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Implication-Based Intuitionistic Fuzzy Subgroup of a Finite Group

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Abstract

The concept of *implication-based intuitionistic fuzzy subgroup* and *implication-based intutitionistic fuzzy normal subgroup* of a group are introduced using the notion of *implication-based fuzzy subgroup*. The external product of *implication-based intuitionistic fuzzy subgroups* is developed. Few fundamental properties concerning them are proved.

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Keywords: *Implication-based fuzzy subgroup*, *Implication-based intuitionistic fuzzy subgroup*, *External Product of Implication-based intuitionistic fuzzy subgroups*.

1. Introduction

In 1965, the concept of *fuzzy* set was first put forth by Zadeh [1]. Later in 1969 the notion of *fuzzy automata* was developed by Wee [2]. In 1971, Rosenfeld [3] applied the concept of fuzzy sets introduced by Zadeh to groups and constituted the elementary theory of groupoids and groups. From then on many research works had been carried out on their algebraic structures. Several studies had also been made on the *fuzzy normal subgroup* by [4], [5] and [6]. Asok Kumar [7] studied about the products of *fuzzy* *subgroups*. In 1986, Atanassov [8] generalised the concept of *fuzzy set* given by Zadeh [1] and introduced the notion of *intuitionistic fuzzy set*. Li Xiaoping [9] introduced the concept of *intuitionistic fuzzy subgroup* and *intuitionistic fuzzy normal subgroup*. In 2003, *Implication-based fuzzy subgroup* was developed by Yuan [10]. In 2015, We [11] studied about *implication-based fuzzy normal subgroup* over a finite group. In this paper, further research has been done and have introuduced the concept of *implication-based intuitionistic fuzzy subgroup*, *inplication-based intuitionitistic fuzzy normal subgroup* and also the external product of *implication-based intuitionistic fuzzy subgroups*. We also proved few properties concerning them.

2. Preliminaries

Definition 2.1. [3] Let $(G, \cdot)$ be a group. Let a fuzzy set in G be a function A from G to $[0, 1]$. A will be called a *fuzzy subgroup* of G, if for all x_1, x_2 in G,

$$A(x_1x_2) \geq min\big(A(x_1), A(x_2)\big)$$

$$A({x_1}^{-1}) \geq A(x_1).$$

Definition 2.2. [8] Let X be a nonempty classical set. The traid formed as $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ on X is called an *intuitionistic fuzzy set* on X, where the functions $\mu_A : A \to (0, 1)$ and $\nu_A : A \to (0, 1)$ denotes the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2.3. [12] Let G be a classical group, *the intuitionistic fuzzy subset* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$ is called an *intuitionistic fuzzy group* on G, if the following conditions are satisfied.

(i) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$, $\nu_A(xy) \leq max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in G$

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$, for all $x \in G$.

Definition 2.4. [9] Let G be a classical group, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in G\}$ be an *intuitionistic fuzzy set* on G, then A is called *intuitionistic fuzzy normal subgroup* on G if $\mu_A(xyx^{-1}) \geq \mu_A(y)$, $\nu_A(xyx^{-1}) \leq \nu_A(y)$ for all $x, y \in G$.

Let X be an universe of discourse and $(G, \cdot)$ be a group. In *fuzzy* logic, $[\alpha]$ is used to denote the truth value of *fuzzy* proposition α. The *fuzzy* logical and the corresponding set theoretical notations used in this paper are

$(x \in A) = A(x);$

$(\alpha \wedge \beta) = min\{[\alpha], [\beta]\};$

$(\alpha \to \beta) = min\{1, 1 - [\alpha] + [\beta]\};$

$(\forall x\ \alpha(x)) = inf_{x\in X}[\alpha(x)]$;

$(\exists x\ \alpha(x)) = sup_{x\in X}[\alpha(x)]$;

$\models \alpha$ if and only if $[\alpha] = 1$ for all valuations.

The truth valuation rules used here are that of the Lukasiewicz system of continuous-valued logic.

Definition 2.5. [10] If a *fuzzy* subset A of a group G satisfies for any $x_1, x_2 \in G$

(i) $\models (x_1 \in A) \wedge (x_2 \in A) \to (x_1x_2 \in A)$

(ii) $\models (x_1 \in A) \to ({x_1}^{-1} \in A)$.

Then A is called a fuzzifying subgroup.

The concept of λ - tautology is $\models_\lambda \alpha$ if and only if $(\alpha) \geq \lambda$ for all valuation by Ying [13].

Definition 2.6. [10] Let A be a *fuzzy* subset of a finite group G and $\lambda \in (0, 1]$ is a fixed number. If for any $x_1, x_2 \in G$

(i) $\models_\lambda (x_1 \in A) \wedge (x_2 \in A) \to (x_1x_2 \in A)$

(ii) $\models_\lambda (x_1 \in A) \to ({x_1}^{-1} \in A)$.

Then A is called an *implication-based fuzzy subgroup* of G.

Definition 2.7. [11] Let A be an *implication-based fuzzy subgroup* of G, $\lambda \in (0, 1]$ is a fixed number and $f : G \to G$ be a function defined on G. Then the *implication-based fuzzy subgroup* B of $f(G)$ is defined by $\models_\lambda \big(\exists x\{(x \in A)\}; x \in f^{-1}(y)\big) \to (y \in B)$, for all $y \in f(G)$.

Similarly if B is an *implication-based fuzzy subgroup* of $f(G)$ then the *implication-based fuzzy subgroup* $A = f \circ B$ in G is defined as $\models_\lambda (f(x) \in B) \to (x \in A)$ for all $x \in G$ and is called the pre-image of B under f.

Definition 2.8. [11] An *implication-based fuzzy subgroup* A of G is called an *implication-based fuzzy normal subgroup* if $\models_\lambda (xy \in A) \to (yx \in A) \quad \forall x, y \in G$ where $\lambda \in (0, 1]$ is a fixed number.

Proposition 2.9. [11] Let A be an implication-based fuzzy subgroup of a finite group G then for any $x \in G$, $\models_\lambda (x \in A) \to (e \in A)$ where e is the identity element of the group G.

Hereafter let G be a finite group with the identity element $'e'$.

3. Implication-Based Intuitionistic Fuzzy Subgroup of a Finite Group

Definition 3.1. An *intuitionistic fuzzy subset* $< g, A, B >$ of a group G is called as an *implication-based intuitionistic fuzzy subgroup* if it satisfies for any $x, y \in G$

(i) $\models_\lambda ((x \in A) \wedge (y \in A)) \to (xy \in A)$

(ii) $\models_\lambda (xy \in B) \to ((x \in B) \vee (y \in B))$

(iii) $\models_\lambda (x \in A) \to (x^{-1} \in A)$

(iv) $\models_\lambda (x^{-1} \in B) \to (x \in B)$

where $(x \in A)$ denotes the degree of membership and $(x \in B)$ denotes the degree of non-membership and is denoted by $IF_{\mathscr{A}} =< A, B >$.

Consider the group $G = \{e, a, b, c, d, f, g, h\}$ whose Cayley's Closure Table is given by

·	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	d	c	g	f	e	h	b
b	b	h	d	a	g	c	e	f
c	c	b	f	d	h	g	a	e
d	d	f	g	h	e	a	b	c
f	f	e	h	b	a	d	c	g
g	g	c	e	f	b	h	d	a
h	h	g	a	e	c	b	f	d

The membership function $A : G \to [0, 1]$ is given by $A(e) = 0.675$, $A(a) = 0.650$, $A(b) = 0.625$, $A(c) = 0.600$, $A(d) = 0.575$, $A(f) = 0.550$, $A(g) = 0.525$ and $A(h) = 0.450$. The non-membership function $B : G \to [0, 1]$is given by $B(e) = 0.125$, $B(a) = 0.150$, $B(b) = 0.175$, $B(c) = 0.200$, $B(d) = 0.225$, $B(f) = 0.250$, $B(g) = 0.275$ and $B(h) = 0.300$. The following table gives the $\wedge$ of the elements of G.

∧	[e]	[a]	[b]	[c]	[d]	[f]	[g]	[h]
[e]	0.675	0.650	0.625	0.600	0.575	0.550	0.525	0.450
[a]	0.650	0.650	0.625	0.600	0.575	0.550	0.525	0.450
[b]	0.625	0.625	0.625	0.600	0.575	0.550	0.525	0.450
[c]	0.600	0.600	0.600	0.600	0.575	0.550	0.525	0.450
[d]	0.575	0.575	0.575	0.575	0.575	0.550	0.525	0.450
[f]	0.550	0.550	0.550	0.550	0.550	0.550	0.525	0.450
[g]	0.525	0.525	0.525	0.525	0.525	0.525	0.525	0.450
[h]	0.450	0.450	0.450	0.450	0.450	0.450	0.450	0.450

The following table gives the $\vee$ of the elements of G.

$\vee$	[e]	[a]	[b]	[c]	[d]	[f]	[g]	[h]
[e]	0.125	0.150	0.175	0.200	0.225	0.250	0.275	0.300
[a]	0.150	0.150	0.175	0.200	0.225	0.250	0.275	0.300
[b]	0.175	0.175	0.175	0.200	0.225	0.250	0.275	0.300
[c]	0.200	0.200	0.200	0.200	0.225	0.250	0.275	0.300
[d]	0.225	0.225	0.225	0.225	0.225	0.250	0.275	0.300
[f]	0.250	0.250	0.250	0.250	0.250	0.250	0.275	0.300
[g]	0.275	0.275	0.275	0.275	0.275	0.275	0.275	0.300
[h]	0.300	0.300	0.300	0.300	0.300	0.300	0.300	0.300

With $\lambda = 0.5$ and the *implication* operator is that of Lukasiewicz, then $IF_{\mathscr{A}} =< A, B >$ is an *implication-based intuitionistic fuzzy subgroup*.

Theorem 3.2. Let $IF_{\mathscr{A}} =< A, B >$ be an implication-based intuitionistic fuzzy subgroup of a group G then for all $x \in G$ we have

(i) $\models_\lambda (x \in A) \to (e \in A)$

(ii) $\models_\lambda (e \in B) \to (x \in B)$.

Proof. Let $x \in G$ then

$$\begin{aligned}\models_\lambda (x \in A) &\to ((x \in A) \wedge (x^{-1} \in A))\\ &\to (xx^{-1} \in A)\\ &\to (e \in A)\end{aligned}$$

Therefore $\models_\lambda (x \in A) \to (e \in A)$

$$\begin{aligned}\models_\lambda (e \in B) &\to (xx^{-1} \in B)\\ &\to ((x \in B) \vee (x^{-1} \in B))\\ &\to (x \in B)\end{aligned}$$

Therefore $\models_\lambda (e \in B) \to (x \in B)$. ■

Theorem 3.3. An intuitionistic fuzzy subset $\mathscr{A} =< g, A, B >$ of a group G is said to be an implication-based intuitionistic fuzzy subgroup $IF_{\mathscr{A}} =< A, B >$ if and only if for all $x, y \in G$

(i) $\models_\lambda ((x \in A) \wedge (y \in A)) \to (xy^{-1} \in A)$

(ii) $\models_\lambda (xy^{-1} \in B) \to ((x \in B) \vee (y \in B))$.

Proof. Assume that the *intuitionistic fuzzy subset* $\mathscr{A} =< g, A, B >$ of a group G be an *implication-based intuitionistic fuzzy subgroup* $IF_{\mathscr{A}} =< A, B >$.

Let $x, y \in G$

$$\vDash_\lambda ((x \in A) \wedge (y \in A)) \to \big((x \in A) \wedge (y^{-1} \in A)\big)$$
$$\to (xy^{-1} \in A) \text{ since } \vDash_\lambda (x \in A) \to (x^{-1} \in A)$$

$$\vDash_\lambda (xy^{-1} \in B) \to ((x \in B) \vee (y^{-1} \in B))$$
$$\to ((x \in B) \vee (y \in B)) \text{ since } \vDash_\lambda (y^{-1} \in B) \to (y \in B)$$

Conversely

Let for all $x, y \in G$

$$\vDash_\lambda ((x \in A) \wedge (y \in A)) \to (xy^{-1} \in A) \quad (3.1)$$

$$\vDash_\lambda (xy^{-1} \in B) \to ((x \in B) \vee (y \in B)) \quad (3.2)$$

Let $y = x$ in equation (1).

$$\vDash_\lambda ((x \in A) \wedge (y \in A)) \to (x \in A) \wedge (x \in A))$$
$$\to (xx^{-1} \in A)$$
$$\to (e \in A)$$

(i.e) $\vDash_\lambda (x \in A) \to (e \in A)$

$$\vDash_\lambda (y \in A) \to ((e \in A) \wedge (y \in A))$$
$$\to (ey^{-1} \in A)$$
$$\to (y^{-1} \in A)$$

(i.e) $\vDash_\lambda (y \in A) \to (y^{-1} \in A)$

$$\vDash_\lambda ((x \in A) \wedge (y \in A)) \to (x \in A) \wedge (y^{-1} \in A))$$
$$\to (x(y^{-1})^{-1} \in A)$$
$$\to (xy \in A)$$

Let $y = x$ in equation (2).

$\vDash_\lambda (xx^{-1} \in B) \to ((x \in B) \vee (x \in B))$

(i.e) $\vDash_\lambda (e \in B) \to (x \in B)$

$$\vDash_\lambda (x^{-1} \in B) \to (ex^{-1} \in B)$$
$$\to ((e \in B) \vee (x \in B))$$
$$\to (x \in B)$$

(i.e) $\vDash_\lambda (x^{-1} \in B) \to (x \in B)$

$$\vDash_\lambda (xy \in B) \to (x(y^{-1})^{-1} \in B)$$
$$\to ((x \in B) \vee (y^{-1} \in B))$$
$$\to ((x \in B) \vee (y \in B))$$

Therefore $IF_{\mathscr{A}} =< A, B >$ is an *implication-based intuitionistic fuzzy subgroup* of G. ■

Definition 3.4. An *implication-based intuitionistic fuzzy subgroup* of G, $IF_{\mathscr{A}} =< A, B >$ is said to be an *implication-based intuitionistic fuzzy normal subgroup* of G if for all $x, y \in G$

$$\models_\lambda (xy \in A) \to (yx \in A)$$

$$\models_\lambda (xy \in B) \to (yx \in B).$$

Theorem 3.5. Let $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ be two implication-based intuitiontistic fuzzy subgroups over a group G. Then $IF_{\mathscr{A}_1 \cap \mathscr{A}_2} =< A_1 \wedge A_2, B_1 \vee B_2 >$ is also an implication-based intuitionistic fuzzy subgroup over the group G. Further if $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ are two implication-based intuitiontistic fuzzy normal subgroups over a group G then $IF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is also an implication-based intuitionistic fuzzy normal subgroup of G.

Proof. Let $x, y \in G$

$$\begin{aligned}\models_\lambda ((x \in A_1 \wedge A_2) \wedge (y \in A_1 \wedge A_2)) \\ &\to (((x \in A_1) \wedge (x \in A_2)) \wedge ((y \in A_1) \wedge (y \in A_2))) \\ &\to (((x \in A_1) \wedge (y \in A_1)) \wedge ((x \in A_2) \wedge (y \in A_2))) \\ &\to ((xy^{-1} \in A_1) \wedge (xy^{-1} \in A_2)) \\ &\quad \text{Since } IF_{\mathscr{A}_1} \text{ and } IF_{\mathscr{A}_2} \text{ are implication-based intuitionistic fuzzy subgroups} \\ &\to (xy^{-1} \in A_1 \wedge A_2)\end{aligned}$$

$$\begin{aligned}\models_\lambda (xy^{-1} \in B_1 \vee B_2) \\ &\to ((xy^{-1} \in B_1) \vee (xy^{-1} \in B_2)) \\ &\to (((x \in B_1) \vee (y \in B_1)) \vee ((x \in B_2) \vee (y \in B_2))) \\ &\quad \text{Since } IF_{\mathscr{A}_1} \text{ and } IF_{\mathscr{A}_2} \text{ are implication-based intuitionistic fuzzy subgroups} \\ &\to (((x \in B_1) \vee (x \in B_2)) \vee ((y \in B_1) \vee (y \in B_2))) \\ &\to ((x \in B_1 \vee B_2) \vee (y \in B_1 \vee B_2))\end{aligned}$$

By theorem 3.3, $IF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is an *implication-based intuitionistic fuzzy subgroup* over a group G.

$$\begin{aligned}\models_\lambda (xy \in A_1 \wedge A_2) &\to ((xy \in A_1) \wedge (xy \in A_2)) \\ &\to ((yx \in A_1) \wedge (yx \in A_2)) \\ &\qquad \text{Since } IF_{\mathscr{A}_1} \text{ and } IF_{\mathscr{A}_2} \text{ are implication-based} \\ &\qquad \text{intuitionistic fuzzy normal subgroups} \\ &\to (yx \in A_1 \wedge A_2)\end{aligned}$$

$$\begin{aligned}\models_\lambda (xy \in B_1 \vee B_2) &\to ((xy \in B_1) \vee (xy \in B_2))\\ &\to ((yx \in B_1) \vee (yx \in B_2))\\ &\qquad \text{Since } IF_{\mathscr{A}_1} \text{ and } IF_{\mathscr{A}_2} \text{ are implication-based}\\ &\qquad \text{intuitionistic fuzzy normal subgroups}\\ &\to (yx \in B_1 \vee B_2)\end{aligned}$$

Thus $IF_{\mathscr{A}_1 \cap \mathscr{A}_2}$ is an *implication-based intuitionistic fuzzy normal subgroup* over a group G. ■

Definition 3.6. Let $IF_{\mathscr{A}_i} =< A_i, B_i >$ be a family of *implication-based intuitionistic fuzzy normal subgroups* over a group G. Then we define $IF_{\cap \mathscr{A}_j} =< \forall j A_j, \exists j B_j >$, by previous theorem $IF_{\cap \mathscr{A}_j}$ is also an *implication-based intuitionistic fuzzy normal subgroup* over the group G.

Theorem 3.7. Let $IF_{\mathscr{A}} =< A, B >$ be an implication-based intuitionistic fuzzy subgroup of a finite group G. Let $x \in G$ then $\models_\lambda (xy \in A) \to (y \in A)$ and $\models_\lambda (y \in B) \to (xy \in B)$ if and only if $\models_\lambda (x \in A) \to (e \in A)$ and $\models_\lambda (e \in B) \to (x \in B)$ for all $y \in G$.

Proof. Let $x \in G$ such that
$\models_\lambda (xy \in A) \to (y \in A)$ and $\models_\lambda (y \in B) \to (xy \in B)$
Put $y = e$ we get
$\models_\lambda (xe \in A) \to (e \in A)$ and $\models_\lambda (e \in B) \to (xe \in B)$
(i.e) $\models_\lambda (x \in A) \to (e \in A)$ and $\models_\lambda (e \in B) \to (x \in B)$
Conversely assume that
$\models_\lambda (x \in A) \to (e \in A)$ and $\models_\lambda (e \in B) \to (x \in B)$ for all $x \in G$
Let $y \in G$
Since $\models_\lambda (x \in A) \to (e \in A)$ for all $x \in G$
This is true in particular for $y \in G$.
$\Rightarrow \models_\lambda (y \in A) \to (e \in A)$
Therefore
$\models_\lambda (y \in A) \to (x \in A)$ (1)
But $\models_\lambda ((x \in A) \wedge (y \in A)) \to (xy \in A)$
$\Rightarrow \models_\lambda (y \in A) \to (xy \in A)$ by (1) (2)
Now

$$\begin{aligned}\models_\lambda ((x \in A) \wedge (xy \in A)) &\to ((x^{-1} \in A) \wedge (xy \in A))\\ &\to ((x^{-1}.xy \in A))\\ &\to (y \in A)\\ &\to (xy \in A) \text{ by (2)}\end{aligned}$$

$\models_\lambda ((x \in A) \wedge (xy \in A)) \to ((x \in A) \wedge (y \in A))$
(i.e) $\models_\lambda (xy \in A) \to (y \in A)$
Now $\models_\lambda (e \in B) \to (x \in B)$ for all $x \in G$
This is true in particular for $y \in G$

Therefore $\models_\lambda (e \in B) \to (y \in B)$
$\Rightarrow \models_\lambda (x \in B) \to (y \in B)$
But $\models_\lambda (xy \in B) \to (x \in B) \vee (y \in B)$
$\Rightarrow \models_\lambda (xy \in B) \to (y \in B)$

$$\begin{aligned}\models_\lambda (y \in B) &\to (x^{-1}.xy \in B)\\ &\to ((x^{-1} \in B) \vee (xy \in B))\\ &\to ((x \in B) \vee (xy \in B))\\ &\to (xy \in B)\end{aligned}$$

■

Theorem 3.8. Let $IF_{\mathscr{A}} =< A, B >$ be an implication-based intuitionistic fuzzy subgroup of a finite group G. Then $IF_{\mathscr{A}}$ is an implication-based intuitionistic fuzzy normal subgroup of G if and only if

(i) $\models_\lambda (y^{-1}xy \in A) \to (x \in A)$

(ii) $\models_\lambda (x \in B) \to (y^{-1}xy \in B)$ for all $x, y \in G$.

Proof. Let $IF_{\mathscr{A}}$ be an implication-based intuitionistic fuzzy normal subgroup of G. Let $x, y \in G$.

$$\models_\lambda (y^{-1}xy \in A) \to (xy^{-1}y \in A) \to (x \in A)$$
$$\models_\lambda (x \in B) \to (xy^{-1}y \in B) \to (y^{-1}xy \in B)$$

Conversely assume that

(i) $\models_\lambda (y^{-1}xy \in A) \to (x \in A)$

(ii) $\models_\lambda (x \in B) \to (y^{-1}xy \in B)$ for all $x, y \in G$

In (i) put $x = yx$ we get

$$\begin{aligned}&\models_\lambda (y^{-1}xy \in A) \to (x \in A)\\ \Rightarrow\ &\models_\lambda (y^{-1}.yx.y \in A) \to (yx \in A)\\ \Rightarrow\ &\models_\lambda (xy \in A) \to (yx \in A)\end{aligned}$$

In (ii) put $x = yx$ we get

$$\begin{aligned}&\models_\lambda (x \in B) \to (y^{-1}xy \in B)\\ \Rightarrow\ &\models_\lambda (yx \in B) \to (y^{-1}.yx.y \in B)\\ \Rightarrow\ &\models_\lambda (yx \in B) \to (xy \in B)\end{aligned}$$

Therefore $IF_{\mathscr{A}}$ is an implication-based intuitionistic fuzzy normal subgroup of G. ■

Theorem 3.9. Let $IF_{\mathscr{A}} =< A, B >$ be an implication-based intuitionistic fuzzy subgroup of G. Then $IF_{\mathscr{A}}$ is an implication-based intuitionistic fuzzy normal subgroup of G if and only if

(i) $\models_\lambda (x \in A) \to (x^{-1}y^{-1}xy \in A)$

(ii) $\models_\lambda (x^{-1}y^{-1}xy \in B) \to (x \in B)$ for all $x, y \in G$.

Proof. Let $IF_{\mathscr{A}}$ be an implication-based intuitionistic fuzzy normal subgroup of G. Let $x, y \in G$.

$$\begin{aligned}\models_\lambda (x \in A) &\to ((x \in A) \wedge (x \in A))\\ &\to ((y^{-1}xy \in A) \wedge (x^{-1} \in A)) \text{ by Theorem 3.8}\\ &\to (y^{-1}xyx^{-1} \in A)\\ &\to (x^{-1}y^{-1}xy \in A) \text{ by assumption}\end{aligned}$$

$$\begin{aligned}\models_\lambda (x^{-1}y^{-1}xy \in B) &\to (y^{-1}xyx^{-1} \in B)\\ &\to ((y^{-1}xy \in B) \vee (x^{-1} \in B))\\ &\to ((x \in B) \vee (x \in B)) \text{ by Theorem 3.8}\\ &\to (x \in B)\end{aligned}$$

Conversely assume that

(i) $\models_\lambda (x \in A) \to (x^{-1}y^{-1}xy \in A)$

(ii) $\models_\lambda (x^{-1}y^{-1}xy \in B) \to (x \in B)$ for all $x, y \in G$

Let $x, z \in G$

$$\begin{aligned}\models_\lambda (z \in A) &\to ((z \in A) \wedge (z^{-1}x^{-1}zx \in A)) \text{ by assumption}\\ &\to (z.z^{-1}x^{-1}zx \in A)\\ &\to (x^{-1}zx \in A)\end{aligned}$$

(i.e) $\models_\lambda (z \in A) \to (x^{-1}zx \in A)$

$$\begin{aligned}\models_\lambda (x^{-1}zx \in B) &\to (zz^{-1}x^{-1}zx \in B)\\ &\to ((z \in B) \vee (z^{-1}x^{-1}zx \in B))\\ &\to ((z \in B) \vee (z \in B))\\ &\to (z \in B)\end{aligned}$$

(i.e) $\models_\lambda (x^{-1}zx \in B) \to (z \in B)$

Moreover $$\begin{aligned}\models_\lambda ((x \in A) \wedge (x^{-1}zx \in A)) &\to ((x \in A) \wedge (x^{-1}zx \in A) \wedge (x^{-1} \in A))\\ &\to (xx^{-1}zxx^{-1} \in A)\\ &\to (z \in A)\end{aligned}$$

(i.e) $\models_\lambda ((x \in A) \wedge (x^{-1}zx \in A)) \rightarrow (z \in A)$

$$\begin{aligned}\text{Also } \models_\lambda (z \in B) &\rightarrow (xx^{-1}zxx^{-1} \in B)\\ &\rightarrow ((x \in B) \vee (x^{-1}zx \in B) \vee (x^{-1} \in B))\\ &\rightarrow ((x \in B) \vee (x^{-1}zx \in B))\end{aligned}$$

(i.e) $\models_\lambda (z \in B) \rightarrow ((x \in B) \vee (x^{-1}zx \in B))$.
Case (i)

$$\begin{aligned}\text{Suppose } &\models_\lambda ((x \in A) \wedge (x^{-1}zx \in A)) \rightarrow (x \in A)\\ \text{And } &\models_\lambda (x \in B) \rightarrow ((x \in B) \vee (x^{-1}zx \in B))\\ \Rightarrow &\models_\lambda (x \in A) \rightarrow (z \in A) \text{ and}\\ &\models_\lambda (z \in B) \rightarrow (x \in B) \text{ for all } x, z \in G\\ \Rightarrow &\models_\lambda (xz \in A) \rightarrow (zx \in A) \text{ and}\\ &\models_\lambda (xz \in B) \rightarrow (zx \in B) \text{ for all } x, z \in G\end{aligned}$$

Thus $IF_{\mathscr{A}}$ is an implication-based intuitionistic fuzzy normal subgroup of G.
Case (ii)

$$\begin{aligned}\text{Suppose } &\models_\lambda ((x \in A) \wedge (x^{-1}zx \in A)) \rightarrow (x^{-1}zx \in A)\\ \text{And } &\models_\lambda (x^{-1}zx \in B) \rightarrow ((x \in B) \vee (x^{-1}zx \in B))\\ \Rightarrow &\models_\lambda (x^{-1}zx \in A) \rightarrow (z \in A)\\ &\models_\lambda (z \in B) \rightarrow (x^{-1}zx \in B) \text{ for all } x, z \in G\end{aligned}$$

By Theorem 3.8, $IF_{\mathscr{A}}$ is an implication-based intuitionistic fuzzy normal subgroup of G. ■

Theorem 3.10. Let $IF_{\mathscr{A}}$ be an implication-based intuitionistic fuzzy subgroup of G. If $x, y \in G$ such that

$$\models_\lambda (xy^{-1} \in A) \rightarrow (e \in A) \text{ and } \models_\lambda (e \in A) \rightarrow (xy^{-1} \in A)$$

$$\models_\lambda (e \in B) \rightarrow (xy^{-1} \in B) \text{ and } \models_\lambda (xy^{-1} \in B) \rightarrow (e \in B) \text{ then}$$

(i) $\models_\lambda (x \in A) \rightarrow (y \in A)$ and $\models_\lambda (y \in A) \rightarrow (x \in A)$

(ii) $\models_\lambda (x \in B) \rightarrow (y \in B)$ and $\models_\lambda (y \in B) \rightarrow (x \in B)$.

Proof. Let $x, y \in G$

$$\begin{aligned}\models_\lambda (x \in A) &\rightarrow (xy^{-1}y \in A)\\ &\rightarrow ((xy^{-1} \in A) \wedge (y \in A))\\ &\rightarrow ((e \in A) \wedge (y \in A))\\ &\rightarrow (y \in A)\end{aligned}$$

$$
\begin{aligned}
Similarly \vDash_\lambda (y \in A) &\rightarrow (yx^{-1}x \in A)\\
&\rightarrow ((yx^{-1} \in A) \wedge (x \in A))\\
&\rightarrow ((e \in A) \wedge (x \in A))\\
&\rightarrow (x \in A)
\end{aligned}
$$

$$
\begin{aligned}
Now \vDash_\lambda (x \in B) &\rightarrow (ex \in B)\\
&\rightarrow ((e \in B) \vee (x \in B))\\
&\rightarrow ((yx^{-1} \in B) \vee (x \in B))\\
&\rightarrow (y \in B)
\end{aligned}
$$

$$
\begin{aligned}
Similarly \vDash_\lambda (y \in B) &\rightarrow (ey \in B)\\
&\rightarrow ((e \in B) \vee (y \in B))\\
&\rightarrow ((xy^{-1} \in B) \vee (y \in B))\\
&\rightarrow (x \in B)
\end{aligned}
$$

■

Definition 3.11. Let $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ be two *implication-based intuitiontistic fuzzy subgroups* over the two finite groups G_1 and G_2 respectively. The external product of $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ in $G_1 \times G_2$ is denoted by $IF_{\mathscr{A}_1 \times \mathscr{A}_2} =< A_1 \times A_2, B_1 \times B_2 >$ and is defined as

$$\vDash_\lambda ((x \in A_1) \wedge (y \in A_2)) \rightarrow ((x, y) \in A_1 \times A_2)$$

$$\vDash_\lambda ((x \in B_1) \vee (y \in B_2)) \rightarrow ((x, y) \in B_1 \times B_2) \text{ for all } x \in G_1, y \in G_2.$$

Theorem 3.12. The external product of two implication-based intuitionistic fuzzy subgroups $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ over two finite groups G_1 and G_2 respectively is again an implication-based intuitionistic fuzzy subgroup of $G_1 \times G_2$. Moreover, if $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ are two implication-based intuitionistic fuzzy normal subgroups over G_1 and G_2 respectively then $IF_{\mathscr{A}_1 \times \mathscr{A}_2}$ is also implication-based intuitionistic fuzzy normal subgroup of $G_1 \times G_2$ respectively.

Proof. Let $x_1, x_2 \in G_1$ and $y_1, y_2 \in G_2$

$$
\begin{aligned}
&\vDash_\lambda (((x_1, y_1) \in A_1 \times A_2) \wedge ((x_2, y_2) \in A_1 \times A_2))\\
&\rightarrow (((x_1 \in A_1) \wedge (y_1 \in A_2)) \wedge ((x_2 \in A_1) \wedge (y_2 \in A_2)))\\
&\rightarrow (((x_1 \in A_1) \wedge (y_1 \in A_2)) \wedge ((x_2{}^{-1} \in A_1) \wedge (y_2{}^{-1} \in A_2)))\\
&\rightarrow (((x_1 \in A_1) \wedge (x_2{}^{-1} \in A_1)) \wedge ((y_1 \in A_2) \wedge (y_2{}^{-1} \in A_2)))\\
&\rightarrow ((x_1x_2{}^{-1} \in A_1) \wedge (y_1y_2{}^{-1} \in A_2))\\
&\rightarrow ((x_1x_2{}^{-1}, y_1y_2{}^{-1}) \in A_1 \times A_2)
\end{aligned}
$$

$$\begin{aligned}
&\models_\lambda ((x_1x_2{}^{-1}, y_1y_2{}^{-1}) \in B_1 \times B_2)\\
&\to ((x_1x_2{}^{-1} \in B_1) \vee (y_1y_2{}^{-1} \in B_2))\\
&\to (((x_1 \in B_1) \vee (x_2{}^{-1} \in B_1)) \vee ((y_1 \in B_2) \vee (y_2{}^{-1} \in B_2)))\\
&\to (((x_1 \in B_1) \vee (y_1 \in B_2)) \vee ((x_2{}^{-1} \in B_1) \vee (y_2{}^{-1} \in B_2)))\\
&\to (((x_1 \in B_1) \vee (y_1 \in B_2)) \vee ((x_2 \in B_1) \vee (y_2 \in B_2)))\\
&\to (((x_1, y_1) \in B_1 \times B_2) \vee ((x_2, y_2) \in B_1 \times B_2))
\end{aligned}$$

By Theorem 3.3, $IF_{\mathscr{A}_1\times\mathscr{A}_2}$ is an *implication-based intuitionistic fuzzy subgroup* of $G_1 \times G_2$.

Let $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ be two implication-based intuitionistic fuzzy normal subgroups over G_1 and G_2 respectively.

$$\begin{aligned}
&\models_\lambda ((x_1x_2, y_1y_2) \in A_1 \times A_2)\\
&\to ((x_1x_2 \in A_1) \wedge (y_1y_2 \in A_2))\\
&\to ((x_2x_1 \in A_1) \wedge (y_2y_1 \in A_2))\\
&\to ((x_2x_1, y_2y_1) \in A_1 \times A_2)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\models_\lambda ((x_1x_2, y_1y_2) \in B_1 \times B_2)\\
&\to ((x_1x_2 \in B_1) \vee (y_1y_2 \in B_2))\\
&\to ((x_2x_1 \in B_1) \vee (y_2y_1 \in B_2))\\
&\to ((x_2x_1, y_2y_1) \in B_1 \times B_2)
\end{aligned}$$

Therefore $IF_{\mathscr{A}_1\times\mathscr{A}_2}$ is an *implication-based intuitionistic fuzzy normal subgroup* of $G_1 \times G_2$. ■

Theorem 3.13. Let $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ be two implication-based intuitionistic fuzzy subgroups over two finite groups G_1 and G_2 respectively. Let e_1 and e_2 be the identity elements of the groups G_1 and G_2 respectively. If $IF_{\mathscr{A}_1\times\mathscr{A}_2}$ is an *implication-based intuitionistic fuzzy subgroup* of $G_1 \times G_2$ then atleast one of the following statements must hold.

(i) $\models_\lambda (x \in A_1) \to (e_2 \in A_2)$ and $\models_\lambda (e_2 \in B_2) \to (x \in B_1)$ for all $x \in G_1$

(ii) $\models_\lambda (y \in A_2) \to (e_1 \in A_1)$ and $\models_\lambda (e_1 \in B_1) \to (y \in B_2)$ for all $y \in G_2$.

Proof. Let $IF_{\mathscr{A}_1\times\mathscr{A}_2}$ be an *implication-based intuitionistic fuzzy subgroup* of $G_1 \times G_2$ and let the statements (i) and (ii) do not hold.

Then there exists $x \in G_1$ and $y \in G_2$ such that

$\models_\lambda (e_2 \in A_2) \to (x \in A_1)$ and $\models_\lambda (x \in B_1) \to (e_2 \in B_2)$ (1)

$\models_\lambda (e_1 \in A_1) \to (y \in A_2)$ and $\models_\lambda (y \in B_2) \to (e_1 \in B_1)$ (2)

Now

$$\begin{aligned}\vDash_\lambda ((e_1, e_2) \in A_1 \times A_2) &\to ((e_1 \in A_1) \wedge (e_2 \in A_2))\\ &\to ((y \in A_2) \wedge (x \in A_1)) \text{ by (1) and (2)}\\ &\to ((x \in A_1) \wedge (y \in A_2))\\ &\to ((x, y) \in A_1 \times A_2)\end{aligned}$$

And

$$\begin{aligned}\vDash_\lambda ((x, y) \in B_1 \times B_2) &\to ((x \in B_1) \vee (y \in B_2))\\ &\to ((e_2 \in B_2) \vee (e_1 \in B_1)) \text{ by (1) and (2)}\\ &\to ((e_1 \in B_1) \vee (e_2 \in B_2))\\ &\to ((e_1, e_2) \in B_1 \times B_2)\end{aligned}$$

$\Rightarrow IF_{\mathscr{A}_1 \times \mathscr{A}_2}$ is not an *implication-based intuitionistic fuzzy subgroup* which is a contradiction to our assumption. Hence either (i) or (ii) should hold. ■

Theorem 3.14. Let $< g_1, A_1, B_1 >$ and $< g_2, A_2, B_2 >$ be two intuitionistic fuzzy subsets over two finite groups G_1 and G_2 respectively such that for all $x \in G_1$

$\vDash_\lambda (x \in A_1) \to (e_2 \in A_2)$ (1)

$\vDash_\lambda (e_2 \in B_2) \to (x \in B_1)$ (2)

where e_2 is the identity element of G_2. If $IF_{\mathscr{A}_1 \times \mathscr{A}_2} =< A_1 \times A_2, B_1 \times B_2 >$ is an implication-based intuitionistic fuzzy subgroup of $G_1 \times G_2$ then $IF_{\mathscr{A}_1}$ is an implication-based intuitionistic fuzzy subgroup of G_1.

Proof. Let $x, y \in G_1$

$\Rightarrow (x, e_2), (y, e_2) \in G_1 \times G_2$

$$\begin{aligned}\vDash_\lambda ((x \in A_1) \wedge (y \in A_1)) &\to ((x \in A_1) \wedge (y^{-1} \in A_1))\\ &\to (((x \in A_1) \wedge (e_2 \in A_2)) \wedge ((y^{-1} \in A_1) \wedge (e_2 \in A_2))) \text{ by (1)}\\ &\to (((x, e_2) \in A_1 \times A_2) \wedge ((y^{-1}, e_2) \in A_1 \times A_2))\\ &\to ((xy^{-1}, e_2e_2) \in A_1 \times A_2)\\ &\to ((xy^{-1} \in A_1) \wedge (e_2e_2 \in A_2))\\ &\to (xy^{-1} \in A_1)\end{aligned}$$

$$\begin{aligned}\vDash_\lambda (xy^{-1} \in B_1) &\to ((xy^{-1} \in B_1) \vee (e_2e_2 \in B_2))\\ &\to ((xy^{-1}, e_2e_2) \in B_1 \times B_2)\\ &\to (((x, e_2) \in B_1 \times B_2) \vee ((y^{-1}, e_2) \in B_1 \times B_2))\\ &\to (((x \in B_1) \vee (e_2 \in B_2)) \vee ((y^{-1} \in B_1) \vee (e_2 \in B_2)))\\ &\to ((x \in B_1) \vee (y^{-1} \in B_1)) \text{ by (2)}\\ &\to ((x \in B_1) \vee (y \in B_1))\end{aligned}$$

By Theorem 3.3, $IF_{\mathscr{A}_1} =< A_1, B_1 >$ is an implication-based intuitionistic fuzzy subgroup of G_1. ■

Corollary 3.15. Let $< g_1, A_1, B_1 >$ and $< g_2, A_2, B_2 >$ be two intuitionistic fuzzy subsets over two finite groups G_1 and G_2 respectively such that for all $y \in G_2$
$\models_\lambda (y \in A_2) \to (e_1 \in A_1)$ (1)
$\models_\lambda (e_1 \in B_1) \to (y \in B_2)$ (2)
where e_1 is the identity element of G_1. If $IF_{\mathscr{A}_1 \times \mathscr{A}_2} =< A_1 \times A_2, B_1 \times B_2 >$ is an implication-based intuitionistic fuzzy subgroup of $G_1 \times G_2$ then $IF_{\mathscr{A}_2}$ is an implication-based intuitionistic fuzzy subgroup of G_2.

Definition 3.16. Let $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ be two *implication-based intuitiontistic fuzzy subgroups* over the finite group G. Then $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ are said to be *implication-based intuitionistic fuzzy conjugate subgroup* of G if for some $y \in G$
$\models_\lambda (y^{-1}xy \in A_2) \to (x \in A_1)$
$\models_\lambda (x \in B_1) \to (y^{-1}xy \in B_2)$ for all $x \in G$.

Theorem 3.17. Let $IF_{\mathscr{A}_1} =< A_1, B_1 >$ and $IF_{\mathscr{A}_2} =< A_2, B_2 >$ be two *implication-based intuitiontistic fuzzy subgroups* over the finite group G_1, and $IF_{\mathscr{B}_1} =< C_1, D_1 >$ and $IF_{\mathscr{B}_2} =< C_2, D_2 >$ be two *implication-based intuitiontistic fuzzy subgroups* over the finite group G_2 respectively such that $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of G_1 and $IF_{\mathscr{B}_1}$ and $IF_{\mathscr{B}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of G_2. Then $IF_{\mathscr{A}_1 \times \mathscr{B}_1}$ and $IF_{\mathscr{A}_2 \times \mathscr{B}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of $G_1 \times G_2$.

Proof. Since $IF_{\mathscr{A}_1}$ and $IF_{\mathscr{A}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of G_1 then for some $u \in G_1$,

$$\models_\lambda (u^{-1}xu \in A_2) \to (x \in A_1)$$
$$\models_\lambda (x \in B_1) \to (u^{-1}xu \in B_2) \text{ for all } x \in G_1$$

Since $IF_{\mathscr{B}_1}$ and $IF_{\mathscr{B}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of G_2 then for some $v \in G_2$,

$$\models_\lambda (v^{-1}yv \in C_2) \to (y \in C_1)$$
$$\models_\lambda (y \in D_1) \to (v^{-1}yv \in D_2) \text{ for all } y \in G_2$$

Now let $(u, v) \in G_1 \times G_2$

$$\begin{aligned}\models_\lambda ((u^{-1}, v^{-1})(x, y)(u, v) \in A_2 \times C_2) &\to ((u^{-1}xu, v^{-1}yv) \in A_2 \times C_2)\\ &\to ((u^{-1}xu \in A_2) \wedge (v^{-1}yv \in C_2))\\ &\to ((x \in A_1) \wedge (y \in C_1))\\ &\to ((x, y) \in A_1 \times C_1) \text{ for all } (x, y) \in G_1 \times G_2\end{aligned}$$

$$\begin{aligned}\vDash_\lambda ((x, y) \in B_1 \times D_1) &\to ((x \in B_1) \vee (y \in D_1)) \\ &\to ((u^{-1}xu \in B_2) \vee (v^{-1}yv \in D_2)) \\ &\to ((u^{-1}xu, v^{-1}yv) \in B_2 \times D_2) \\ &\to ((u^{-1}, v^{-1})(x, y)(u, v) \in B_2 \times D_2) \text{ for all } (x, y) \in G_1 \times G_2\end{aligned}$$

Therefore $IF_{\mathscr{A}_1 \times \mathscr{B}_1}$ and $IF_{\mathscr{A}_2 \times \mathscr{B}_2}$ are implication-based intuitionistic fuzzy conjugate subgroups of $G_1 \times G_2$. ■

References

[1] L. A. Zadeh, Fuzzy Sets, *Information Control*, 8, (1965), 338–353.

[2] William G Wee, K.S. FU, A Formation of Fuzzy Automata and its Application as a Model of Learning Systems, *IEEE Transactions on Systems Science and cybernetics*, 3, (1969), 215–223.

[3] Azriel Rosenfeld, Fuzzy Groups, *Journal of Mathematical Analysis and applications*, 35, (1971), 512–517.

[4] Dib K.A. and Hassan A.A., The Fuzzy Normal Subgroup, *Fuzzy Sets and Systems*, 98, (1998), 393–402.

[5] D.S. Malik, John N. Mordeson and P.S. Nair, Fuzzy Normal Subgroups into Fuzzy Subgroups, *J. Korean Math. Soc.*, 12, (1992), 1–8.

[6] Mukherjee N.P. and Prabir Bhattacharya, Fuzzy Normal Subgroups and Fuzzy Cosets, *Information Sciences*, 34, (1984), 225–239.

[7] Asok Kumar Ray, On Product of Fuzzy Subgroups, *Fuzzy Sets and Systems*, 105, (1999), 181–183.

[8] Antanassov K., Intuituitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 20, (1986), 87–96.

[9] Li Xiaoping, The intuitionisitic fuzzy normal subgroup and its equivalent propositions, *BUSEFAL*, 82, (2000), 40–44.

[10] Xuehai Yuan, Cheng Zhang, Yonghong Ren,Generalized fuzzy groups and many-valued implications, *Fuzzy Sets and System*, 138, (2003), 205–211.

[11] Selva Rathi M. and Michael Anna Spinneli J., Implication-Based Fuzzy Normal Subgroup of a Finite Group, *International Journal of Applied Engineering Research*, Vol. 10, No. 80, (2015), 5–8.

[12] Li Xiaoping, Wang Guijun, The intuitionistic fuzzy group and its homomorphic image, *Fuzzy Systems and Math.*, 14, 1, (2000).

[13] Mingsheng Ying, A New Approach For Fuzzy Topology (I), *Fuzzy Sets and Systems*, 39, (1997), 303–321.