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Common Fixed Point Theorem for Six Mappings

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Abstract

Using the notion of reciprocally continuous we obtained a result common fixed point theorem for six mappings which generalized the results of M.Kulkarni and V.H. Badshah, Bijendra Singh and others.

Keywords: Compatible mappings, weakly compatible, reciprocally continuous, fixed point.

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1. INTRODUCTION

In this note we prove a common fixed point theorem for the help of compatible mappings [3] and reciprocally continuous mappings [8]. Here we enlist the some definitions which are uses in our result.

Definition 1.1 [3] Two self- mappings S and T of a metric spaces (X, d) is said to be compatible if$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.2[8] Two self -mappings S and T of a metric spaces (X, d) is said to be reciprocally continuous if $\lim_{n\to\infty} STx_n = Sz$ and $\lim_{n\to\infty} TSx_n = Tz$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.3[5] Two self- mappings S and T of a metric spaces (X, d) is said to be compatible of type (A), if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$, and

$\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.4[9] Two self- mappings S and T of a metric spaces (X, d) is said to be compatible of type (B) if

$\lim_{n\to\infty} d(STx_n, TTx_n) \le \frac{1}{2} [\lim_{n\to\infty} d(STx_n, Sz) + \lim_{n\to\infty} d(Sz, SSx_n)]$,

and

$\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{2} [\lim_{n\to\infty} d(TSx_n, Tz) + \lim_{n\to\infty} d(Tz, TTx_n)]$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.5[10] Two self- mappings S and T of a metric spaces (X, d) is said to be compatible of type (p) on X if $\lim_{n\to\infty} d(SSx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.6 [2] Two self- mappings S and T of a metric spaces (X, d) is said to be weakly compatible of type(p)

$$\lim_{n\to\infty} d(SSx_n, TTx_n) \le d(Sz,Tz) \le \lim_{n\to\infty} d(Tz, TTx_n)$$

$$\lim_{n\to\infty} d(SSx_n, TTx_n) \le d(Sz,Tz) \le \lim_{n\to\infty} d(Sz, SSx_n)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_{n\to\infty} = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Definition 1.7[4] A pair of self- mappings (S,T) of a metric space (X,d) is said to weakly compatible if (S,T) commutes only at their coincidence points of S and T.

Kulkarni [6] given the followings result.

Theorem 1.8 [6] Let S, A and T be the three continuous self-mappings of a complete metric space (X, d) such that the following condition are satisfy.

$$ST = TS,\ SA = AS,\ S(X) \subset A(X),\ S(X) \subset T(X)$$

$$[d(Sx,Sy)]^2 \le a[d(Ax,Sx)d(Ty,Sx) + d(Ax,Sy)d(Ty,Sx)]$$

$$+ b[d(Ax,Sx)d(Ty,Sy) + d(Ax,Sy)d(Ty,Sy)]$$

for all x, y in X where a, b are non- negative reals satisfying, $a + b < 1$, $a \ge 0$, $b \ge 0$. Thus S, A, and T have a unique common fixed point.

2. MAIN RESULT

Theorem2.1. Let A, B, S, T, I and J be the self-mappings of a complete metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ and for each $x,y \in X$, where a, b are non- negative reals satisfying, $a+b < 1$, $a \ge 0$, $b \ge 0$ either

$$[d(ABx,STy)]^2 \le a[d(Ix,ABx)d(Jy,ABx) + d(Ix,STy)d(Jy,ABx)]$$

$$+ b(Ix,ABx)d(Jy,STy) + d(Ix,STy)d(Jy,STy)] \qquad (2.1.1)$$

(a) Moreover if {AB,I} are compatible, (AB, I) is reciprocally continuous and (ST, J) are weakly compatible or

(a') If {ST, J} are compatible, (ST, J) is reciprocally continuous and (AB, I) are weakly compatible.

Then AB, ST, I and J have a unique common fixed point. Furthermore, if the pairs (A,B), (A,I), (B,I), (S,J) and (T,J) commute at the earlier common fixed point then A, B, S, T, I and J also have the same unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $AB(X) \subset J(X)$, we can find a point x_1 in X, such that $ABx_0 = Jx_1$. Also since $ST(X) \subset I(X)$, we can choose a point, x_2 with $STx_1 = Ix_2$. Using this argument repeatedly one can construct a sequence $\{z_n\}$ such that $z_{2n} = ABx_{2n} = Jx_{2n+1}$, $z_{2n+1} = STx_{2n+2}$ for n = 0,1,2……….. for the sake of brevity let us put

$$d^2(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$

$$\leq a\left[d\left(Ix_{2n+2}, ABx_{2n+2}\right)d\left(Jx_{2n+1}, ABx_{n+2}\right) + d\left(Ix_{2n+2}, STx_{2n+1}\right)d\left(Jx_{2n+1}, ABx_{2n+2}\right)\right]$$
$$+ b\left[d\left(Ix_{2n+2}, ABx_{2n+2}\right)d\left(Jx_{2n+1}\right) + d\left(Ix_{2n+2}, STx_{2n+1}\right)d\left(Jx_{2n+1}, STx_{2n+1}\right)\right]$$

$$\leq a\left[d\left(z_{2n+2}, z_{2n+1}\right)d\left(z_{2n}, z_{2n+2}\right)\right] + b\left[d\left(z_{2n+2}, z_{2n+1}\right)d\left(z_{2n}, z_{2n+1}\right)\right]$$
$$\left[d\left(z_{2n+1}, z_{2n+2}\right)\right]^2 \leq \frac{a+b}{(1-a)}\left[d\left(z_{2n}, z_{2n+1}\right)\right]^2$$

Similarly one can show that

$$\left[d\left(z_{2n+1}, z_{2n+2}\right)\right]^2 \leq \frac{a+b}{(1-a)}\left[d\left(z_{2n}, z_{2n+1}\right)\right]^2$$

Thus for every n we have

$$d^2\left(z_n, z_{n+1}\right) \leq kd^2\left(z_{n-1}, z_n\right),$$

which shows that $\{z_n\}$ is a Cauchy sequence in the complete metric space (X,d) and so has a limit point z in X. Now it follows that the sequence $ABx_{2n} = Jx_{2n+1}$ and $STx_{2n-1} = Ix_{2n}$ which are subsequence, also converge to the point z.

Since the mappings (AB, I) are reciprocally continuous therefore $\lim_{n\to\infty} ABIx_n = ABz$ and $\lim_{n\to\infty} IABx_n = Iz.$ Compatibility of (AB,I) yields that $\lim_{n\to\infty} d(ABIx_n, IABx_n) = 0,$ i.e. d (ABz, Iz) = 0. Hence ABz=Iz.

Now,

$$\left[d\left(ABz, STx_{2n+1}\right)\right]^2 \leq a\left[d\left(Iz, ABz\right)d\left(Jx_{2n+1}, ABz\right) + d\left(Iz, STx_{2n+1}\right)d\left(Jx_{2n+1}, ABz\right)\right]$$

$$+ b\left[d\left(Iz, ABz\right)d\left(Jx_{2n+1}, STx_{2n+1}\right) + d\left(Iz, STx_{2n+1}\right)d\left(Jx_{2n+1}, STx_{2n+1}\right)\right]$$

taking limit $n\to\infty$, reduces to

$$\left[d\left(ABz, z\right)\right]^2 \leq a\left[d\left(ABz, z\right)\right]^2$$

yielding thereby ABz = z = Iz.

Since AB (X) $\subset$ J(X) always exists a point z' in X such that Jz' = z. Now

$$[d(ABz, STz') \leq ad(Iz, ABz)d(Jz', ABz) + d(Iz, STz')[d(Jz', ABz)]$$

$$+ b[d(Iz, ABz)d(Jz', STz') + d(Iz, STz')d(Jz', STz')]$$

$$[d(z, STz')]^2 \leq (a+b)[d(z, STz')]^2$$

yielding thereby STz' = Jz'. But since (ST,J) are weakly compatible therefore

STz = ST(Jz') = J (STz') = Jz.

Now

$$[d(ABz, STz)]^2 \leq a[d(Iz, ABz)d(Jz, ABz) + d(Iz, STz)d(Jz, ABz)]$$

$$+ b[d(Iz, ABz)d(Jz, STz) + d(Iz, STz)d(Jz, STz)]$$

$$[d(z, STz)]^2 \leq (a+b)[d(z, STz)]^2$$

yielding thereby STz = Jz = z.

Let v be another common fixed point of AB, ST, I and J.

$$[d(ABz, STv)]^2, \leq a[d(Iz, ABz)d(Jv, ABz) + d(Iz, STv)d(Jv, ABz)]$$

$$+ b[(Iz, ABz)d(Jv, STv) + d(Iz, STv)d(Jv, STv)].$$

yielding thereby z = v.

Finally we need to show that z is also a common fixed point of A,B,S,T,I and J.For this let z be a common fixed point of the pair (AB, I) then

Az = A (ABz) = A (BAz) = AB (Az), Az = A (Iz) = I (Az)

Bz = B (ABz) = BA (Bz) = AB (Bz), Bz = B (Iz) = I (Az)

Which shows that Az and Bz is a common fixed point of (AB, I) yielding thereby Az = Bz = Iz = ABz = z in view of the uniqueness of the common fixed point of the pair (AB,I).

Similarly we can show by using commutativity of (S,T), (S,J), (T,J), it can show that Sz = z = Tz = Jz = STz. Thus z is the unique common fixed point of A, B, S, T, I and J.

Theorem 2.2. Theorem 2.1 remains true if we replace 'compatibility' condition by

(a) Compatibility of type (A) or
(b) Compatibility of type (B) or

(c) Compatibility of type (P) or
(d) Weak compatibility of type (P).

Proof. The proof is identical except minor changes where the particular compatibility condition is be used which is also straightforward by using particular definition. Hence we omit the proof.

Related Example.

We construct example to demonstrate the validity of the hypotheses and degree of generality of our main result.

Example: Consider X = [0,1] with the usual metric. Define self-mappings A,B,S,T,I and J as Ax = 2x/3,.Bx = 3x/4,Sx = x/4,Tx = 4x/3,Ix = x/4, and Jx= 3x/4.Clearly AB(X) = [0,1/2] ⊂ J(X) = [0,3/4],ST(X) = [0,1/5] ⊂ IX) = [0,1/4].Also, the pairs of mappings (AB),(ST),(A,B)(B,I)(S,J) and(T,J) are commuting hence compatible or weak compatible.

For all x, y in X(x> y) with a = 1/20 and b = 1/5

$$\left(\frac{x}{2}-\frac{y}{5}\right)^2 \leq a\left[\left(\frac{x}{4}-\frac{x}{2}\right)\left(\frac{3y}{4}-\frac{x}{2}\right)+\left(\frac{x}{4}-\frac{y}{5}\right)\left(\frac{3y}{4}-\frac{x}{2}\right)\right]$$
$$+b\left[\left(\frac{x}{4}-\frac{x}{2}\right)\left(\frac{3y}{4}-\frac{x}{2}\right)+\left(\frac{x}{4}-\frac{y}{5}\right)\left(\frac{3y}{4}-\frac{x}{2}\right)\right]$$
$$a\left(\frac{3y}{4}-\frac{x}{2}\right)\left(\frac{x}{2}-\frac{y}{5}\right)+b\left(\frac{3y}{5}-\frac{y}{5}\right)\left(\frac{x}{2}-\frac{y}{5}\right)$$

x/2> x/4 > y/5 and 3y /4 > y/5 , one can get

$$\left(\frac{x}{2}-\frac{y}{5}\right)^2 \leq (a+b)\left(\frac{x}{2}-\frac{y}{5}\right)^2$$

which verifies the contraction condition 2.1.1 . Clearly 0 is the unique common fixed point of A, B, S, T, I and J.

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