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Q-fuzzy ideals of regular ordered Γ-semirings

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Abstract

The aim of this paper is to study ordered Γ-semiring and its several types of regularity using the concept of Q-fuzziness. To make it possible, at first we define Q-fuzzy ideal (interior ideal, bi-ideal) of an ordered Γ-semiring. Then we define compositions of two Q-fuzzy sets and use this to define Q-fuzzy quasi-ideal and study some related properties of these mentioned ideals. After that we define regular, intra-regular and quasi-regular ordered Γ-semiring and investigate their properties and inter-relations using the above mentioned ideals and obtain some of their characterizations.

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1. Introduction

The fundamental concept of fuzzy set, introduced by Zadeh [12], provides a natural frame-work for generalizing several basic notions of algebra. Jun and Lee [6] applied the concept of fuzzy sets to the theory of Γ-rings. The notion of Γ-semiring was introduced by Rao [11] as a generalization of Γ-ring as well as of semiring [3].

The concept of Q-fuzzifcation of ideals of Γ-semigroups was introduced and studied by Majumder [9]. Akram et al. [1], Lekkoksung [7, 8], Mandal [10] extended this concept in case of Γ-semigroup, ordered semigroups [4], ordered Γ-semiring and investigated some important properties.

The main object of the present paper is to study ordered Γ-semiring and its several types of regularity using Q-fuzzy ideals, Q-fuzzy bi-ideals and Q-fuzzy quasi-ideals.

2. Preliminaries

Definition 2.1. A semiring is a system consisting of a non-empty set S on which operations addition and multiplication (denoted in the usual manner) have been defined such that $(S, +)$ is a semigroup, $(S, \cdot)$ is a semigroup and multiplication distributes over addition from either side.

A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring S is zerosumfree if and only if $s + s' = 0$ implies that $s = s' = 0$.

Definition 2.2. Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ($(a, \alpha, b) \mapsto a\alpha b$) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

(v) $0_S\alpha a = 0_S = a\alpha 0_S$,

(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Example 2.3. Let S be the set of all $m \times n$ matrices over $\mathbf{Z_0^-}$(the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbf{Z_0^-}$, then S forms a Γ-semiring with usual addition and multiplication of matrices.

Definition 2.4. A subset A of a Γ-semiring S is called a left (resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset A of a semiring S is called an ideal if it is both left and right ideal of S. A subset A of a Γ-semiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$. A subset A of a Γ-semiring S is called a bi-ideal(resp. interior ideal) if A is closed under addition and $A\Gamma S\Gamma A \subseteq A$(resp. $S\Gamma A\Gamma S \subseteq A$).

As a generalization of the definition of ordered semiring and ordered ideal of [2], we define ordered Γ-semiring and its ordered ideal as follows:

Definition 2.5. An ordered Γ-semiring is a Γ-semiring S equipped with a partial order $\leq$ such that the operation is monotonic and constant 0 is the least element of S.

Definition 2.6. A left (resp. right) ideal I of S is called a left (resp. right) ordered ideal, if for any $a \in S, b \in I, a \leq b$ implies $a \in I$ (i.e., $(I] \subseteq I$). I is called an ordered ideal of S if it is both a left and a right ordered ideal of S.

Definition 2.7. A fuzzy subset f of a non-empty set S is defined as a mapping from S to [0,1].

Definition 2.8. A function μ from $S \times Q$ to the real closed interval [0, 1] is called Q-fuzzy subset of S, where Q is a non-empty set.

Definition 2.9. Let μ be a Q-fuzzy subset of a set S and $t \in [0,1]$. The set

$$\mu_t = \{(x, q) \in S \times Q | \mu(x, q) \geq t\}$$

is called the level subset of μ. Clearly, $\mu_t \subseteq \mu_s$, whenever $t \geq s$.

Definition 2.10. The characteristic function χ_A of A, is the mapping of $S \times Q$ to [0, 1] defined by

$$\begin{aligned} \chi_A(x, q) &= 1 \text{ if } (x, q) \in A \times Q \\ &= 0, \text{ if } (x, q) \notin A \times Q \end{aligned}$$

Definition 2.11. The union and intersection of two Q-fuzzy subsets μ and σ of a set S, denoted by $\mu \cup \sigma$ and $\mu \cap \sigma$ respectively, are defined as

$$(\mu \cup \sigma)(x, q) = \max\{\mu(x, q), \sigma(x, q)\} \text{ for all } x \in S, \ q \in Q$$

$$(\mu \cap \sigma)(x, q) = \min\{\mu(x, q), \sigma(x, q)\} \text{ for all } x \in S, \ q \in Q.$$

3. Q-Fuzzy ideals and regularity criterion

Throughout this paper unless otherwise mentioned S denote the ordered Γ-semiring.

Definition 3.1. Let μ be a non empty Q-fuzzy subset of an ordered Γ-semiring S (i.e., $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a Q-fuzzy left ideal [resp. Q-fuzzy right ideal] of S if

(i) $\mu(x + y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$,

(ii) $\mu(x\gamma y, q) \geq \mu(y, q)$ [resp. $\mu(x\gamma y, q) \geq \mu(x, q)$] and

(iii) $x \leq y$ implies $\mu(x, q) \geq \mu(y, q)$

for all $x, y \in S, \gamma \in \Gamma$ and $q \in Q$.
By a Q-fuzzy ideal we mean, it is both a Q-fuzzy left ideal as well as a Q-fuzzy right ideal.

Definition 3.2. A Q-fuzzy subset μ of an ordered Γ-semiring S is called Q-fuzzy interior-ideal if for all $x, y, z \in S, \alpha, \beta \in \Gamma$ and $q \in Q$ we have

(i) $\mu(x+y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$

(ii) $\mu(x\alpha y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$

(iii) $\mu(x\alpha y\beta z,q) \geq \mu(y,q)$

(iv) $x \leq y \Rightarrow \mu(x,q) \geq \mu(y,q)$.

Definition 3.3. An ordered Γ-semiring S is called regular (resp. intra-regular) if for each $a \in S$ there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a$ (resp. $a \leq x\alpha a\gamma a\beta y$).

Proposition 3.4. Every Q-fuzzy interior ideal of a regular (resp. intra-regular) ordered Γ-semiring S is an Q-fuzzy ideal of S.

Proof. Let μ be a Q-fuzzy interior ideal of a regular ordered Γ-semiring S and $a, b \in S$, $\gamma \in \Gamma$ and $q \in Q$. It is sufficient to prove that $\mu(a\gamma b,q) \geq \mu(a,q)$ and $\mu(a\gamma b,q) \geq \mu(b,q)$.

Since S is regular there exist $x \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq a\alpha x\beta a$ and hence we have

$$\begin{aligned}\mu(a\gamma b,q) &\geq \mu(a\alpha x\beta a\gamma b,q) \text{ [since } a\gamma b \leq a\alpha x\beta a\gamma b \text{ and } \mu \text{ is an ordered ideal]}\\ &\geq \mu(a,q) \text{ [since } \mu \text{ is an Q}-\text{fuzzy interior ideal]}\end{aligned}$$

Similarly for $b \in S$ there exist $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $b \leq b\alpha y\beta b$ and hence $a\gamma b \leq a\gamma b\alpha y\beta b$.

Now $\mu(a\gamma b,q) \geq \mu(a\gamma b\alpha y\beta b,q) \geq \mu(b,q)$. Therefore μ is a Q-fuzzy ideal of S. Similarly, we can prove the result for intra-regular ordered Γ-semiring. ■

Definition 3.5. Let μ and ν be two Q-fuzzy subsets of an ordered Γ-semiring S and $x, y, z \in S, \gamma \in \Gamma, q \in Q$. We define composition of μ and ν as follows:

$$\begin{aligned}\mu \circ_1 \nu(x,q) &= \sup_{x \leq y\gamma z}\{\min\{\mu(y,q), \nu(z,q))\}\}\\ &= 0, \text{ if } x \text{ cannot be expressed as } x \leq y\gamma z\end{aligned}$$

We can generalize the above composition as follows:

$$\begin{aligned}(\mu o_2 \nu)(x,q) &= \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i}[\min_i\{\min\{\mu(a_i,q), \nu(b_i,q)\}\}]\\ &= 0, \text{ if } x \text{ cannot be expressed as above}\end{aligned}$$

Definition 3.6. A Q-fuzzy subset μ of a ordered Γ-semiring S is called Q-fuzzy bi-ideal if for all $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$ we have

(i) $\mu(x+y,q) \geq \min\{\mu(x,q), \mu(y,q)\}$

(ii) $\mu(x\alpha y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$

(iii) $\mu(x\alpha y\beta z, q) \geq \min\{\mu(x, q), \mu(z, q)\}$

(iv) $x \leq y \Rightarrow \mu(x, q) \geq \mu(y, q)$.

Definition 3.7. A Q-fuzzy subset μ of a ordered Γ-semiring S is called Q-fuzzy quasi-ideal if for all $x, y \in S$ and $q \in Q$ we have

(i) $\mu(x + y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$

(ii) $\min\{(\mu o_2 \chi_S)(x, q), (\chi_S o_2 \mu)(x, q) \leq \mu(x, q)$

(iii) $x \leq y \Rightarrow \mu(x, q) \geq \mu(y, q)$.

Theorem 3.8. A Q-fuzzy subset μ of a ordered Γ-semiring S is a Q-fuzzy left(resp. right) ideal of S if and only if for all $x, y \in S$ and $q \in Q$, we have

(i) $\mu(x + y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$

(ii) $\chi_S o_2 \mu(x, q) \leq \mu(x, q)(resp.\ \mu o_2 \chi_S(x, q) \leq \mu(x, q))$

(iii) $x \leq y \Rightarrow \mu(x, q) \geq \mu(y, q)$.

Proof. Assume that μ is a Q-fuzzy left ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$, $q \in Q$. If $(\chi_S o_2 \mu)(x, q) = 0$, it is clear that $(\chi_S o_2 \mu)(x, q) \leq \mu(x, q)$. Otherwise, there exist elements $a_i, b_i \in S$ and $\gamma_i \in \Gamma$, for $i = 1, \ldots, n$ such that $x \leq \sum_{i=1}^{n} a_i \gamma_i b_i$. Then we have

$$
\begin{aligned}
(\chi_S o_2 \mu)(x, q) &= \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i}[\min_i\{\min\{\chi_S(a_i, q), \mu(b_i, q)\}\}] \\
&= \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i}[\min_i\{\min\{\mu(b_i, q)\}\}] \leq \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i}[\min_i\{\min\{\mu(a_i\gamma_i b_i, q)\}\}] \\
&\leq \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i}[\min\{\mu(\sum_{i=1}^{n} a_i\gamma_i b_i, q)\}] \leq \sup_{x \leq \sum_{i=1}^{n} a_i\gamma_i b_i} \mu(x, q) = \mu(x, q).
\end{aligned}
$$

This implies that $\chi_S o_2 \mu \subseteq \mu$. Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of ideal. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned} \mu(x\gamma y, q) &\geq (\chi_S o_2 \mu)(x\gamma y, q) = \sup_{x\gamma y \leq \sum_{i=1}^{n} a_i \gamma_i b_i} [\min_i \{\min\{\mu(b_i)\}\}] \\ &\geq \mu(y)(since\ x\gamma y \leq x\gamma y). \end{aligned}$$

Hence μ is a Q-fuzzy left ideal of S. The case for Q-fuzzy right ideal can be proved similarly. ■

Proposition 3.9. Let μ and ν be a Q-fuzzy right ideal and a Q-fuzzy left ideal of a ordered Γ-semiring S, respectively. Then $\mu \cap \nu$ is a Q-fuzzy quasi-ideal of S.

Proof. Proof follows by routine verification. ■

Lemma 3.10. Any Q-fuzzy quasi-ideal of S is a Q-fuzzy bi-ideal of S.

Proof. Let μ be any Q-fuzzy quasi-ideal of S. It is sufficient to show that $\mu(x\alpha y\beta z, q) \geq \min\{\mu(x, q), \mu(z, q)\}$ and $\mu(x\alpha y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$ for all $x, y, z \in S,\ q \in Q$ and $\alpha, \beta \in \Gamma$.
In fact, by the assumption, we have

$$\begin{aligned} \mu(x\alpha y\beta z, q) &\geq ((\mu o_2 \chi_S) \cap (\chi_S o_2 \mu))(x\alpha y\beta z, q) \\ &= \min\{(\mu o_2 \chi_S)(x\alpha y\beta z, q), (\chi_S o_2 \mu)(x\alpha y\beta z, q)\} \\ &= \min\{ \sup_{x\alpha y\beta z \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min(\mu(a_i)))\ ,\ \sup_{x\alpha y\beta z \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min(\mu(b_i)))\ \} \\ &\geq \min\{\mu(x), \mu(z)\}(since\ x\alpha y\beta z \leq x\alpha y\beta z) \end{aligned}$$

Similarly, we can show that $\mu(x\alpha y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$ for all $x, y \in S,\ q \in Q$ and for all $\alpha \in \Gamma$. ■

Theorem 3.11. An ordered Γ-semiring S is regular if and only if for any Q-fuzzy right ideal μ and Q-fuzzy left ideal ν, we have $\mu o_1 \nu = \mu \cap \nu$.

Proof. Let S be a regular ordered Γ-semiring and $x \in S$. Then there exist $a \in S$, $\alpha, \beta \in \Gamma$ such that $x \leq x\alpha a\beta x$. Then

$$\begin{aligned} (\mu o_1 \nu)(x, q) &= \sup_{x \leq y\gamma z} \{\min\{\mu(y, q), \nu(z, q)\}\} \geq \min_{x \leq x\alpha a\beta x} \{\mu(x\alpha a, q), \nu(x, q)\} \\ &\geq \min\{\mu(x, q), \nu(x, q)\} = (\mu \cap \nu)(x, q) \quad \cdots (i) \end{aligned}$$

Now since $x \leq y\gamma z$, $\mu(x, q) \geq \mu(y\gamma z, q) \geq \mu(y, q)$ and $\nu(x, q) \geq \nu(y\gamma z, q) \geq \nu(z, q)$ which implies

$$(\mu \cap \nu)(x, q) = \min\{\mu(x, q), \nu(x, q)\} \geq \min\{\mu(y, q), \nu(z, q)\}$$

and this relation is true for all representations of x. Therefore

$$(\mu \cap \nu)(x, q) \geq \sup_{x \leq y\gamma z} \{\min\{\mu(y, q), \nu(z, q)\}\} = (\mu o_1 \nu)(x, q) \quad \cdots (ii).$$

Therefore (i) and (ii) implies that $\mu \cap \nu = \mu o_1 \nu$.

Conversely, let C and D be respectively right and left ideals of S. Then χ_C and χ_D are respectively Q-fuzzy right ideal and Q-fuzzy left ideal. Moreover, $C\Gamma D \subseteq C \cap D$.

Let $a \in C \cap D$. Then $\chi_C(a, q) = 1 = \chi_D(a, q)$.
Thus $(\chi_C o_1 \chi_D)(a, q) = (\chi_C \cap \chi_D)(a, q) = \min\{\chi_C(a, q), \chi_D(a, q)\} = 1$.
So, $\min\{\chi_C(a_1, q), \chi_D(a_2, q)\} = 1$, for some $a_1, a_2 \in S$ satisfying $a \leq a_1 \gamma a_2$ i.e. $a \in C\Gamma D$. Hence $C \cap D = C\Gamma D$ and so S is regular [see [5]]. ∎

Theorem 3.12. Let S be a regular ordered Γ-semiring. Then

(i) $\mu \subseteq \mu o_2 \chi_S o_2 \mu$ for every Q-fuzzy bi-ideal μ of S.

(ii) $\mu \subseteq \mu o_2 \chi_S o_2 \mu$ for every Q-fuzzy quasi-ideal μ of S.

Proof. (i) Let μ be any Q-fuzzy bi-ideal of S and x be any element of S. Since S is regular there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq x\alpha a\beta x$.

$$\begin{aligned}
(\mu o_2 \chi_S o_2 \mu)(x, q) &= \sup_{x \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min\{(\mu o_2 \chi_S)(a_i, q), \mu(b_i, q)\}) \\
&\geq \min\{(\mu o_2 \chi_S)(x\alpha a, q), \mu(x, q)\} \\
&= \min\{\sup_{x\alpha a \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min\{(\mu(a_i, q), \chi_S(b_i, q))\}), \mu(x, q)\} \\
&\geq \min\{\mu(x, q), \mu(x, q)\} (\text{since } x\alpha a \leq x\alpha a\beta x\alpha a) \\
&= \mu(x, q).
\end{aligned}$$

This implies that $\mu \subseteq \mu o_2 \chi_S o_2 \mu$.
(i) ⇒ (ii) This is straight forward from Lemma 3.10. ∎

Theorem 3.13. Let S is a regular ordered Γ-semiring. Then

(i) $\mu \cap \nu \subseteq \mu o_2 \nu o_2 \mu$ for every Q-fuzzy bi-ideal μ and every Q-fuzzy ideal ν of S.

(ii) $\mu \cap \nu \subseteq \mu o_2 \nu o_2 \mu$ for every Q-fuzzy quasi-ideal μ and every Q-fuzzy ideal ν of S.

Proof. (i) Let μ and ν be any Q-fuzzy bi-ideal and Q-fuzzy ideal of S, respectively and x be any element of B. Since S is regular, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that

$x \leq x\alpha a\beta x$.

$$\begin{aligned}(\mu o_2 \nu o_2 \mu)(x,q) &= \sup_{x \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min\{(\mu o_2 \nu)(a_i,q), \mu(b_i,q)\}) \\ &\geq \min\{(\mu o_2 \nu)(x\alpha a,q), \mu(x,q)\} \\ &= \min\{\sup_{x\alpha a \leq \sum_{i=1}^{n} a_i \gamma_i b_i}(\min\{(\mu(a_i,q), \nu(b_i,q))\}), \mu(x,q)\} \\ &\geq \min\{\min\{\mu(x,q), \nu(a\beta x\alpha a,q)\}, \mu(x,q)\} (\text{since } x\alpha a \leq x\alpha a\beta x\alpha a) \\ &\geq \min\{\mu(x,q), \nu(x,q)\} = (\mu \cap \nu)(x,q)\end{aligned}$$

$(i) \Rightarrow (ii)$ This is straight forward from Lemma 3.10. ■

Theorem 3.14. Let S be a intra-regular ordered Γ-semiring. Then $\mu \cap \nu \subseteq \mu o_2 \nu$ for every Q-fuzzy left ideal μ and every Q-fuzzy right ideal ν of S.

Proof. Suppose S is intra-regular and μ and ν be any Q-fuzzy left ideal and Q-fuzzy right ideal of S respectively. Now let $a \in S$, $q \in Q$. Then by hypothesis there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq x\alpha a\gamma a\beta y$. Therefore

$$\begin{aligned}(\mu o_2 \nu)(a,q) &= \sup_{x \leq \sum_{i=1}^{n} a_i \gamma_i b_i}[\min_i\{\min\{\mu(a_i,q), \nu(b_i,q))\}\}] \\ &\geq \min\{\mu(x\alpha a,q), \nu(a\beta y,q)\} \\ &\geq \min\{\mu(a,q), \nu(a,q)\} = (\mu \cap \nu)(a,q).\end{aligned}$$

■

We now obtain the following result for regular and intra-regular ordered Γ-semiring.

Theorem 3.15. Let S is both regular and intra-regular ordered Γ-semiring. Then

(i) $\mu = \mu o_2 \mu$ for every Q-fuzzy bi-ideal μ of S.

(ii) $\mu = \mu o_2 \mu$ for every Q-fuzzy quasi-ideal μ of S.

Proof. (i) Let $x \in S$ and μ be any Q-fuzzy bi-ideal of S. Since S is both regular and intra-regular there exist $a, b \in S$ and $\alpha, \beta, \gamma, \delta, \eta \in \Gamma$ such that $x \leq x\eta a\alpha x\gamma x\beta b\delta x$.

Therefore

$$\begin{aligned}(\mu o_2 \mu)(x,q) &= \sup_{x \leq \sum_{i=1}^{n} a_i \gamma_i b_i}[\min_i\{\min\{\mu(a_i,q),\mu(b_i,q)\}\}] \\ &\geq \min_{x \leq x\eta a \alpha x \gamma x \beta b \delta x}\{\mu(x\eta a\alpha x,q),\mu(x\beta b\delta x,q)\} \\ &\geq \mu(x,q).\end{aligned}$$

Now $\mu o_2 \mu \subseteq \mu o_2 \chi_S \subseteq \mu$. Hence $\mu o_2 \mu = \mu$ for every Q-fuzzy bi-ideal μ of S.
(i) ⇒ (ii) This is straightforward from the Lemma 3.10. ■

Definition 3.16. A ordered Γ-semiring S is called (left, right) quasi-regular if every (left, right) ordered ideal of S is idempotent.

Then it is easily seen that, a ordered Γ-semiring S is left(right) quasi-regular if and only if $a \in S\Gamma a\Gamma S\Gamma a(resp.\ a \in a\Gamma S\Gamma a\Gamma S)$.

Theorem 3.17. A ordered Γ-semiring S is left(right) quasi-regular if and only if every Q-fuzzy left(right) ideal is idempotent.

Proof. Let S be a left quasi-regular ordered Γ-semiring and μ be any Q-fuzzy left ideal of S. Let $a \in S$. Since S is left quasi-regular $a \in (S\Gamma a)\Gamma(S\Gamma a)$. So, there exist $x, y \in S$ and $\alpha, \beta\gamma \in \Gamma$ such that $a \leq x\alpha a\beta y\gamma a$. Therefore

$$\begin{aligned}(\mu o_2 \mu)(a,q) &= \sup_{a \leq \sum_{i=1}^{n} a_i \gamma_i b_i}[\min_i\{\min\{\mu(a_i,q),\ \mu(b_i,q)\}\}] \\ &\geq \min_{a \leq x\alpha a\beta y\gamma a}\{\mu(x\alpha a,q),\ \mu(y\gamma a,q)\} \\ &\geq \min\{\mu(a,q),\ \mu(a,q)\} = \mu(a,q).\end{aligned}$$

Hence $\mu \subseteq \mu o_2 \mu$. Since μ is a Q-fuzzy left ideal of S, $\mu o_2 \mu \subseteq \mu$. Therefore $\mu = \mu o_2 \mu$.

Conversely, assume that every Q-fuzzy left ideal is idempotent. Let $a \in S$. Then $< a|$ is a left ideal of S. So, $\chi_{<a|}$ is a Q-fuzzy left ideal of S. Now, $\chi_{<a|\Gamma<a|}(a) = (\chi_{<a|} o_2 \chi_{<a|})(a) = \chi_{<a|}(a) = 1$. So, $a \in < a|\Gamma < a| = S\Gamma a\Gamma S\Gamma a$. Thus S is left quasi-regular. ■

We now obtain the following result regarding intra-regular and quasi-regular ordered Γ-semiring.

Theorem 3.18. Let S be a ordered Γ-semiring which is both intra-regular and left quasi-regular. Then

(i) $\mu \cap \nu \cap \omega \subseteq \mu o_2 \nu o_2 \omega$, for every Q-fuzzy bi-ideal ω, every Q-fuzzy left ideal μ and every right ideal ν of S.

(ii) $\mu \cap \nu \cap \omega \subseteq \mu o_2 \nu o_2 \omega$, for every Q-fuzzy quasi-ideal ω, every Q-fuzzy left ideal μ and every right ideal ν of S.

Proof. (i) Let μ, ν, ω be Q-fuzzy left ideal, Q-fuzzy right ideal and Q-fuzzy bi-ideal respectively. Since S is intra-regular and left quasi-regular, any element $a \in S$ can be expressed as $a \leq (x\alpha a)\beta(a\eta a)\delta(y\gamma a)$ for $x, y \in S$ and $\eta, \alpha, \beta, \gamma, \delta \in \Gamma$. Therefore

$$\begin{aligned}(\mu o_2 \nu o_2 \omega)(a, q) &= \sup_{a \leq \sum_{i=1}^{n} a_i \gamma_i b_i}[\min_i\{\min\{\mu(a_i, q), (\nu o_2 \omega)(b_i, q)\}\}] \\ &\geq \min_{a \leq (x\alpha a)\beta(a\eta a)\delta(y\gamma a)}\{\mu(x\alpha a\beta a, q), (\nu o_2 \omega)(a\delta y\gamma a, q)\} \\ &\geq \min_{a\delta y\gamma a \leq a\delta y\gamma x\alpha a\beta(a\eta a)\delta y\gamma a}\{\mu(a, q), \nu(a\delta y\gamma x\alpha a\beta a, q), \omega(a\delta y\gamma a, q)\} \\ &\geq \min\{\mu(a, q), \nu(a, q), \omega(a, q)\} = (\mu \cap \nu \cap \omega)(a, q).\end{aligned}$$

$(i) \Rightarrow (ii)$ follows from Lemma 3.10. ■

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