}. \tag{1.6}$$

The generalized Euler polynomials associated with χ, $E_{n,\chi}(x)$, are also defined by

$$F_\chi(t,x) = \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a e^{at}}{e^{dt}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}. \tag{1.7}$$

2. Generalized twisted q-Euler numbers and polynomails with weight (α, β)

In this section, our primary goal is to define the generalized twisted q-Euler numbers and polynomials with weight (α, β). We also find generating functions of the generalized twisted q-Euler numbers and polynomials with weight (α, β).

Definition 2.1. We define the generalized twisted q-Euler numbers $E_{n,\chi,q,w}^{(\alpha,\beta)}$ with weight (α, β). Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $w \in \mathrm{T_p}$ and $\alpha, \beta \in \mathbb{Q}$.

$$E_{n,\chi,q,w}^{(\alpha,\beta)} = \int_X \chi(x) w^x [x]_{q^\alpha}^n d\mu_{-q^\beta}(x).$$

By using p-adic q-integral, we get below theorem.

Theorem 2.2. For $\alpha, \beta \in \mathbb{Q}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, we have

$$E_{n,\chi,q,w}^{(\alpha,\beta)} = [2]_{q^\beta} \sum_{l=0}^{\infty} (-1)^l \chi(l) q^{\beta l} w^l [l]_{q^\alpha}^n.$$

Proof. Let $\alpha, \beta \in \mathbb{Q}$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We obtain

$$\begin{aligned}
E_{n,\chi,q,w}^{(\alpha,\beta)} &= \int_X \chi(x) w^x [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_{q^\beta}} \sum_{x=0}^{dp^N-1} \chi(x) w^x [x]_{q^\alpha}^n (-q^\beta)^x \\
&= [2]_{q^\beta} \sum_{i=0}^{d-1} (-1)^i q^{i\beta} w^i \chi(i) \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha i l} \frac{1}{1+q^{(\alpha l+\beta)d} w^d} \\
&= [2]_{q^\beta} \sum_{m=0}^{\infty} \left(\sum_{i=0}^{d-1} (-1)^{i+dm} \chi(i+dm) q^{\beta(i+dm)} w^{i+dm} [i+dm]_{q^\alpha}^n \right) \\
&= [2]_{q^\beta} \sum_{l=0}^{\infty} (-1)^l \chi(l) q^{\beta l} w^l [l]_{q^\alpha}^n .
\end{aligned}$$

Hence, we can get the above theorem. ■

By using Definition 2.1, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,\chi,q,w}^{(\alpha,\beta)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_X \chi(x) w^x [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \frac{t^n}{n!} \\
&= \int_X \sum_{n=0}^{\infty} \chi(x) w^x [x]_{q\alpha}^n \frac{t^n}{n!} d\mu_{-q^\beta}(x) \qquad (2.1) \\
&= \int_X \chi(x) w^x e^{[x]_{q^\alpha} t} d\mu_{-q^\beta}(x).
\end{aligned}$$

Next, we define the generalized twisted q-Euler polynomials $E_{n,\chi,q,w}^{(\alpha,\beta)}(x)$ with weight (α, β).

Definition 2.3. For $w \in T_p$, $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $\alpha, \beta \in \mathbb{Q}$, we define

$$E_{n,\chi,q,w}^{(\alpha,\beta)}(x) = \int_X \chi(y) w^y [x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y).$$

By using p-adic q-integral, we obtain

$$\begin{aligned}
&\int_X \chi(y) w^y [x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) \\
&= [2]_{q^\beta} \sum_{i=0}^{d-1} \chi(i) w^i (-1)^i q^{i\beta} \left(\frac{1}{1-q^\alpha}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l(x+i)} \frac{1}{1+w^d q^{(\alpha l+\beta)d}} . \qquad (2.2)
\end{aligned}$$

Thus we get

$$E^{(\alpha,\beta)}_{n,\chi,q,w}(x) = [2]_{q^\beta}\sum_{i=0}^{d-1}\chi(i)w^i(-1)^i q^{i\beta}\left(\frac{1}{1-q^\alpha}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l(x+i)}\frac{1}{1+w^d q^{(\alpha l+\beta)d}}. \tag{2.3}$$

The generating function of the generalized twisted q-Euler polynomials $E^{(\alpha,\beta)}_{n,\chi,q,w}(x)$ with weight (α,β) is defined analogously as follows.

Definition 2.4. For $w \in T_p,\ q \in \mathbb{C}_p$ with $|1-q|_p < 1, \alpha, \beta \in \mathbb{Q}$, we have

$$F^{(\alpha,\beta)}_{\chi,q,w}(t,x) = [2]_{q^\beta}\sum_{n=0}^{\infty}(-1)^n q^{\beta n}w^n\chi(n)e^{[x+n]_{q^\alpha}t} = \sum_{n=0}^{\infty}E^{(\alpha,\beta)}_{n,\chi,q,w}(x)\frac{t^n}{n!}.$$

By using Definition 2.3 and Definition 2.4, we get the following theorem.

Theorem 2.5. Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1, \alpha, \beta \in \mathbb{Q}$. Then we have

$$E^{(\alpha,\beta)}_{n,\chi,q,w}(x) = \sum_{l=0}^{n}\binom{n}{l}q^{\alpha(n-l)y}[x]^{n-l}_{q^\alpha}E^{(\alpha,\beta)}_{n,\chi,q,w}.$$

Proof. By Definition 2.3 and Definition 2.4, we obtain

$$\int_X \chi(y)w^y e^{[x+y]_{q^\alpha}t}d\mu_{-q^\beta}(y) = [2]_{q^\beta}\sum_{n=0}^{\infty}(-1)^n q^{\beta n}w^n\chi(n)e^{[x+n]_{q^\alpha}t}. \tag{2.4}$$

Since $[x+y]_{q^\alpha} = [y]_{q^\alpha} + q^y[x]_{q^\alpha}$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}E^{(\alpha,\beta)}_{n,\chi,q,w}(x)\frac{t^n}{n!} &= \int_X \chi(y)w^y e^{[x+y]_{q^\alpha}t}d\mu_{-q^\beta}(y)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}q^{\alpha(n-l)y}[x]^{n-l}_{q^\alpha}E^{(\alpha,\beta)}_{n,\chi,q,w}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

■

Theorem 2.6. Let $q \in \mathbb{C}_p, \alpha, \beta \in \mathbb{Q}$. Then we obtain an addition theorem as below

$$E^{(\alpha,\beta)}_{n,\chi,q,w}(x+y) = \sum_{l=0}^{n}\binom{n}{l}q^{\alpha x}E^{(\alpha,\beta)}_{l,\chi,q,w}(y)[x]^{n-l}_{q^\alpha}.$$

By using p-adic q-integral, we get

$$\begin{aligned}\int_X \chi(x)w^x[x]_{q^\alpha}^n d\mu_{-q^\beta}(x) &= \lim_{N\to\infty}\frac{1}{[dp^N]_{-q^\beta}}\sum_{x=0}^{dp^N-1}\chi(x)w^x[x]_{q^\alpha}^n(-q^\beta)^x\\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}}\sum_{i=0}^{d-1}\chi(i)w^i(-1)^i q^{a\beta}E_{n,q^d,w^d}^{(\alpha,\beta)}(\frac{i}{d}),\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\int_X \chi(y)w^y[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) &= \lim_{N\to\infty}\frac{1}{[dp^N]_{-q^\beta}}\sum_{y=0}^{dp^N-1}\chi(y)w^y[x+y]_{q^\alpha}^n(-q^\beta)^y\\ &= \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}}\sum_{i=0}^{d-1}\chi(i)w^i(-1)^i q^{i\beta}E_{n,q^d,w^d}^{(\alpha,\beta)}(\frac{i+x}{d}).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we have following theorem.

Theorem 2.7. Let $q \in \mathbb{C}_p$, $\alpha, \beta \in \mathbb{Q}$. Then we have

$$E_{n,\chi,q,w}^{(\alpha,\beta)} = \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}}\sum_{i=0}^{d-1}\chi(i)w^i(-1)^i q^{i\beta}E_{n,q^d,w^d}^{(\alpha,\beta)}(\frac{i}{d}).$$

$$E_{n,\chi,q,w}^{(\alpha,\beta)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_{-q^\beta}}\sum_{i=0}^{d-1}\chi(i)w^i(-1)^i q^{i\beta}E_{n,q^d,w^d}^{(\alpha,\beta)}(\frac{i+x}{d}).$$

From (1.5), we have following equation.

$$w^n q^{\beta n}I_{-q^\beta}(g_n) + (-1)^{n-1}I_{-q^\beta}(g) = [2]_{q^\beta}\sum_{l=0}^{n-1}(-1)^{n-1-l}q^{\beta l}w^l g(l). \tag{2.8}$$

Let $g(x) = \chi(x)w^x e^{[x]_{q^\alpha}^m}$. By using Equation (2.8) and the above function, we obtain following formula.

$$w^{nd}q^{\beta nd}E_{m,\chi,q,w}^{(\alpha,\beta)}(nd) + (-1)^{n-1}E_{m,\chi,q,w}^{(\alpha,\beta)} = [2]_{q^\beta}\sum_{l=0}^{nd-1}(-1)^{n-1-l}\chi(l)q^{\beta l}w^l[l]_{q^\alpha}^m. \tag{2.9}$$

Hence we have the following result.

Theorem 2.8. Let $q \in \mathbb{C}_p$, $\alpha, \beta \in \mathbb{Q}$, $m \in \mathbb{Z}_+$.

If $n = 0(mod\ 2)$, then

$$w^{nd}q^{\beta nd}E^{(\alpha,\beta)}_{m,\chi,q,w}(nd) - E^{(\alpha,\beta)}_{m,\chi,q,w} = [2]_{q^\beta}\sum_{l=0}^{nd-1}(-1)^{l+1}\chi(l)q^{\beta l}w^l[l]^m_{q^\alpha}.$$

If $n = 1(mod\ 2)$, then

$$w^{nd}q^{\beta nd}E^{(\alpha,\beta)}_{m,\chi,q,w}(nd) + E^{(\alpha,\beta)}_{m,\chi,q,w} = [2]_{q^\beta}\sum_{l=0}^{nd-1}(-1)^{l}\chi(l)q^{\beta l}w^l[l]^m_{q^\alpha}.$$

3. The analogue of the Euler zeta function

In the section, we assume that w be the p^N-th root of unity, $\alpha, \beta \in \mathbb{Q}$ and $q \in \mathbb{C}$ with $|q| < 1$. By using the generalized twisted q-Euler numbers and polynomials with weight (α, β), we define the generalized twisted q-Euler zeta function and Hurwitz type of the generalized q-Euler zeta function with weight (α, β), respectively.

We set

$$F^{(\alpha,\beta)}_{\chi,q,w}(t) = \sum_{n=0}^{\infty} E^{(\alpha,\beta)}_{n,\chi,q,w}\frac{t^n}{n!}. \tag{3.1}$$

By using Theorem 2.2 and (3.1), we have the generating function of the generalized twisted q-Euler numbers with weight (α, β).

For $\alpha, \beta \in \mathbb{Q}$, $q \in \mathbb{C}$ with $|1-q|_p < 1$, we have

$$\begin{aligned} F^{(\alpha,\beta)}_{\chi,q,w}(t) &= \sum_{n=0}^{\infty} E^{(\alpha,\beta)}_{n,\chi,q,w}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left([2]_{q^\beta}\sum_{l=0}^{\infty}(-1)^l\chi(l)q^{\beta l}w^l[l]^n_{q^\alpha}\right)\frac{t^n}{n!} \\ &= [2]_{q^\beta}\sum_{l=0}^{\infty}(-1)^l\chi(l)w^lq^{\beta l}e^{[l]_{q^\alpha}t}. \end{aligned} \tag{3.2}$$

Hence, we have the following theorem.

Theorem 3.1. For $\alpha, \beta \in \mathbb{Q}$, $q \in \mathbb{C}$ with $|q| < 1$, we have

$$F^{(\alpha,\beta)}_{\chi,q,w}(t) = [2]_{q^\beta}\sum_{n=0}^{\infty}(-1)^n\chi(n)q^{\beta n}w^ne^{[n]_{q^\alpha}t}.$$

From Theorem 3.1, we note that

$$\left.\frac{d^k}{dt^k}F^{(\alpha,\beta)}_{\chi,q,w}(t)\right|_{t=0} = [2]_{q^\beta}\sum_{n=0}^{\infty}\chi(n)(-1)^nw^nq^{\beta n}[n]^k_{q^\alpha},\ (k \in \mathbb{N}). \tag{3.3}$$

Definition 3.2. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. We define the generalized twisted q-Euler zeta function $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s)$ with (α, β) by

$$\zeta^{(\alpha,\beta)}_{\chi,q,w}(s) = [2]_{q^\beta} \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n) w^n q^{\beta n}}{[n]^s_{q^\alpha}}.$$

Note that $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s)$ and $E^{(\alpha,\beta)}_{k,\chi,q,w}$ is given by the following theorem.

Theorem 3.3. For $k \in \mathbb{N}$, we have

$$\zeta^{(\alpha,\beta)}_{\chi,q,w}(-k) = E^{(\alpha,\beta)}_{k,\chi,q,w}.$$

Observe that $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s)$ function interpolates $E^{(\alpha,\beta)}_{k,\chi,q,w}$ numbers at non-negative integers.

By using (2.17), we note that

$$\left.\frac{d^k}{dt^k} F^{(\alpha,\beta)}_{\chi,q,w}(t,x)\right|_{t=0} = [2]_{q^\beta} \sum_{n=0}^{\infty} (-1)^n \chi(n) w^n q^{\beta n} [n+x]^k_{q^\alpha}, \ (k \in \mathbb{N}). \tag{3.4}$$

By (3.4), we are now ready to define the Hurwitz type of the generalized twisted q-Euler zeta functions with weight (α, β).

Definition 3.4. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta^{(\alpha,\beta)}_{\chi,q,w}(s,x) = [2]_{q^\beta} \sum_{n=0}^{\infty} \frac{(-1)^n \chi(n) w^n q^{\beta n}}{[n+x]^s_{q^\alpha}}.$$

Note that $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta^{(\alpha,\beta)}_{\chi,q,w}(s,x)$ and $E^{(\alpha,\beta)}_{k,\chi,q,w}(x)$ is given by the following theorem.

Theorem 3.5. For $k \in \mathbb{N}$, we get

$$\zeta^{(\alpha,\beta)}_{\chi,q,w}(-k,x) = E^{(\alpha,\beta)}_{k,\chi,q,w}(x).$$

Observe that $\zeta^{(\alpha,\beta)}_{\chi,q,w}(-k,x)$ function interpolates $E^{(\alpha,\beta)}_{k,\chi,q,w}(x)$ polynomials at non-negative integers.

4. Symmetric property of the generalized twisted q-Euler polynomials with weight (α, β)

In this section, by using the similar method of [1, 11], expect for obvious modifications, we investigate some symmetric identities for the generalized twisted q-Euler polynomials with weight (α, β) and the generalized twisted q-Euler zeta function with weight (α, β).

Theorem 4.1. Let $d \in \mathbb{N}$ be a conductor and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and let a, b, d be odd positive integers. Then we have

$$\sum_{i=0}^{bd-1} [2]_{q^{\beta a}}[a]_{q^\alpha}^s(-1)^i\chi(i)w^{ai}q^{\beta ai}\zeta_{\chi,q^b,w^b}^{(\alpha,\beta)}(s, ax + \frac{ai}{b})$$
$$= \sum_{i=0}^{bd-1}[2]_{q^{\beta b}}[b]_{q^\alpha}^s(-1)^j\chi(j)w^{bj}q^{\beta bj}\zeta_{\chi,q^a,w^a}^{(\alpha,\beta)}(s, bx + \frac{bj}{a}).$$

Proof. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In Definition 3.4, we derive next result by substitute $ax + \dfrac{ai}{b}$ for x in and replace q and w by q^b and w^b, respectively.

$$\begin{aligned}\zeta_{\chi,q^b,w^b}^{(\alpha,\beta)}(s, ax + \frac{ai}{b}) &= [2]_{q^{\beta b}}\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)w^{bn}q^{\beta bn}}{[n + ax + \frac{ai}{b}]_{q^{\alpha b}}^s}\\ &= [2]_{q^{\beta b}}[b]_{q^\alpha}^s\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)w^{bn}q^{\beta bn}}{[abx + ai + bn]_{q^\alpha}^s}.\end{aligned} \tag{4.1}$$

Since for any non-negative integer n and odd positive integer a, there exist unique non-negative integer r, j such that $m = (ad)r + j$ with $0 \le j \le ad - 1$. So, the equation (4.1) can be written as

$$\begin{aligned}\zeta_{\chi,q^b,w^b}^{(\alpha,\beta)}(s, ax + \frac{ai}{b}) &= [2]_{q^{\beta b}}[b]_{q^\alpha}^s\sum_{\substack{adr+j=0\\ 0\le j\le ad-1}}^{\infty}\frac{(-1)^{adr+j}\chi(adr + j)w^{b(adr+j)}q^{\beta b(adr+j)}}{[abdr + abx + ai + bj]_{q^\alpha}^s}\\ &= [2]_{q^{\beta b}}[b]_{q^\alpha}^s\sum_{j=0}^{ad-1}\sum_{r=0}^{\infty}\frac{(-1)^j\chi(j)w^{b(adr+j)}q^{\beta b(adr+j)}}{[ab(dr + x) + ai + bj]_{q^\alpha}^s}.\end{aligned} \tag{4.2}$$

In similarly, we can see that

$$\begin{aligned}\zeta_{\chi,q^a,w^a}^{(\alpha,\beta)}(s, bx + \frac{bj}{a}) &= [2]_{q^{\beta a}}\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)w^{an}q^{\beta an}}{[n + bx + \frac{bj}{a}]_{q^{\alpha b}}^s}\\ &= [2]_{q^{\beta a}}[a]_{q^\alpha}^s\sum_{n=0}^{\infty}\frac{(-1)^n\chi(n)w^{an}q^{\beta an}}{[abx + an + bj]_{q^\alpha}^s}.\end{aligned} \tag{4.3}$$

Using the method in (4.2), we obtain

$$\begin{aligned}\zeta^{(\alpha,\beta)}_{\chi,q^a,w^a}(s, bx+\frac{bj}{a}) &= [2]_{q^{\beta a}}[a]^s_{q^\alpha}\sum_{\substack{bdr+i=0\\0\le i\le bd-1}}^{\infty}\frac{(-1)^{bdr+i}\chi(bdr+i)w^{a(bdr+i)}q^{\beta a(bdr+i)}}{[abdr+abx+ai+bj]^s_{q^\alpha}}\\ &= [2]_{q^{\beta a}}[a]^s_{q^\alpha}\sum_{i=0}^{bd-1}\sum_{r=0}^{\infty}\frac{(-1)^i\chi(i)w^{a(bdr+i)}q^{\beta a(bdr+i)}}{[ab(dr+x)+ai+bj]^s_{q^\alpha}}.\end{aligned} \tag{4.4}$$

From (4.2) and (4.4), we have

$$\begin{aligned}&\sum_{i=0}^{bd-1}[2]_{q^{\beta a}}[a]^s_{q^\alpha}(-1)^i\chi(i)w^{ai}q^{\beta ai}\zeta^{(\alpha,\beta)}_{\chi,q^b,w^b}\left(s, ax+\frac{ai}{b}\right)\\ &= \sum_{i=0}^{bd-1}[2]_{q^{\beta a}}[a]^s_{q^\alpha}(-1)^i\chi(i)w^{ai}q^{\beta ai}\\ &\quad\times[2]_{q^{\beta b}}[b]^s_{q^\alpha}\sum_{j=0}^{ad-1}\sum_{r=0}^{\infty}\frac{(-1)^j\chi(j)w^{b(adr+j)}q^{\beta b(adr+j)}}{[ab(dr+x)+ai+bj]^s_{q^\alpha}}\\ &= \sum_{j=0}^{ad-1}[2]_{q^{\beta b}}[b]^s_{q^\alpha}(-1)^j\chi(j)w^{bj}q^{\beta bj}\zeta^{(\alpha,\beta)}_{\chi,q^a,w^a}\left(s, bx+\frac{bj}{a}\right).\end{aligned} \tag{4.5}$$

■

Next, we derive the symmetric results by using definition and theorem of the generalized twisted q-Euler polynomials with weight (α, β).

Theorem 4.2. Let $d \in \mathbb{N}$ be a conductor and odd integer, let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, and a, b be odd positive integers. For i, j and n be non-negative integer, we have

$$\begin{aligned}&\frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}}\sum_{i=0}^{bd-1}(-1)^i\chi(i)w^{ai}q^{\beta ai}E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(ax+\frac{ai}{b}\right)\\ &= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}}\sum_{j=0}^{ad-1}(-1)^j\chi(j)w^{bj}q^{\beta bj}E^{(\alpha,\beta)}_{n,\chi,q^a,w^a}\left(bx+\frac{bj}{a}\right).\end{aligned}$$

Proof. By substitute $ax+\dfrac{ai}{b}$ for x in Theorem 3.5 and replace q and w by q^b and w^b, respectively, we derive

$$\begin{aligned}E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(ax+\frac{ai}{b}\right) &= [2]_{q^{\beta b}}\sum_{m=0}^{\infty}(-1)^m\chi(m)w^{bm}q^{\beta bm}\left[ax+\frac{ai}{b}+m\right]^n_{q^{\alpha b}} \\ &= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}}\sum_{m=0}^{\infty}(-1)^m\chi(m)w^{bm}q^{\beta bm}[abx+ai+bm]^n_{q^\alpha}.\end{aligned} \tag{4.6}$$

Since for any non-negative integer m and odd positive integer a and d, there exist unique non-negative integer r, j such that $m = adr + j$ with $0 \le j \le ad-1$. Hence, the equation (4.6) is written as

$$\begin{aligned}&E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(ax+\frac{ai}{b}\right) \\ &= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}}\sum_{\substack{adr+j=0\\0\le j\le ad-1}}^{\infty}(-1)^{adr+j}\chi(adr+j)w^{b(adr+j)}q^{\beta b(adr+j)}[abx+ai+b(adr+j)]^n_{q^\alpha} \\ &= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}}\sum_{i=0}^{ad-1}\sum_{r=0}^{\infty}(-1)^{adr+j}\chi(j)w^{b(adr+j)}q^{\beta b(adr+j)}[ab(x+dr)+ai+bj)]^n_{q^\alpha}.\end{aligned} \tag{4.7}$$

In similar, we get

$$\begin{aligned}E^{(\alpha,\beta)}_{n,\chi,q^a,w^a}\left(bx+\frac{bj}{a}\right) &= [2]_{q^{\beta a}}\sum_{m=0}^{\infty}(-1)^m\chi(m)w^{am}q^{\beta am}\left[bx+\frac{bj}{a}+m\right]^n_{q^{\alpha a}} \\ &= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}}\sum_{m=0}^{\infty}(-1)^m\chi(m)w^{am}q^{\beta am}[abx+bj+am]^n_{q^\alpha},\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&E^{(\alpha,\beta)}_{n,\chi,q^a,w^a}\left(bx+\frac{bj}{a}\right) \\ &= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}}\sum_{\substack{bdr+i=0\\0\le i\le bd-1}}^{\infty}(-1)^{bdr+i}\chi(bdr+i)w^{a(bdr+i)}q^{\beta a(bdr+j)}[abx+ai+b(adr+j)]^n_{q^\alpha} \\ &= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}}\sum_{i=0}^{bd-1}\sum_{r=0}^{\infty}(-1)^{bdr+i}\chi(i)w^{a(bdr+i)}q^{\beta a(bdr+i)}[ab(x+dr)+ai+bj)]^n_{q^\alpha}.\end{aligned} \tag{4.9}$$

It follows from the above equation that

$$
\begin{aligned}
&\frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}} \sum_{i=0}^{bd-1} (-1)^i \chi(i) w^{ai} q^{\beta ai} E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(ax+\frac{ai}{b}\right) \\
&= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}} \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}} \sum_{j=0}^{ad-1} \sum_{i=0}^{bd-1} \sum_{r=0}^{\infty} (-1)^{i+j} \chi(i)\chi(j) w^{abdr+ai+bj} \\
&\times q^{\beta abdr+\beta ai+\beta bj} [ab(x+dr)+ai+bj)]^n_{q^\alpha} \\
&= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}} \sum_{j=0}^{ad-1} (-1)^j \chi(j) w^{bj} q^{\beta bj} E^{(\alpha,\beta)}_{n,\chi,q^a,w^a}\left(bx+\frac{bj}{a}\right).
\end{aligned}
\tag{4.10}
$$

From (4.8) and (4.9), the proof of the Theorem 4.2 is complete. ■

By Theorem 2.7 and Theorem 4.2, we have the following theorem.

Theorem 4.3. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $d \in \mathbb{N}$ be a conductor and odd integer. Let a, b be odd positive integers. For i, j and n be non-negative integer, then we have

$$
\begin{aligned}
&[2]_{q^{\beta a}} \sum_{k=0}^{n} \binom{n}{k} [a]^k_{q^\alpha} [b]^{n-k}_{q^\alpha} E^{(\alpha,\beta)}_{n-k,\chi,q^b,w^b}(ax) \sum_{i=0}^{bd-1} (-1)^i \chi(i) w^{ai} q^{(\beta+n-k)ai} [i]^k_{q^{\alpha a}} \\
&= [2]_{q^{\beta b}} \sum_{k=0}^{n} \binom{n}{k} [a]^{n-k}_{q^\alpha} [b]^k_{q^\alpha} E_{n-k,\chi,q^a,w^a}(bx) \sum_{j=0}^{ad-1} (-1)^j \chi(j) w^{bj} q^{(\beta+n-k)bj} [j]^k_{q^{\alpha b}}.
\end{aligned}
$$

Proof. After some elementary calculations, we obtain

$$
\begin{aligned}
&\frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}} \sum_{i=0}^{bd-1} (-1)^i \chi(i) w^{ai} q^{\beta ai} E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(ax+\frac{ai}{b}\right) \\
&= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}} \sum_{i=0}^{bd-1} (-1)^i \chi(i) w^{ai} q^{\beta ai} \sum_{l=0}^{n} \binom{n}{l} q^{lai} E^{(\alpha,\beta)}_{l,\chi,q^b,w^b}(ax) \left[\frac{ai}{b}\right]^{n-l}_{q^{\alpha b}} \\
&= \frac{[2]_{q^{\beta a}}}{[a]^n_{q^\alpha}} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a}{b}\right]^k_{q^{\alpha b}} E^{(\alpha,\beta)}_{n-k,\chi,q^b,w^b}(ax) \sum_{i=0}^{bd-1} (-1)^i \chi(i) w^{ai} q^{(\beta+n-k)ai} [i]^k_{q^{\alpha a}},
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
&\frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}} \sum_{j=0}^{ad-1} (-1)^j \chi(j) w^{bj} q^{\beta bj} E^{(\alpha,\beta)}_{n,\chi,q^b,w^b}\left(bx+\frac{bj}{a}\right) \\
&= \frac{[2]_{q^{\beta b}}}{[b]^n_{q^\alpha}} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{a}{b}\right]^k_{q^{\alpha a}} E^{(\alpha,\beta)}_{n-k,\chi,q^a,w^a}(bx) \sum_{j=0}^{ad-1} (-1)^j \chi(j) w^{bj} q^{(\beta+n-k)bj} [j]^k_{q^{\alpha b}}.
\end{aligned}
\tag{4.12}
$$

From (4.11), (4.12) and Theorem 4.2, we obtain that

$$[2]_{q^{\beta a}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[a]_{q^{\alpha}}^{n-k}[b]_{q^{\alpha}}^{k}} E_{n-k,\chi,q^{b},w^{b}}^{(\alpha,\beta)}(ax) \sum_{i=0}^{bd-1} (-1)^{i} \chi(i) w^{ai} q^{(\beta+n-k)ai} [i]_{q^{\alpha a}}^{k}$$

$$= [2]_{q^{\beta b}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{[a]_{q^{\alpha}}^{k}[b]_{q^{\alpha}}^{n-k}} E_{n-k,\chi,q^{a},w^{a}}(bx) \sum_{j=0}^{ad-1} (-1)^{j} \chi(j) w^{bj} q^{(\beta+n-k)bj} [j]_{q^{\alpha b}}^{k}.$$

Hence, above theorem is proved. ■

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