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On Commutativity of Prime Assosymmetric Rings

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Abstract

In this paper we show that a2- and 3- divisible prime assosymmetric ring R is commutative.

1. INTRODUCTION

E. Kleinfeld [1] introduced a class of non-associative rings called as assosymmetri rings in which the associator (x,y,z) = (xy)z - x(yz) has the property (x,y,z) = (p(x),p(y), p(z)) for each permutation p of x,y and z. These rings are neither flexible nor power associative. In [1] it is proved that the commutator and the associator are in the nucleus of this ring. In 2000, K. Suvarna and G.R.B. Reddy[3] proved that a non-associative 2- ad 3-divisile prime assosymmetric ring is flexible. By using these properties we show that a 2- and 3- divisible prime assosymmetric ring R is commutative.

2. PRELIMINARIES

Throughout this paper R will denote a non-associative 2- and 3- divisible assosymmetric ring. The commutator (x,y) of two elements x and y in a ring is defined by (x,y) = xy-yx. The nucleus N in R is the set of elements $n \in R$ such that (n,x,y) = (x,n,y) = (x,y,n) = 0 for all x,y in R. The center C of R is the set of elements $c \in N$ such that (c,x) = 0 for all x,y in R. A non-associative ring R is called flexible if (x,y,x)=0 for all x,y in R. A ring is said to be power-associative if every subring of it generated by a single element is associative if every subring of it generated by a single element is associative Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators. R is called k-divisible if kx=0 implies x=0, $x \in R$ and k is a natural number.

In an arbitrary ring the following identities hold :

(1) (wx,y,z) - (w,xy,z) + (w,x,yz) = w (x,y,z) + (w,x,y)z
f(w,z,y,z) = (wx,y,z) - x(w,y,z) - (x,y,z) w
and

(2) (x,y,z) + (y,z,x) + (z,x,y) = (xy,z) + (yz,x) + (zx,y)
(xy,z) - x(y,z) - (x,z)y = (x,y,z) - (x,z,y) + (z,x,y).
In any assosymetric ring (2) becomes

(3) (xy,z) - x(y,z) - (x,z)y = (x,y,z)
(xy,x) + x(x,y) = (x,y,x)
It is proved in [1] that in a 2- and 3-divisible assosymmetric ring R the following identities hold for all w,x,y,z,t in R

(4) f(w,x,y,z) = 0, that is, (wx,y,z) = x(w,y,z) + (x,y,z)w,

(5) ((w,x),y,z) = 0
and

(6) ((w,x,y),z,t) = 0
That is, every commutator and associator is in the nucleus N.
From (3), (5) and (6), we obtain

(7) $x(y,z) + (x,z)y \subset N$.
Suppose that $n \in N$. Then with w=n in (1) we get (nx,y,z) = n(x,y,z).
Combining this with (5) yields.

(8) (nx,y,z) = n(x,y,z) = (xn,y,z)
From (7) and (8) we obtain

(9) (y,z) (x,r,s) = -(x,z) (y,r,s).

3. MAIN RESULTS.

Lemma 1. Let $S = \{s \in N / s(R,R,R)=0\}$. Then S is an ideal of R and S (R, R, R) = 0

Proof. By substituting s for n in (8), we have (sx,y,z) = s(x,y,z) = (xs,y,z) = 0. Thus $sR \subset N$ and $Rs \subset N$. From (6), sw(x,y,z)=sw(x,y,z)=s. w(x,yz). But (1) multiplied on the left by s yields s.w(x,y,z) = -s(w,x,y)z = -s(w,x,y).z = 0. Thus sw. (x,y,z)=0, From (9), we have (s,w) (x,y,z) = -(x,w)(s,y,z) =0. Combining this with sw.(x,y,z)=0, we obtain ws.(x,y,z)=0. Thus S is an ideal of R. The rest is obvious. This completes the proof of the lemma.

Lemma 2. $(x,y,x) \in S$.

Proof. By forming the associators of both sides of (1) with u and v, and using (6), we obtain

(10) (w(x,y,z), u,v) + ((w,x,y) z,u,v) = 0
Interchanging y and x in (10) and subtracting the result from (10), we get

(11) ((w,x,y) z,u,v) = ((w,x,z) y,u,v).
But ((w,x,z) y,u,v) = (y(w,x,z), u,v), because of (5). So that

(12) ((w,x,y) z,u,v) = (y(w,x,z),u,v), as result of (11).
Also by permuting w and y in (10), we obtain (y(w,x,z),u,v) + ((w,x,y)z,u,v) = 0.

(13) This identity with (12) yields 2((w,x,y)z,u,v) = 0 Thus

(14) ((w,x,y)z,u,v) = 0.
From (6) we have (x,y,x) $\subset$ N. Using (13) and (8),
we get 0 = ((x,y,x)z,u,v) = (x,y,x)(z,u,v) for all x,y,z,u,v in R. Hence (x,y,z) $\in$ S. This complete the proof of the lemma.

Lemma 3. In an assosymmetric ring R, ((a,b,c),d)$\in$S.

Proof. Using (9) we see that ((a,b,c),d) (x,y,z) = - (x, d) ((a,b,c),y,z)=0 because (6). Hence ((a,b,c),d)$\in$ S

Lemma 4. If R is a non-associative 2- and 3-divisible prime assosymmetric ring then R is a Thedy ring.

Proof : Using lemma 1 and the identity (1) we establish S.V = 0. Since R is prime, either S = 0 or V = 0. If V = 0, R is associative. But we have assumed that R is not associative. Therefore V$\neq$0. Hence S = 0. From lemma 3, ((a, b, c), d)$\in$S. Thus

(15) ((a, b, c), d) = 0
and R is a Thedy ring.

Lemma 5. : In a 2-divisible prime assosymmetric ring, all the commutators are in the center.

Proof : By forming associators on each side of (3) and using (5) gives

((x, y, x), r, s) = (x (x, y), r, s) = ((x, y) x, r, s).

Using (1) and (5) we have ((x, y) x, r, s) = (x, y) (x, r, s).

We conclude that ((x, y, x), r, s) = (x, y) (x, r, s). Linearizing (replacing x by x+t), we obtain ((x, y, t) + (t, y, x), r, s) = (x, y) (t, r, s) + (t, y) (x, r, s). If we substitute a commutator v for t, we see that (v, y) (x, r, s) = 0.

This can be restated as ((R, R), R) (R, R, R) = 0. But now the ideal generated by double commutators ((R, R), R) (which can be characterized as all sums of double commutators plus right multiples of double commutators, because of (5)) annihilates the associator ideal. Since R is prime and not associative, we conclude that

(16) ((R, R), R) = 0.

Thus the commutators are in the center.

Lemma 6 : If R is a 2-divisible assosymmetric ring, then R(R, N) + (R, N)R $\subseteq$ N.

Proof : It is sufficient to show that R(R, N) or (R,N) R $\subseteq$ N.

Since N is a subalgebra of R containing (R, R), this follows from (2) and from (3).

Lemma 7 : If R is a 2-divisible assosymmetric ring,

then the set W = {v/v$\in$N, Rv$\subseteq$N} is an ideal of R contained in N, such that (R, N) + (R, R, N) + (R, N, R) + (N, R, R) $\subseteq$ W.

Proof : We use lemma 4 in the form (R, N) $\subseteq$ W $\subseteq$ N and (3) by substituting z = v $\in$ N. First we see that (x, y, v) $\in$ N. Consequently W is an ideal and again from (3), we get (x, y, v), (y, v, x) and (v, x, y) are in W.

The center of R is the set of all x $\in$ N such that (x, y) = 0 for all y $\in$ R.

Corollary 1 : The canonical homomorphism of R into R/W maps N into the center of R/W.

Corollary 2 : If W = 0, then N equals the center of R.

Corollary 3 : If R satisfies (1) (x, (y,x, y)) = 0

and (2) (R, R) $\subseteq$ N,

then $(x, y)^2 \in W$ for $x, y \in R$.

Proof : Since $(yx, y) - y\,(x, y) = (y, x,y)$ in every ring , the condition (1) is equivalent to $(x, (yx, y)) = (x, y\,(x, y))$.

Hence by lemma 7, $(x, y)^2 \equiv (x, y\,(x, y)) \equiv (x, (yx, y)) \equiv 0$ modulo W.

Let U be an ideal of R contained in N. Then trivially $U \subseteq W$ and hence equals the sum of all ideals of R contained in N. We define $U^{\perp} = \{ x / x \in R, Ux = 0 = x U\}$.

Lemma 8 : Let R be a 2-divisible assosymmetric ring and U be an ideal of R contained in N. Then $U^{\perp}$ is an ideal of R such that $U U^{\perp} = 0$ or $U^{\perp} U = 0$.

Proof : Let $p \in U^{\perp}$ and $v \in U \subseteq N$.

We have $v\,(px) = -(v, p, x) = 0$ and $v\,(xp) = -(v, x, p) = 0$,

$(xp)\,v = (x, p, v) = 0$ and $(px)\,v = (p, x, v) = 0$ proving that $U^{\perp}$ is an ideal of R such that

$U U^{\perp} = U^{\perp} U = 0$.

From $U \subseteq W$, we get $W^{\perp} \subseteq U^{\perp}$. The importance of $W^{\perp}$ comes from

Lemma 9 : If R is a 2-divisible assosymmetric ring R, then $(R, R,R) \subseteq W^{\perp}$.

Proof : Let (x, y, z) be an associator and $v \in W$.

By the definition of W, we have $R\,v + v\,R \subseteq N$. In case $v \in N$, $v\,R \subseteq N$ and $R\,v \subseteq N$, we see that $v\,(x, y,z) = 0$ and $(x, y, z)v =0$, from (1).

Hence $(x, y, z) \in W^{\perp}$.

Theorem 1 : If R is a non-associative 2-and 3-divisible prime assosymmetric ring, then R is flexible.

Proof : Using lemma 1 and the identity (1) we establish that S.I = 0. Since R is prime, either S = 0 or I = 0. If I = 0, R is associative. But we have assumed that R is not associative. Therefore $I \neq 0$. Hence S = 0. From lemma 2, $(x, y, x) \in S$. Thus (x, y, x)=0. That is, R is flexible.

Theorem 2 : A 2- and 3- divisible prime assosymmetric R is power-associative, that is (x, x, x)=0.

Proof : By commuting each term in (1) with r, and using (14) we obtain

(r, w(x, y, z)) + (r, (w, x, y)z) = 0.

So that (r, w(x, y, z)) = – (r, (w, x, y) z) = – (r, z (w, x, y)) using (14).

By permuting cyclically (wzyx), we get

(17) (r, w (x, y, z)) = – (r, z (w, x, y)) = (r, y (z, w, x)) = – (r, x (y, z, w).

We know that in an assosymmetric ring (x, x, x) is in the nucleus of R. This combined with (14) prove that (x, x, x) is in the center of R.

Next applying (16) to (z, x (x, x, x)), we obtain

(z, x (x, x, x)) = – (z, x (x, x, x)).

This leads to 2 (z, x (x, x, x)) = 0. So that (z, x (x, x, x)) = 0.

Expanding (x, (x, x, x), z) = 0 by using (2), we have

0 = x ((x, x, x), z) + (x, z) (x, x, x) + (x, (x, x, x), z).

However (x, x, x) is in the center of R. Thus only one term servives and we obtain (x, z) (x, x, x) = 0. Since R is prime and not commutative, by similar argument in the proof of theorem 1, we obtain (x, x, x) = 0.

Theorem 3 : A 2- and 3 – divisible prime assosymmetric ring R is commutative.

Proof : By forming the commutators of each side of (2) with w and using (15) it follows that 3((x, y, z), w) = 0. Since R is 3-divisible, we have

(18) ((x, y, z) , w) = 0.

From lemma 5, commutators are in the center. So commuting equation (3) with y and using (17), we obtain $0 = ((x, y, x), y) = (x\,(x, y), y) = (x, y)^2$. A center element, which squares to zero generates an ideal which squares to zero. Hence (x, y) = 0 and R is commutative.

Now since R is commutative, from (3) it follows that R is also associative.

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