Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 2 (2017), pp. 393-403

http://www.ripublication.com

A Unified Presentation of Generalized Fractional Integral Operators and H-Function

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Abstract

We propose a unified approach to the so-called special functions of fractional calculus, recently enjoying increasing interest from both theoretical mathematicians and applied scientists. It is mainly due to its vast potential demonstrated applications in field of science, engineering, chemical, biological, earth science etc. Our approach is based on the use of generalized fractional calculus operators. Motivated mainly by those earlier works, we establish some fractional integral formulas the H-function and M-series, by using certain general pair of fractional integral operators involving Gauss hypergeometric ${}_2F_1$.The results are expressed in terms of H-function, which are in compact form suitable for numerical computations. Special cases of the results are also pointed out in the form of the corollaries.

Keywords and phrases: H-function, M-series, Generalized hypergeometric function, Fractional integral operators.

2010 Mathematical Subject Classification: 33C60, 26A33, 33C20, 33B15, 33C05.

INTRODUCTION AND PRELIMINARIES:

Special functions constitute a very old branch of mathematics; the origins of their unified and rather complete theory date to the nineteenth century. From the point of view of the applied scientists and engineers dealing with the practical application of differential equations, the role of Special functions as an important tool of mathematical analysis rests on the following fact. Most of these named functions (among them: the Bessel and all cylindrical function, the incomplete gamma, kummer, confluent and generalized hypergeometric functions, the classical orthogonal polynomials, the Beta-functions and Error functions, the Airy, Whittaker functions etc.) provided solution of integer order (!) differential equations and systems, used as mathematical models. The "Special Function of Fractional Calculus" are the necessary tool, related to the theory of differential and integration fractional order (arbitrary order), that needs now its unified presentations and extensive development.

The subject of fractional calculus, which deals with investigations of integrals and derivatives, has gained importance and popularity the last four decades. It is mainly due to its vast potential demonstrated applications in field of engineering, biological and biomedical, chemical, Earth science, economics, etc. Different extensions of various fractional integration operators are studied by Kalla [1,2], Mc Biride [3], Kilbas [4,5], Kirya korva [6,9], Purohit and Kalla [7], Kumbhat and Khan [8].The several interesting and useful extensions of some familiar special functions such as Beta and Gauss hypergeometric functions and their properties have been investigated by many authors (see, e.g.[10,11,12,13,14,15] and see also very recent works, [16,37,38]).

For our purpose, we beginning by recalling some known function are earlier works. In 1997, chaudhary et al. [12] presented the following extension of Euler's Beta function $B(\alpha,\beta)$:

$$B_p\,(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,exp\left(\frac{-p}{t(t-1)}\right)\,dt \quad (Re(p)>0), \quad \ldots(1)$$

where the Beta function $B(\alpha,\beta)$ is a function of two complex variables α and β defined by

$$B(\alpha,\beta)=\begin{cases}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\;\;dt & (Re(\alpha)>0, Re(\beta)>0),\\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in C/z_0^-)\end{cases} \quad \ldots(2)$$

and Γ denotes the familiar gamma function.

The generalized hypergeometric series ${}_pF_q$ is defined by (see [17, p.73]):

$${}_pF_q\begin{bmatrix}\alpha_{1,\ldots,}\alpha_p\,; \\ \beta_{1,\ldots,}\beta_q\,;\end{bmatrix} z\Bigg]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n,\ldots,(\alpha_p)_n}{(\beta_1)_n,\ldots,(\beta_q)_n}\frac{z^n}{n!}={}_pF_q(\alpha_{1,\ldots,}\alpha_p;\beta_{1,\ldots,}\beta_q;z). \quad \ldots(3)$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that variable z, the numerator parameters $\alpha_{1,\ldots,}\alpha_p$ and the denominator parameters $\beta_{1,\ldots,}\beta_q$ take on complex values, provided that no zeros appear in the denominator of eq.(3), that is ($\beta_j\in C/z_0^-\,; j=1,\ldots,q$).The special case ${}_pF_q$ of (3) is called (Gauss) hypergeometric series.

$(\lambda)_n$ is the Pochhammer symbol defined $(for\ \lambda \in C)$ by

$$(\lambda)_n = \begin{cases} 1 & (n=0) \\ \lambda\,(\lambda+1)\ldots(\lambda+n-1) & (n \in N) \end{cases} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}\ (a \in C/z_0^-) \quad \ldots(4)$$

The importance of H-function and M-series are realized by scientists, engineers and statisticians(see caputo [18],Glöckle and Nonnenmacher [19], Mainardi et al. [20], Hilfer [21] etc.) due to vast potential of their applications in diversified fields of science and engineering such as fluid flow, rheology, diffusion in porous media, propagation of seismic waves and turbulence etc.

The Fox H-function [22] is generalized hypergeometric function, defined by means of the Mellin-bernes type contour integral,

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_i, A_i)_1^p \\ (b_j, B_j)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_{\ell} \mathcal{H}_{p,q}^{m,n}(z)\, z^{-s}\, ds\,, \qquad z \neq 0 \qquad \ldots(5)$$

where ℓ is suitable contour(of three possible types in C: $\ell_{-\infty}, \ell_{\infty}, (\gamma - i\infty, \gamma + i\infty)$, the order $(m, n, p, q)$are non negative integers such that $0 \le m \le q, 0 \le n \le p$,the parameters $A_i > 0, B_i > 0$ are positive and a_i, $b_j = 1, \ldots, p, j = 1, \ldots, q$ can be arbitrary complex such that $A_i\left(b_j + l\right) \neq B_j(a_i - l' - 1), l\,, l' = 0,1,2, \ldots;\ i = 1, \ldots, n; j = 1, \ldots, m$ and the integrand (i.e. the mellin transform of (5)) has the form

$$\mathcal{H}_{p,q}^{m,n}[z] = \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_j s\right)\prod_{i=1}^{n}\Gamma(1 - a_i - A_i s)}{\prod_{j=m+1}^{q}\Gamma\left(1 - b_j - B_j s\right)\prod_{i=n+1}^{p}\Gamma(a_i + A_i s)}. \qquad \ldots(6)$$

Define also

$$\rho = \prod_{i=1}^{p} A_i^{-A_i} \prod_{j=1}^{q} B_j^{-B_j}; \qquad \Delta = \prod_{j=1}^{q} B_j - \prod_{i=1}^{p} A_i;$$

$$\mu = \prod_{j=1}^{q} b_j - \prod_{i=1}^{p} a_i + \frac{p-q}{2}\ ; \Omega = \prod_{i=1}^{n} A_i - \prod_{i=n+1}^{p} A_i + \prod_{j=1}^{m} B_j - \prod_{j=m+1}^{q} B_j \qquad \ldots(7)$$

Then (5) is an analytic function of z in circle domain $|z| < \rho$ (or sector of them or in the whole C), depending on the parameters in (6) and the contours. Asymptotic expansions and analytic continuations together with converges conditions of H-function have been discussed by Braaksma [23], Mathai [24].

When all $A_i = B_j = 1;\ i = 1, \ldots, p,\ j = 1, \ldots, q$, in eq. (6) the H-function (5) reduces to the simpler (yet very wide ranged) Meijer's G-function, denoted by

$G_{p,q}^{m,n}\left[z \left| \begin{matrix} (a_i)_1^p \\ (b_j)_1^q \end{matrix} \right. \right]$; (see [22]).

Wright [25] defined generalized hypergeometric function by means of the series representation in the form

$$ {}_p\Psi_q = {}_p\Psi_q \left[z \left| \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix} \right. \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!} $$

$$ = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + nA_1), \ldots, \Gamma(a_p + nA_p)}{\Gamma(b_1 + nB_1), \ldots, \Gamma(b_q + nB_q)} \frac{z^n}{n!} \qquad \ldots (8) $$

$$ = H_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1, A_1), \ldots, (1 - a_p, A_p) \\ (0,1), (1 - b_1, B_1), \ldots, (1 - b_q, B_q) \end{matrix} \right. \right] \qquad \ldots (9) $$

where

$$ z, a_i, b_j \in C, A_i, B_j \in \Re_+ , A_i \neq 0, B_j \neq 0; i = 1, \ldots, p, j = 1, \ldots, q, \prod_{j=1}^{q} B_j - \prod_{i=1}^{p} A_i > -1. $$

For $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$ in eq. (8) and (9), the functions reduce to a ${}_p\mathrm{F}_q$ function and a G-function, respectively:

$$ {}_p\Psi_q \left[z \left| \begin{matrix} (a_1, 1), \ldots, (a_p, 1) \\ (b_1, 1), \ldots, (b_q, 1) \end{matrix} \right. \right] = \mathbb{C}\ {}_p\mathrm{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1)_n, \ldots, (a_p)_n}{(b_1)_n, \ldots, (b_q)_n} \frac{z^n}{n!} = G_{p,q+1}^{1,p} \left[-z \left| \begin{matrix} (1 - a_1), \ldots, (1 - a_p) \\ 0, (1 - b_1), \ldots, (1 - b_q) \end{matrix} \right. \right]. \qquad \ldots (10) $$

where $\mathbb{C} = \left[\prod_{i=1}^{p} \Gamma(a_i) \,/\, \prod_{j=1}^{q} \Gamma(b_j) \right]$

Sharma and Jain [26] introduced the generalized $\mathbb{M}$ -series as the function defined by means of the power series:

$$ {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta} = {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}\left(a_1, \dots, a_p; b_1, \dots, b_q; z\right) = {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}\left[(a_i)_1^p; \left(b_j\right)_1^q; z\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(a_1)_n, \dots, \left(a_p\right)_n}{(b_1)_n, \dots, \left(b_q\right)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}. \qquad (z, \alpha, \beta \in C, Re(\alpha) > 0) \quad \dots (11) $$

where $(a_i)_n$, $\left(b_j\right)_n$are the known Pochammer symbols.The series (11) is defined when none of the parameters $b_j^{\prime}s,\ j = 1, \dots, q$ is a negative integer or zero; if any numerator parameter a_j is negative integer or zero, then the series terminates to a polynomial in z. The series (11) is converges for all z if $p \leq q$, it is convergent for $|z| < \delta = \alpha^{\alpha}$if $p = q + 1$ and divergent, if $p > q + 1$.When p = q+1 and $|z| = \delta$, the series can converge on conditions depending on the parameters. Properties of $\mathbb{M}$ series are further studied by Sexena [27], Chouhan and Saraswat [28] etc.

The generalized $\mathbb{M}$ series (11) can be represented as a special case of Fox H-function (6) and Wright generalized hypergeometric function (8) as:

$$ {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}\left(a_1, \dots, a_p; b_1, \dots, b_q; z\right) = \mathbb{C}\ {}_{p+1}\Psi_{q+1}\left[z\left|\begin{matrix}(a_1,1), \dots, \left(a_p,1\right), (1,1)\\(b_1,1), \dots, \left(b_q,1\right), (\beta,\alpha)\end{matrix}\right.\right] $$

$$ = \mathbb{C}\ H_{p+1,q+2}^{1,p+1}\left[-z\left|\begin{matrix}(1-A_i,1;1)_1^p, (0,1)\\(0,1), (1-B_i,1;1)_1^q, \dots, (1-\beta,\alpha)\end{matrix}\right.\right]. \qquad \dots (12) $$

Where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma\left(b_j\right) / \prod_{i=1}^{p}\Gamma(a_i)\right]$

The generalized Mittag-leffler function, introduced by Prabhakar [29], may be obtained from (11) for $p = q = 1; a = \gamma \in C; b = 1$, as

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha m + \beta)} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{(1)_m} \frac{z^m}{\Gamma(\alpha m + \beta)} = {}_{1}^{\alpha}\mathbb{M}_{1}^{\beta}\,(\gamma; 1; z). \qquad \dots (13) $$

OPERATORS OF GENERALIZED FRACTIONAL CALCULUS

Recently fractional integral operators involving the various special functions have been considered by many authors (see, e.g. [30, 31, 32, 33, and 34], [36, 37], also see [39]).

In this paper, we established image formulas for M-series, involving some operators of fractional integrals in terms of Fox H-function. For our purpose, we begin by recalling the following pairs of Saigo hypergeometric operators of fractional integrations. For $x > 0, \mu, v \in C$and $Re(\alpha) > 0$, we have

$$\left(I_{0,x}^{\mu,v,\eta}\, f(t)\right)(x) = \frac{x^{-\mu-v}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\ {}_2\mathrm{F}_1\left(\mu+v, -\eta; \mu; 1-\frac{t}{x}\right) f(t)dt, \quad \dots (14)$$

and $\left(J_{x,\infty}^{\mu,v,\eta}\, f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1}\, t^{-\mu-v}\ {}_2\mathrm{F}_1\left(\mu+v, -\eta; \mu; 1-\frac{x}{t}\right) f(t)dt \quad \dots (15)$

where ${}_2\mathrm{F}_1(.)$ is Gamma hypergeometric series which is a special case of the generalized hypergeometric series ${}_p\mathrm{F}_q$ in (3).

The operator $I_{0,x}^{\mu,v,\eta}(.)$ contains both the Reimann-Liouville and Erdelyi-Kobar fractional integral operators, by means of the following relationships:

$$\left(R_{0,x}^{\mu}\, f(t)\right)(x) = \left(I_{0,x}^{\mu,-\mu,\eta}\, f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\ f(t)dt, \quad \dots (16)$$

and

$$\left(E_{0,x}^{\mu,\eta}\, f(t)\right)(x) = \left(I_{0,x}^{\mu,0,\eta}\, f(t)\right)(x) = \frac{x^{-\mu-\eta}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1}\, t^{\eta}\ f(t)dt \quad \dots (17)$$

whereas the operator (15) unifies the Weyl type and Erdélyi-Kobar fractional integral operators as follows:

$$\left(W_{x,\infty}^{\mu}\, f(t)\right)(x) = \left(J_{x,\infty}^{\mu,-\mu,\eta}\, f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1}\ f(t)dt, \quad \dots (18)$$

and

$$\left(K_{x,\infty}^{\mu,\eta}\, f(t)\right)(x) = \left(J_{x,\infty}^{\mu,0,\eta}\, f(t)\right)(x) = \frac{x^{\eta}}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1}\, t^{-\mu-\eta} f(t)dt, \quad \dots (19)$$

We use the following image formulas which are easy consequences of the operators (14) and (15) (see [41, 42]):

$$\left(I_{0,x}^{\mu,v,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\,\Gamma(\lambda-v+\eta)}{\Gamma(\lambda-v)\,\Gamma(\lambda+\mu+\eta)} x^{\lambda-v-1} \quad (\lambda > 0, \lambda-v+\eta > 0) \quad \dots (20)$$

$$\left(J_{x,\infty}^{\mu,v,\eta}\, t^{\lambda-1}\right)(x) = \frac{\Gamma(v-\lambda+1)\,\Gamma(\eta-\lambda+1)}{\Gamma(1-\lambda)\,\Gamma(v+\mu-\lambda+\eta+1)} x^{\lambda-v-1} \quad (v-\lambda+1 > 0, \eta-\lambda+1 > 0) \quad \dots (21)$$

MAIN RESULTS

In this section, we established image formulas for M-series (11), involving operators (14), (15), (16), (17), (18) and (19) in terms of Fox H-function.

Theorem 1. Let $x > 0, \sigma > 0$ and parameters $\mu, v, \eta, z, \rho, \alpha, \beta \in C$. Such that $Re(\mu) > 0, Re(\rho) > 0$ and $Re(\mu) > \max\{0, Re(v - \eta)\}$. Then there holds the integral formula

$$\left(I_{0,x}^{\mu,v,\eta}\left[t^{\rho-1}\,{}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{\sigma})\right]\right)(x) = x^{\rho-v-1}\mathbb{C}$$
$$\times H_{p+3,q+4}^{1,p+3}\left[-zx^{\sigma}\left|\begin{matrix}(1-\rho,\sigma),(1-\rho+v-\eta,\sigma),(0,1);(1-a_i,1)_1^p\\(1-\rho+v,\sigma),(1-\rho-\mu-\eta,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right.\right]. \quad \dots (22)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j) \,/\, \prod_{i=1}^{p}\Gamma(a_i)\right]$

Proof. By applying (11) to the L.H.S of (22), we find that

$$\left(I_{0,x}^{\mu,v,\eta}\left[t^{\rho-1}\,{}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{\sigma})\right]\right)(x) == \sum_{n=0}^{\infty}\frac{(a_1)_n,\dots,(a_p)_n}{(b_1)_n,\dots,(b_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta)}\times\, I_{0,x}^{\mu,v,\eta}(\{t^{\rho+\sigma n-1}\})(x)$$

Now applying the Saigo fractional integral operator (14) and then changing the order of integration and summation, which is valid under the condition of Theorem 1.Hence, by virtue of H-function (6), we finally obtain R.H.S of (22).

On setting $v = 0$ and using the relation (17), then result (22) yields Corollary 1.

Corollary 1. Let $x > 0, \sigma > 0$ and parameters $\mu, \eta, \rho, \alpha, \beta \in C$. Such that $Re(\mu) > 0, Re(\rho) > 0$ and $Re(\rho) > Re(-\eta)$. Then the following fractional integral formula holds true:

$$\left(E_{0,x}^{\mu,\eta}\left[t^{\rho-1}\,{}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{\sigma})\right]\right)(x)$$
$$= x^{\rho-1}\mathbb{C}\; H_{p+2,q+3}^{1,p+2}\left[-zx^{\sigma}\left|\begin{matrix}(1-\rho-\eta,\sigma),(0,1);(1-a_i,1)_1^p\\(1-\rho-\mu-\eta,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right.\right]. \dots (23)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j) \,/\, \prod_{i=1}^{p}\Gamma(a_i)\right]$

When we replace v by $-\mu$ and making use of the relation (16), in Theorem 1, we obtain yet another Corollary 2, providing Riemann-Liouville fractional integral.

Corollary 2. Let $x > 0, \sigma > 0$ and parameters $\mu, \eta, \rho, \alpha, \beta \in C$. Such that $Re(\mu) > 0, Re(\rho) > 0$. Then we get following result

$$\left(R_{0,x}^{\mu}\left[\, t^{\rho-1}\, {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{\sigma})\right]\right)(x) = x^{\rho+\mu-1}\mathbb{C}$$
$$\times H_{p+2,q+3}^{1,p+2}\left[-zx^{\sigma}\middle|\begin{matrix}(1-\rho,\sigma),(0,1);(1-a_i,1)_1^p\\(1-\rho-\mu,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right]. \quad \dots(24)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j) \,/\, \prod_{i=1}^{p}\Gamma(a_i)\right]$

Theorem 2. Let $x > 0, \sigma > 0$ and parameters $\mu, v, \eta, z, \rho, \alpha, \beta \in C$. Such that $Re(\mu) > 0, Re(\alpha) > 0$ and $Re(\rho) < 1 + \min\{Re(\eta), Re(v)\}$. Then the following fractional integral formula holds true:

$$\left(J_{x,\infty}^{\mu,v,\eta}\left[\, t^{\rho-1}\, {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{-\sigma})\right]\right)(x) = x^{\rho-v-1}\mathbb{C}$$
$$\times H_{p+3,q+4}^{1,p+3}\left[\frac{-z}{x^{\sigma}}\middle|\begin{matrix}(\rho-v,\sigma),(\rho-\eta,\sigma),(0,1);(1-a_i,1)_1^p\\(\rho,\sigma),(\rho-v-\mu-\eta,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right]. \quad \dots(25)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j) \,/\, \prod_{i=1}^{p}\Gamma(a_i)\right]$

Proof. The same argument as the proof of Theorem 1, taking the Saigo hypergeometric operator of fractional integral (15) and interchanging the order of integrations and summation ,which is valid under the condition of Theorem 1.Hence, by virtue of H-function(6),we finally arrive at (25).

Considering the relation (19) and setting $v = 0$, we are led to the following formula given in Corollary (3).

Corollary 3. With all assumptions and conditions on parameters, as stated in Theorem 2, with $Re(\rho) < 1 + Re(\eta)$.The following result holds:

$$\left(K_{x,\infty}^{\mu,\eta}\left[\, t^{\rho-1}\, {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{-\sigma})\right]\right)(x)$$
$$= x^{\rho-1}\mathbb{C}\; H_{p+2,q+3}^{1,p+2}\left[\frac{-z}{x^{\sigma}}\middle|\begin{matrix}(\rho-\eta,\sigma),(0,1);(1-a_i,1)_1^p\\(\rho-\mu-\eta,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right]. \quad \dots(26)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j) \,/\, \prod_{i=1}^{p}\Gamma(a_i)\right]$

If we replace v by $-\mu$ and making use of the relation (18) in Theorem 2, we are led to the following formula given in Corollary 4, concerning the Weyl fractional integral operator.

Corollary 4. With all assumptions and conditions on parameters, as stated in Theorem 2, with $Re(\mu) > 0, Re(\rho) > 0$.The following result holds:

$$\left(W_{x,\infty}^{\mu,\eta}\left[\, t^{\rho-1}\, {}_{p}^{\alpha}\mathbb{M}_{q}^{\beta}(zt^{-\sigma})\right]\right)(x)$$
$$= x^{\rho+\mu-1}\mathbb{C}\ H_{p+2,q+3}^{1,p+2}\left[\frac{-z}{x^{\sigma}}\middle|\begin{matrix}(\rho+\mu,\sigma)\,,(0,1);(1-a_i,1)_1^p\\(\rho,\sigma),(0,1),(1-\beta,\alpha);(1-b_j,1)_1^q\end{matrix}\right]. \quad \dots(27)$$

where $\mathbb{C} = \left[\prod_{j=1}^{q}\Gamma(b_j)\,/\prod_{i=1}^{p}\Gamma(a_i)\right]$

CONCLUSION

Recently, fractional operator theory was recognized to be great tool for modeling, complex problems, kinetic equation, fractional reaction, fractional diffusion equations and so forth. In this work, the authors investigated and studied Saigo hypergeometric operators, Riemann-Liouville, Erd*é*lyi -Kober, Weyl type fractional integral operators associated with Fox's H-function which are applied to M-series. Recent derived in this paper are very significant and may find applications in the solution of fractional order differential equations that are arising in certain areas of turbulence, propagation of seismic waves and diffusion process. On account of general nature of H-function and M-series a number of results involving special functions can be obtained merely by specializing the parameter.

REFERENCES

[1] S.L.Kalla, Integral operator involving Fox's H-function I, Acta Mexicana de Cienciay technologia Vol.3 (1969), pp.117-122.
[2] S.L.Kalla, Integral operator involving Fox's H-function II, Acta Mexicana de Cienciay technologia Vol.7 (1969), pp.72-79.
[3] A.C. McBride, Fractional power of a class of ordinary differential operators, Proceeding of the London mathematical society, Vol.3 (3) (1982), pp.519-546.
[4] A.A.Kilbas, Fractional calculus of the generalized Wright-function, Fractional calculus and Applied analysis, Vol.8 (2) (2005), pp.113-126.
[5] A.A. Kilbas, N. Sebastian ,Generalized fractional integration of the Bessel function of the first kind, Integral Transform and Special Functions, Vol.19,no.11-12 (2008),pp.869-883.

[6] V.Kiryakova, On two Saigo's fractional integral operators in the class of univalent functions, Fractional Calculus and Applied analysis, Vol.9 (2) (2006), pp.159-176.

[7] S.D.Purohit and S.L.Kalla, On fractional partial differential equations related to quantum mechanics, Journal of physics A: Mathematical and Theoretical, Vol.44 (4) (2011) Article ID 045202, 8 pages.

[8] R.K.Kumbhat and A.M.Khan ,A regularized approximate solution of the fractional integral operators,In proceeding of the 2nd international conference of the society for special functions and their application, Lucknow,India,Vol.II(2001),pp.99-105.

[9] V.Kiryakova,The special functions of fractional calculus as generalized fractional calculus operators of some basic function, Computers and Mathematics with Applications,Vol.59(2010),pp.1128-1141.

[10] P.Agarwal, M.Chand and S.D. Purohit, A note on generating function involving generalized Gauss hypergeometric functions, Nat. Acad.Sci.Lett. in press.

[11] M.A.Chaudhry, A.Qadir, M.Rafique and S.M.Zubair, Extension of Euler's beta function, J.Comput.Appl.Math. Vol.78 (1997), pp.19-32.

[12] M.A.Chaudhry, A.Qadir, H.M.Srivastava and R.B.Paris, Extended hypergeometric and confluent hypergeometric functions, Appl.Math.Comput. Vol.159 (2004), pp.589-602.

[13] D.M.Lee, A.K.Rathie, R.K.Parmar and Y.S.Kim, Generalization of extended beta function, hypergeometric and confluent hypergeometric functions, Honam Math.J.,Vol.33(2)(2011),pp.187-196.

[14] E.Özergin, Some properties of hypergeometric functions, Ph.D Thesis, Eastern Mediterranean University, North Cyprus, February (2011).

[15] E.Özergin, M.A.Ozarslam and A.Altin, Extension of gamma, beta and hypergeometric functions, J.Comput.Appl.Math.,Vol.235(2011),pp.4601-4610.

[16] J.Choi, P.Agarwal and S.Jain, Certain Fractional Integral Operators and Extended Generalized Gauss Hypergeometric Function, Vol.55 (2015), pp.695-703.

[17] E.D.Rainville, Special functions Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, New York, 1971.

[18] M.Caputo, Elasticitia E Dissipazione, Zainchelli, Bologana, 1969.

[19] W.G. Glöckle and T.F. Nonnenmacher, Fox function representation on Non-Debye Relaxation Processes, J.Stat. Physics, Vol.71 (1993), pp.741-757.

[20] F.Mainardi,Y.Luchko and G.Pogini, The fundamental solution of the space time fractional diffusion equation,Frac.Calc. Appl.Anal. Vol.4 (2) (2001), pp.153-192.

[21] R.Hilfer (Ed.), Application of fractional calculus in physics, World Scientific Publishing Company, Singapore, New Jersey, London and Honkong,2000.

[22] C.Fox, The G and H-functions as symmetrical Fourier Kernels, Trans.Amer.Math.Soc. Vol.98 (1961), pp.395-429.

[23] B.L.J.Braaksma, Asymptotic expansion and analytic continuations for a class of Bernes integrals, Comp. Math. Vol.15 (1964), pp.239-341.

[24] A.M.Mathai, A handbook of generalized special functions for statistical and physical sciences, Oxford University Press, Oxford 1993.

[25] E.M. Wright, The Asymptotic Expansion of the Generalized Hypergeometric Function, J.London Math. Soc., Vol.10 (1935), pp.286-293.

[26] M.Sharma and R.Jain, A note on a Generalized M-series as a special function of fractional calculus, Fract.Calc.Appl. anal., Vol.12 (4) (2009), pp.449-452.

[27] R.K.Sexena, A remark on a paper on M-series, Fract.Calc.Appl. anal., Vol.12 (1) (2009), pp.109-110.

[28] A.Chouhan and S.Saraswat, Certain properties of fractional calculus operators associated with M-series, Scietia: Series A:Mathematical sciences ,Vol.22 (2012),pp.25-30.

[29] T.R.Prabhakar, A singular integral equation with a generalized Mittag-leffler function in the kernel, Yokohama math.J. , Vol.19 (1971), pp.7-15.

[30] R.K.Parmar, A new generalization of gamma, beta, hypergeometric and confluent hypergeometric functions, Le Mathematice, Vol. LX Vlll (2013), pp.33-52.

[31] M.Saigo, On generalization fractional calculus operators, In: Recent Advances in Applied Mathematics (Proc.Inernat.workshop held at Kuwait Univ.Kuwait), (1996), pp.441-450.

[32] M.Saigo,A remark on integral operators involving the gauss hypergeometric function, Math.Rep.Kyushu univ.,Vol.11(1978),pp.135-143.

[33] M.Saigo, A certain boundary value problem for the Euler-darboux equation I, Math.Japonica, Vol.24 (4) (1979), pp.377-385.

[34] A.A.Kilbas, Fractional,Calculus of the generalized Wright function, Fract.Calc.Appl.Anal., Vol.8 (2) (2005),pp.113-126.

[35] A.A.Kilbas, H.M.Srivastava and J.J.Trujiillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, 204,Elsevier Science Publisher, Amsterdam, London and New York ,2006.

[36] P.Agarwal, Certain properties of the generalized gauss hypergeometric functions, Appl.Math.Inform.Sci.in press.

[37] P.Agarwal, Fractional integration of the product of two multivariable H-function and general class of polynomial, In: Adva.Appl.Math. Approx. Theo.(2011).

[38] A.M.Khan, R.K.Kumbhat, Amit Chouhan and Anita Alsaria, Generalized Fractional Integral Operators and M-Series, Hindwai Publishing Corporation, Journal of Mathematics, Article ID 2872185 ,Vol.(2005) 10 pages.