Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 2 (2017), pp. 331–345

http://www.ripublication.com/gjpam.htm

A New Class of Generalized Closed Sets in Topological Spaces

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Abstract

The aim of this paper is to introduce a new class of sets called delta generalized β-closed sets and a new class of functions called delta generalized β-continuous functions in topological spaces. Some of their properties and characterizations are studied.

AMS subject classification: $54A05$, $54C05$, $54C08$.
Keywords: β-closed sets, $\delta g\beta$-closed sets, $\delta g\beta$-continuous, $\delta g\beta$-irresolute.

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1. Introduction

Among various generalized open sets, the notion of β-open sets introduced by Abd El-Monsef et al. [1] which is equivalent to the notion of semi-preopen sets due to Andrijevic [3], plays a significant role in General Topology and Real Analysis. Many results have been obtained by using the concept of β-closed sets. Dontchev [9]introduced and established the concept of generalized semi-preclosed sets as a generalization of semi-preclosed sets which is equivalent to the notion of generalized β-closed sets due to Tahiliani [26]. In this paper, the concepts of $\delta g\beta$-closed sets, $\delta g\beta$-continuous, $\delta g\beta$-irresolute and pre $\delta g\beta$-continuous functions are introduced and studied their properties and characterizations.

Throughout this paper, (X, τ), (Y,σ) and (Z,η) (or simply X,Y and Z) represent topological spaces (or simply spaces) on which no separation axioms are assumed unless explicitly stated.

2. Preliminaries

Let us recall the following definitions which are useful in the sequel:

Definition 2.1. A subset A of a topological space X is called a

(i) β-closed [1] (or semi-preclosed [3]) if int(cl(int(A))) $\subseteq$ A.

(ii) pre-closed [16] if cl(int(A)) $\subseteq$ A.

(iii) b-closed [4] if cl(int(A)) $\cap$ int(cl(A)) $\subseteq$ A.

(iv) regular-closed [25] if A=cl(int(A)).

(v) α-closed [18] if cl(int(cl(A))) $\subseteq$ A.

(vi) semi-closed [17] if int(cl(A)) $\subseteq$ A.

(vii) δ-closed [27] if A=$cl_\delta(A)$ where $cl_\delta(A) = \{x \in$ X:int(cl(U)) $\cap$ A $\neq \phi$, U $\in \tau$ and x $\in$ U$\}$.

Definition 2.2. A subset A of a topological space X is called,

(i) generalized β-closed (briefly, gβ-closed) [26] if βcl(A)$\subseteq$G whenever A$\subseteq$G and G is open in X.

(ii) δ generalized b-closed (briefly, δgb-closed) [6] if βcl(A)$\subseteq$ G whenever A$\subseteq$G and G is open in X.

(iii) generalized pre regular closed (briefly, gpr-closed) [11] if pcl(A)$\subseteq$G whenever A$\subseteq$G and G is regular open in X.

(iv) generalized δ-semiclosed (briefly, gδs-closed) [5] if scl(A)⊆G whenever A⊆G and G is δ-open in X.

(v) gδ-closed [8] if cl(A)⊆U whenever A⊆G and G is δ-open in X.

(vi) gδ^*-closed [8] if $\mathrm{cl}_\delta(A)$⊆G whenever A⊆G and G is δ-open in X.

(vii) regular generalized b-closed (briefly, rgb-closed) [15] if bcl(A)⊆G whenever A⊆G and G is regular open in X.

(viii) generalized b-closed (briefly, gb-closed) [2] if bcl(A)⊆G whenever A⊆G and G is open in X.

The complements of the above mentioned closed sets are their respective open sets.

Definition 2.3. A function f:X→Y from a topological space X into a topological space Y is called a

(i) β-continuous [1] (resp, β-irresolute [14], δ-continuous [20], gβ-continuous [22], β-gβ-continuous [22]) if f^{-1}(G) is β-closed (resp, β-closed, δ-closed, gβ-closed and gβ-closed) in X for every closed (resp, β-closed, δ-closed, closed and β-closed) set G of Y.

(ii) pre β-closed [14] (resp, pre β-open [14], δ-closed[19] and δ-open [21])if for every β-closed (resp, β-open, δ-closed and δ-open) subset A of X, f(A) is β-closed (resp, β-open, δ-closed and δ-open) in Y.

Definition 2.4. A topological space X is said to be a

(i) extremely disconnected [12] if the closure of every open set of X is open in X.

(ii) submaximal [23]if every dense set of X is open in X.

3. Delta Generalized β-Closed Sets

Definition 3.1. A subset A of a space X is called a delta generalized β-closed (briefly, δgβ-closed) set if βcl(A) ⊆ G whenever A ⊆ G and G is δ-open in X.

The complement of a δgβ-closed set is called δgβ-open.

From the above definition and known results, we have the following diagram of implications and none of its implications is reversible.

$$\begin{array}{ccccccc} \text{b-closed} & \longrightarrow & \text{gb-closed} & \longrightarrow & \delta\text{gb-closed} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \beta\text{-closed} & \longrightarrow & g\beta\text{-closed} & \longrightarrow & \delta g\beta\text{-closed} & \longrightarrow & \text{rgb-closed.} \end{array}$$

Example 3.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

The subset $\{a, b, c\}$ is $\delta g\beta$-closed but neither β-closed nor gβ-closed.

Example 3.3. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. The subset $\{a, b, c\}$ is δgβ-closed but not δgb-closed and the subset $\{a, b, c, d\}$ is rgb-closed but not δgβ-closed.

Remark 3.4. The following examples show that δgβ-closed set is independent of gpr-closed set.

Example 3.5. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $\{a\}$ is δgβ-closed but not gpr-closed and the subset $\{a, b\}$ is gpr-closed but not δgβ-closed.

Theorem 3.6. Let X be a semi-regular space. Then A $\subseteq$ X is gβ-closed if and only if A is δgβ-closed.

Proof. In a semi-regular space X, $\delta O(X) = O(X)$ and hence proof follows. ■

Lemma 3.7. [28] Let X be a space. Then bcl(A)=scl(A)=βcl(A) for every semi open set A in X.

Theorem 3.8. The following are equivalent for any semi-open set A $\subseteq$ X:

(i) A is gδs-closed

(ii) A is δgb-closed

(iii) A is δgβ-closed.

Lemma 3.9. [10]If X is extremely disconnected and submaximal space then $\beta cl(A) = cl(A)$ for every subset A of X.

Theorem 3.10. The following are equivalent for any subset A of extremely disconnected and submaximal space X:

(i) A is gδ-closed

(ii) A is δgb-closed

(iii) A is gδs-closed

(iv) A is δgβ-closed.

Theorem 3.11. A subset A of a space X is δgβ-open if and only if V $\subseteq$ βint(A) whenever V is δ-closed and V $\subseteq$ A.

Proof. Let V be a δ-closed of X and V$\subseteq$A.Then (X-V) is δ-open and (X-A) $\subseteq$ (X-V).Since (X-A) is δgβ-closed, then βcl(X-A)$\subseteq$(X-M) which implies M$\subseteq$$\beta$int(A).

Conversely, let U be an δ-open set of X and (X-A)$\subseteq$U. Since (X-U) is a δ-closed set contained in A,by hypothesis (X-U)$\subseteq$$\beta$int(A). That is, X-$\beta$int(A)=$\beta$cl(X-A)$\subseteq$U. Hence X-A is δgβ-closed and so A is δgβ-open. ■

Theorem 3.12. Let B⊆X be δgβ-closed then βcl(B)-B contains no non empty δ-closed set.

Proof. Suppose there exists a non empty δ-closed set G of X such that G⊆βcl(B)-B, then G⊆βcl(B) and G⊆X-B implies B⊆X-G. Then βcl(B)⊆X-G as B is δgβ-closed.

Hence G ⊆ βcl(B) ∩ (X-βcl(B))=ϕ. This shows $G = \phi$. ■

Theorem 3.13. Let A⊆X be a δgβ-closed set. Then A is β-closed if and only if βcl(A)-A is δ-closed.

Proof. Let A be β-closed, then βcl(A)=A and so βcl(A)-A=ϕ which is δ-closed.

Conversely,let A be a δgβ-closed subset set of X and βcl(A)-A is a δ-closed. Then by Theorem 3.12, βcl(A)-A=ϕ and hence A is β-closed.

Theorem 3.14. If A ⊆ X is both δ-open and δgβ-closed then A is β-closed in X.

Proof. Let A be δ-open δgβ-closed set of X then βcl(A)⊆A. But always A ⊆ βcl(A).

Therefore $\beta cl(A) = A$ and hence A is β-closed. ■

Theorem 3.15. If A is δgβ-closed and A⊆B⊆βcl(A). Then:

(i) B is δgβ-closed.

(ii) βcl(B)-B contains no non empty δ-closed set.

Proof.

(i) Let G be a δ-open set in X such that B⊆G then A⊆G. Since A is δgβ-closed, then βcl(A)⊆G. Now, βcl(B)⊆βcl(βcl(A))= βcl(A)⊆G. Therefore βcl(B)⊆G.

(ii) Follows from Theorem 3.12. ■

Theorem 3.16. If A is δgβ-open and B is any set in X such that βint(A)⊆B⊆A then B is δgβ-open in X.

Remark 3.17. The intersection of two δgβ-closed sets need not be δgβ-closed in general as seen from the following example.

Example 3.18. In Example 3.2, the subsets {a,b,c} and {a,b,d} are δgβ-closed but their intersection $\{a, b, c\} \cap \{a, b, d\} = \{a, b\}$ is not δgβ-closed.

Remark 3.19. The union of two δgβ-closed sets need not be δgβ-closed set in general as seen from the following example.

Example 3.20. In Example 3.2, the subsets {a} and {b} are δgβ-closed but their union {a}∪{b}={a, b} is not δgβ-closed.

Theorem 3.21. If A and B are $\delta g\beta$-closed sets in extremely disconnected and submaximal space X then A∪B is $\delta g\beta$-closed in X.

Proof. Let A ∪ B ⊆ U where U is δ-open in X, then A ⊆ U and B ⊆ U. Then βcl(A)⊆U and βcl(B)⊆U since A and B are $\delta g\beta$-closed sets. By Lemma 3.9, βcl(A)=cl(A) and βcl(B)=cl(B). Now, βcl(A ∪ B) ⊆ cl(A∪B) =cl(A) ∪ cl(B) =βcl(A) ∪ βcl(B) ⊆ U ∪ U=U. Thus βcl(A∪B)⊆U whenever A ∪ B ⊆ U and U is δ-open in X and hence A∪B is $\delta g\beta$-closed. ■

Theorem 3.22. [24] Let A and B be two subsets of X with A is semi-closed then βcl(A∪B)=βcl(A)∪βcl(B).

Theorem 3.23. If A and B are $\delta g\beta$-closed sets with A is semi-closed, then A∪B is $\delta g\beta$-closed in X.

Proof. Follows from Definition 3.1 and Theorem 3.22. ■

Theorem 3.24. The intersection of a $\delta g\beta$-closed set and a δ-closed set of X is always $\delta g\beta$-closed.

Proof. Let A be $\delta g\beta$-closed and let F be δ-closed.If G is a δ-open set with A∩F⊆G.Then A⊆G∪F^c and G∪F^c is δ-open. Since A is $\delta g\beta$-closed, then βcl(A)⊆G∪F^c which implies βcl(A)∩F⊆G.Now βcl(A∩F)⊆βcl(A)∩βcl(F) ⊆βcl(A)∩δcl(F)⊆βcl(A)∩F ⊆G. Hence A∩F is $\delta g\beta$-closed. ■

Theorem 3.25. Let A⊆X be δ-open $\delta g\beta$-closed and M⊆X is β-closed then A∩M is $\delta g\beta$-closed.

Proof. Let A⊆X be δ-open and $\delta g\beta$-closed. Then by Theorem 3.14, A is β-closed.Hence A∩M is β-closed which implies that A∩M is $\delta g\beta$-closed. ■

Theorem 3.26. [13] Let A⊆Y⊆X and Y be α-open in X, then βcl_Y(A)=βcl_X(A)∩Y.

Theorem 3.27. Let A⊆Y⊆X and Y be α-open β-closed then βcl_Y(A)=βcl_X(A).

Theorem 3.28. Let Y be α-open subspace of a space X and A⊆Y. If A is $\delta g\beta$-closed in X then A is $\delta g\beta$-closed in Y.

Proof. Let U be a δ-open set of Y such that A⊆U.Then U=Y∩H for some δ-open set H of X. Since A is $\delta g\beta$-closed in X, we have βcl(A)⊆H and βcl_Y(A)=Y∩βcl(A)⊆Y∩H=U. Hence A is $\delta g\beta$-closed in Y.

Converse of the above theorem need not be true as seen from the following example. ■

Example 3.29. Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let $Y = \{a, b, c\}$ and $A = \{a, b\}$, then $A \subseteq Y \subseteq X$ and A is $\delta g\beta$-closed relative to Y but it is not $\delta g\beta$-closed relative to X.

Theorem 3.30. Let $A \subseteq Y \subseteq X$ and Y be α-open and β-closed. If A is $\delta g\beta$-closed in Y then A is $\delta g\beta$-closed in X.

Proof. Let U be a δ-open set of X such that $A \subseteq U$. Then $A = Y \cap A \subseteq Y \cap U$ where Y∩U is δ-open in Y.Since A is $\delta g\beta$-closed in Y, we have $\beta \mathrm{cl}_Y(A) \subseteq Y \cap U$ and by Theorem 3.27, $\beta \mathrm{cl}_X(A) \subseteq Y \cap U \subseteq U$. ■

Theorem 3.31. For a space X, the following statements are equivalent:

(i) Every $\delta g\beta$-closed set is δgb-closed and

(ii) Every β-closed set is δgb-closed.

Proof. Clearly (i) → (ii).
(ii) → (i): Let A be a $\delta g\beta$-closed set in X such that A⊆G where G is δ-open in X, then βcl(A)⊆G. As βcl(A) is β-closed, by (ii), βcl(A) is δgb-closed,
bcl(A)⊆bcl(βcl(A)⊆G. Therefore bcl(A)⊆G. ■

Theorem 3.32. If $\beta O(X) = \beta C(X)$, then $\delta G\beta C(X) = P(X)$.

Proof. Let $A \subset V$ where V is δ-open in X, then V is β-open. By hypothesis V is β-closed. Hence βcl(A)⊆V and so A is $\delta g\beta$-closed. ■

Theorem 3.33. For any $x \in X$, the set X-{x} is $\delta g\beta$-closed or δ-open.

Proof. Suppose X-$\{x\}$ is not δ-open, then X is the only δ-open set containing X-$\{x\}$. This implies $\beta cl(X - \{x\}) \subseteq X$. Hence X-$\{x\}$ is $\delta g\beta$-closed. ■

Theorem 3.34. If A is $\delta g\beta$-closed, then $\mathrm{cl}_\delta\{x\} \cap A \neq \phi$, for every $x \in \beta cl(A)$.

Proof. Let $x \in \beta cl(A)$. Suppose $\mathrm{cl}_\delta\{x\} \cap A = \phi$, then $A \subset X - cl_\delta\{x\}$ and X-$\mathrm{cl}_\delta\{x\}$ is δ-open. Since A is $\delta g\beta$-closed, then $\beta cl(A) \subseteq X - cl_\delta\{x\}$ so $x \notin \beta cl(A)$ which is a contradiction. Therefore $\mathrm{cl}_\delta\{x\} \cap A \neq \phi$. ■

Definition 3.35. [7] The intersection of all δ-open subsets of X containing A is called the δ kernel of A and it is denoted by $\ker_\delta$(A).

Theorem 3.36. A subset A of X is $\delta g\beta$-closed if and only if $\beta cl(A) \subseteq ker_\delta(A)$.

Proof. Suppose A is $\delta g\beta$-closed set in X such that $x \in \beta cl(A)$. If possible, let $x \notin ker_\delta(A)$, then there exists a δ-open set G in X such that $A \subseteq G$ and $x \notin G$. Since A is $\delta g\beta$-closed in X, βcl(A)⊆G implies x∉βcl(A) which is a contradiction.

Conversely, let βcl(A)⊆$\ker_\delta$(A) be true and G is a δ-open set containing A, then $\ker_\delta$(A)⊆G which implies βcl(A)⊆G. Hence A is $\delta g\beta$-closed. ■

Lemma 3.37. For any set $A \subseteq X$, βint(βcl(A)-A)=ϕ.

Theorem 3.38. Let $A \subseteq X$ be $\delta g\beta$-closed, then βcl(A)-A is $\delta g\beta$-open.

Proof. Suppose that A is $\delta g\beta$-closed and M is δ-closed set in X such that M⊆βcl(A)-A Then by Theorem 3.12, $M = \phi$ and hence $M \subseteq \beta$int(βcl(A)-A). Therefore by Theorem 3.11, βcl(A)-A is $\delta g\beta$-open. ■

Definition 3.39. For a subset A of a space X, $\delta g\beta$cl(A)=∩{F:A⊂F, F is $\delta g\beta$-closed in X}.

Theorem 3.40. Let A and B be subsets of a topological space X. Then:

(i) $\delta g\beta$cl(X)=X and $\delta g\beta$cl(Φ)= Φ.

(ii) If A⊆B, then $\delta g\beta$cl(A)⊆ $\delta g\beta$cl(B).

(iii) $\delta g\beta$cl(A)∪$\delta g\beta$cl(B)⊆ $\delta g\beta$cl(A∪B).

(iv) $\delta g\beta$cl(A∩B)⊆ $\delta g\beta$cl(A)∩$\delta g\beta$cl(B).

(v) If A is $\delta g\beta$ closed, then $\delta g\beta$cl(A)=A.

(vi) $A \subseteq \delta g\beta$cl(A)⊂$g\beta$cl(A)⊂$\beta$cl(A).

Remark 3.41. The equalities do not hold in results (iii) and(iv). Also converse of (v) need not be true in general as seen from the following examples.

Example 3.42. (iii) In Example 3.3, let A={a} and B={b}. Then $\delta g\beta$cl(A)={a}, $\delta g\beta$cl(B)={b} and $\delta g\beta$cl(A∪B)={a,b,e}.

Thus we have $\delta g\beta$cl(A∪B)={a,b,e}≠{a,b}= $\delta g\beta$cl(A)∪$\delta g\beta$cl(B).
(iv) Consider X={a,b,c,d} with the topology τ = {X, ϕ,{b},{c}, {a,b}, {b,c},{a,b,c}, {b,c,d}}.
Let A={b} and B={c}, then $\delta g\beta$cl(A)={b,d}, $\delta g\beta$cl(B)={c,d} and $\delta g\beta$cl(A∩B)=ϕ. Thus we have $\delta g\beta$cl(A∩B)=ϕ≠{d}= $\delta g\beta$cl(A)∩$\delta g\beta$cl(B).
(v) In Example 3.2, let A={a}. Then $\delta g\beta$cl(A)=A but it is not a $\delta g\beta$-closed set in X.

Theorem 3.43. If $\delta G\beta$C(X) is closed under finite unions, then $\delta g\beta$cl(A∪B)=$\delta g\beta$cl(A)∪$\delta g\beta$cl(B) for all $A, B \in \delta G\beta C(X)$.

Proof. Let A and B be $\delta g\beta$ closed sets in X. Then by hypothesis, A∪B is $\delta g\beta$-closed.Thus $\delta g\beta$cl(A∪B)=A∪B= $\delta g\beta$cl(A)∪$\delta g\beta$cl(B). ■

Theorem 3.44. Let A be a subset of a space X. Then $x \in \delta g\beta$cl(A) if and only if G∩A≠ϕ for every $\delta g\beta$-open set G containing x.

Proof. Let $x \in \delta g\beta$cl(A). Suppose that there exists a $\delta g\beta$open set G containing x such that G∩A=ϕ then A⊆X-G and X-G is $\delta g\beta$-closed. Therefore $\delta g\beta$cl(A)⊆X-G which implies $x \notin \delta g\beta$cl(A), a contradiction.

Conversely, suppose that x∉$\delta g\beta$cl(A). Then there exists a $\delta g\beta$-closed set F containing A such that x∉F. Hence F^c is a $\delta g\beta$-open set containing x. Therefore F^c∩A=ϕ which contradicts the hypothesis. ■

Definition 3.45. For a subset A of a space X, $\delta g\beta$int(A)=$\cup${G:G$\subset$A,G is $\delta g\beta$-open in X}.

Theorem 3.46. Let A and B be subsets of a space X. Then:

(i) $\delta g\beta$int(X)=X and $\delta g\beta$int(Φ)= Φ.

(ii) If A$\subset$B, then $\delta g\beta$int(A)$\subseteq$ $\delta g\beta$int(B).

(iii) $\delta g\beta$int(A)$\cup\delta g\beta$int(B)$\subseteq\delta g\beta$int(A$\cup$B).

(iv) $\delta g\beta$int(A$\cap$B)$\subseteq\delta g\beta$int(A)$\cap\delta g\beta$int(B).

(v) If A is $\delta g\beta$-open, then $\delta g\beta$int(A)=A.

4. Delta Generalized β-Continuous Functions

In this section, the concepts of $\delta g\beta$-continuous, pre $\delta g\beta$-continuous and $\delta g\beta$-irresolute functions in topological spaces are introduced.Some of their properties and characterizations are established.

Definition 4.1. A function f:X$\to$Y is called a

(i) $\delta g\beta$-continuous if the inverse image of every closed set in Y is $\delta g\beta$-closed in X.

(ii) pre $\delta g\beta$-continuous if the inverse image of every β-closed set in Y is $\delta g\beta$-closed in X.

(ii) $\delta g\beta$-irresolute if the inverse image of every $\delta g\beta$-closed set in Y is $\delta g\beta$-closed in X.

Clearly, (i) f is $\delta g\beta$-continuous if and only if f^{-1}(V) is $\delta g\beta$-open in X for each open set V of Y.

(ii) pre $\delta g\beta$-continuous if and only if f^{-1}(V) is $\delta g\beta$-open in X for each β-open set V of Y.

(iii) f is $\delta g\beta$-irresolute if and only if f^{-1}(V) is $\delta g\beta$-open in X for each $\delta g\beta$-open set V of Y.

From the above definitions, we have the following:

Theorem 4.2.

(i) Every β-continuous function is $\delta g\beta$-continuous.

(ii) Every β-irresolute function is pre $\delta g\beta$-continuous.

(iii) Every pre $\delta g\beta$-continuous function is $\delta g\beta$-continuous.

(iv) Every $\delta g\beta$-irresolute function is pre $\delta g\beta$-continuous.

(v) Every gβ-continuous function is $\delta g\beta$-continuous.

(vi) Every β-gβ-continuous function is pre $\delta g\beta$-continuous.
Reverse implications of above theorem need not be true in general:

Example 4.3. Let $X = Y = \{a, b, c\}$. Let τ={X,ϕ,{a},{b}, {a,b},{a,c}} and σ={Y, ϕ, {a}} be topologies on X and Y respectively.Let f:X→Y be a function defined by f(a)=b,f(b)=c and f(c)=a. Then f is $\delta g\beta$-continuous and pre $\delta g\beta$-continuous but neither β-continuous nor β-irresolute, since {b,c} is closed and hence β-closed in Y but f^{-1}({b,c})={a,b} is not β-closed in X.

Example 4.4. Let $X = Y = \{a, b, c, d\}$. Let $\tau = \{X, \phi, \{a\}, \{b\}$, {a,b}, {a,b,c}} and σ={Y,ϕ,{a},{b}, {a,b}, {a,c}, {a,b,c}} be topologies on X and Y respectively. Let $f : X \to Y$ be a function defined by $f(a) = f(b) = b$, $f(c) = c$ and $f(d) = d$. Then f is $\delta g\beta$-continuous but not pre $\delta g\beta$-continuous. Since {b} is β-closed in Y but f^{-1}({b}) ={a,b} is not $\delta g\beta$-closed in X.

Example 4.5. Let $X = Y = \{a, b, c, d\}$.
Let $\tau = \{X, \phi, \{a\}$, {b}, {a,b}, {a,b,c}} and σ={Y,ϕ, {a}, {b}, {a,b}, {a,c}, {a,b,c}} be topologies on X and Y respectively. Let f:X→Y be a function defined by $f(a) = f(b) = f(c) = d$ and $f(d) = a$. Then f is $\delta g\beta$-continuous but not gβ-continuous, since {d} is closed in Y but f^{-1}({d})={a,b,c} is not gβ-closed in X.

Example 4.6. Let $X =$ {a,b,c,d,e}, $Y =$ {a,b,c,d}.
Let $\tau = \{X, \phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$ and $\sigma = \{Y, \phi$, {b},{c}, {a,b}, {b,c}, {a,b,c}, {b,c,d}} be topologies on X and Y respectively. Let $f : X \to Y$ be a function defined by $f(a) = a$, $f(b) = f(d) = f(e) = b$ and $f(c) = d$. Then f is pre $\delta g\beta$-continuous but neither β-gβ-continuous nor $\delta g\beta$-irresolute, since {a,d} is β-closed in Y but f^{-1}({a,d})={a,c} is not gβ-closed in X.

Theorem 4.7. If $f : X \to Y$ is $\delta g\beta$-continuous, then for each $x \in X$ and for each open set V in Y with f(x)∈V, there exists a $\delta g\beta$-open set U in X containing x such that f(U)⊆V.

Proof. Let $x \in X$ and V is an open set in Y with $f(x) \in V$, then $x \in f^{-1}(V)$. Since f is $\delta g\beta$-continuous, f^{-1}(V) is $\delta g\beta$-open in X. Put $U = f^{-1}(V)$, then $x \in U$ and $f(U) = f(f^{-1}(V)) \subseteq V$. ∎

Theorem 4.8. If the bijective function f:X→Y is pre $\delta g\beta$-continuous and δ-open, then it is $\delta g\beta$-irresolute.

Proof. Let V be a $\delta g\beta$-closed set in Y and F be a δ-open set in X such that f^{-1}(V)⊆F, then V⊆f(F) and f(F) is δ-open in Y as f is δ-open. Since V is $\delta g\beta$-closed in Y, βcl(V)⊆f(F). This implies f^{-1}(βcl(V))⊆F.

Since f is pre $\delta g\beta$-continuous and βcl(V) is β-closed in Y it follows that $f^{-1}(\beta cl(V))$ is $\delta g\beta$-closed in X. Therefore $\beta cl(f^{-1}(\beta cl(V))) \subseteq F$ which implies $\beta cl(f^{-1}(V)) \subseteq \beta cl(f^{-1}(\beta cl(V))) \subset F$. That is, $\beta cl(f^{-1}(V)) \subseteq F$ and hence $f^{-1}(V)$ is $\delta g\beta$-closed set in X. Thus f is $\delta g\beta$-irresolute. ■

Theorem 4.9. If the bijective function $f : X \to Y$ is δ-continuous and pre β-closed, then f^{-1} is $\delta g\beta$-irresolute.

Proof. Let G be a $\delta g\beta$-closed set in X and U be a δ-open set in Y such that $(f^{-1})^{-1}(G) = f(G) \subseteq U$, then $G \subseteq f^{-1}(U)$. Since f is δ-continuous, f^{-1}(U) is δ-open in X. Since G is $\delta g\beta$-closed in X, βcl(G)$\subseteq f^{-1}$(U) which implies f(βcl(G))$\subseteq$U. Since f is pre β-closed and βcl(G) is β-closed in X it follows that f(βcl(G)) is β-closed in Y.

Now, βcl(f(G))$\subseteq\beta$cl(f(βcl(G)))= f(βcl(G))$\subseteq$U which implies βcl(f(G))$\subseteq$U. Therefore f(G)=$(f^{-1})^{-1}$(G) is is $\delta g\beta$-closed set in Y. ■

Remark 4.10. The composition of two $\delta g\beta$-continuous functions need not be $\delta g\beta$-continuous as seen from the following examples.

Example 4.11. Let $X = Y = Z = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a\}$, {b}, {a,b}}, $\sigma = \{Y, \phi, \{a\}\}$ and $\eta = \{Z, \phi$, {b}} be topologies on X,Y and Z respectively. Define a function f:X→Y as $f(a) = a$, $f(b) = b$ and $f(c) = c$ and a function g:Y→Z as $g(a) = a$, $g(b) = c$ and $g(c) = b$. Then f and g are $\delta g\beta$-continuous but g∘f:X→Z is $\delta g\beta$-continuous,since there exists a closed set {a,c} in Z such that(g∘f)$^{-1}${a,c}={a,b} is not $\delta g\beta$-closed in X.

Theorem 4.12. Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then;

(i) If f is $\delta g\beta$-continuous and g is continuous then g∘f is $\delta g\beta$-continuous.

(ii) If f and g are $\delta g\beta$-irresolute, then g∘f is $\delta g\beta$-irresolute.

Proof. (i) Let $h = g \circ f$ and U be a closed set in Z.
Since g is continuous, g^{-1}(U) is closed in Y. Therefore f^{-1}[g^{-1}(U)]=h^{-1}(U) is $\delta g\beta$-closed in X because f is $\delta g\beta$-continuous. Hence g∘f is $\delta g\beta$-continuous.
(iii) Similar to (i). ■

Theorem 4.13. If $f : (X, \tau) \to (Y, \sigma)$ is a $\delta g\beta$-continuous and A is δ-open $\delta g\beta$-closed subset of a space X and assume the class $\delta G\beta C(X,\tau)$ is closed under finite intersections, then the restriction f/A:(A,τ/A)→(Y,σ) is $\delta g\beta$-continuous.

Proof. Let U be a closed subset of Y. By hypothesis, f^{-1}(U)∩A=V(say) is $\delta g\beta$-closed in X. Since $(f/A)^{-1}$(U)=V, then it is sufficient to show that V is $\delta g\beta$-closed in A. Let V⊆M where M is a δ-open set in A. Then there exists a δ-open set N in X such that M=N∩A.Then V⊆N implies pcl_X(V)⊆N and βcl_X(V)∩A⊆N∩A which implies βcl_A(V)⊆M. Hence V is $\delta g\beta$-closed in A. ■

Theorem 4.14. Pasting lemma for $\delta g\beta$-continuous functions:
Let X=A∪B be a space with topology τ and Y be a space with topology σ.
Let $\delta G\beta C(X)$ be closed under finite unions and let f:(A,τ/A)→(Y,σ) and g:(B,τ/B)→(Y,σ) be $\delta g\beta$-continuous functions such that f(x)=g(x) for every x∈A∩B. Suppose that both A and B are α-open and β-closed in X. Then their combination α:(X,τ)→(Y,σ)defined by α(x)=f(x) if x∈A and α(x)=g(x) if x∈B is $\delta g\beta$-continuous.

Proof. Let F be any closed set in Y. Then α^{-1}(F)=f^{-1}(F)∪g^{-1}(F)=M∪N where M=f^{-1}(F) and N=g^{-1}(F). Since M is $\delta g\beta$-closed in A and A is α-open and β-closed in X. Then by Theorem 3.30,M is $\delta g\beta$-closed in X. Similarly N is $\delta g\beta$-closed in X. Also M∪N is $\delta g\beta$-closed in X by hypothesis. Therefore α^{-1}(F) $\delta g\beta$-closed in X. Hence, α is $\delta g\beta$-closed in X. ■

Definition 4.15. A space X is said to be $\delta g\beta T_{\frac{1}{2}}$-space if every $\delta g\beta$-closed subset of X is β-closed.

Theorem 4.16. A space X is $\delta g\beta T_{\frac{1}{2}}$-space if and only if every singleton set is either δ-closed or β-open.

Proof. Let $x \in X$. Suppose {x} is not δ-closed, then X-{x} is not δ-open.
By Theorem 3.33,X-{x} is $\delta g\beta$-closed. Since X is $\delta g\beta T_{\frac{1}{2}}$, then X-{x} is β-closed and so {x} is β-open.
Conversely, let A⊂X be $\delta g\beta$-closed and x∈βcl(A).
Suppose that x∉A,thus A⊆X-{x}.We have the following two cases:
Case i) If {x} is δ-closed, then X-{x} is δ-open. Since A is $\delta g\beta$-closed, βcl(A)⊂X-{x}, then x∉βcl(A) which is a contradiction.
Case ii) If {x} is β-open, then X-{x} β-closed. Therefore βcl(A)⊆X-{x} and so x∉βcl(A), also a contradiction. Thus βcl(A)⊂A. ■

Theorem 4.17. If the bijective f:X→Y is pre β-open and δ-closed and X is $\delta g\beta T_{\frac{1}{2}}$-space , then Y is also $\delta g\beta T_{\frac{1}{2}}$-space.

Proof. Suppose y∈Y,then y=f(x) for some x∈X as f is bijective. Since X is $\delta g\beta T_{\frac{1}{2}}$-space,by Theorem 4.16 it follows that {x} is δ-closed or β-open.

If {x} is δ-closed, then f({x})={y} as f is δ-closed. Also, if {x} is β-open, then f({x})={y} is β-open as f is β-open. Therefore by Theorem 4.16, Y is $\delta g\beta T_{\frac{1}{2}}$-space. ■

Definition 4.18. A space X is said to be $T_{\delta g\beta}$-space if every $\delta g\beta$-closed subset of X is closed.

Theorem 4.19. Every $T_{\delta g\beta}$-space is $\delta g\beta T_{\frac{1}{2}}$-space but not conversely.

Proof. Let X be $T_{\delta g\beta}$-space and A be $\delta g\beta$-closed, then A is closed. Therefore A is β-closed and hence X is $\delta g\beta T_{\frac{1}{2}}$-space. ■

Example 4.20. In Example 3.3, X is $\delta g\beta T_{\frac{1}{2}}$-space but not $T_{\delta g\beta}$-space,since {a} is $\delta g\beta$-closed but not closed.

Theorem 4.21.

(i) If f:X→Y is $\delta g\beta$-continuous and X is $T_{\delta g\beta}$-space, then f is continuous.

(ii) If f:X→Y is $\delta g\beta$-continuous and X is $\delta g\beta T_{\frac{1}{2}}$-space, then f is β-continuous.

Theorem 4.22. If f:X→Y and g:Y→Z are $\delta g\beta$-continuous with Y is $T_{\delta g\beta}$-space, then g∘f is $\delta g\beta$-continuous.

Proof. Let $h = g \circ f$ and V be a closed set in Z. Since g is $\delta g\beta$-continuous, $g^{-1}(V)$ is $\delta g\beta$-closed in Y. Therefore $g^{-1}(V)$ is closed in Y because Y is $T_{\delta g\beta}$-space. Since f is $\delta g\beta$-continuous, $f^{-1}[g^{-1}(V)]=h^{-1}(V)$ is $\delta g\beta$-closed in X and hence $g \circ f$ is $\delta g\beta$-continuous. ■

Acknowledgement

The authors are grateful to the University Grants Commission, New Delhi, India for financial support under UGC SAP DRS-III: F-510/3/DRS-III/2016(SAP-I) dated 29th Feb 2016 to the Department of Mathematics, Karnatak University, Dharwad, India.

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