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Haar Wavelet Collocation Method for the Numerical Solution of Nonlinear Volterra-Fredholm-Hammerstein Integral Equations

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Abstract

In this paper, we proposed Haar wavelet collocation method for the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations. Properties of Haar wavelet and its operational matrices are utilized to convert the integral equation into a system of algebraic equations, solving these equations using MATLAB to compute the Haar coefficients. Numerical results are compared with exact solution through error analysis, which shows the efficiency of this technique.

Keywords: Nonlinear Volterra-Fredholm-Hammerstein integral equations, Haar wavelet collocation method, Operational matrix, Leibnitz rule.

INTRODUCTION

Integral equations find its applications in various fields of mathematics, science and technology have motivated a large amount of research work in recent years. In particular, integral equations arise in fluid mechanics, biological models, solid state physics, kinetics in chemistry etc. In most of the cases, it is difficult to solve them, especially analytically. Anticipating exact solution for integral equations is not possible always. Due to this fact, several numerical methods have been developed for finding solutions of integral equations. Nonlinearity is one of the most interesting topics among the physicists, mathematicians, engineers, etc.

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [1, 2]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations. The solutions are often quite complicated and the advantages of the wavelet method get lost. Therefore any kind of simplification is welcome. One possibility for it is to make use of the Haar wavelets, which are mathematically the simplest wavelets. In the previous work, system analysis via Haar wavelets was led by Chen and Hsiao [3], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the applications for the Haar analysis into the dynamic systems. Recently, Haar wavelet method is applied for different type of problems. Namely, Siraj-ul-Islam et al. [4] proposed for the numerical solution of second order boundary value problems. Shiralashetti et al. [5-8] applied for the numerical solution of Klein–Gordan equations, multi-term fractional differential equations, singular initial value problems, Riccati and Fractional Riccati Differential Equations. Shiralashetti et al. [9] have introduced the adaptive gird Haar wavelet collocation method for the numerical solution of parabolic partial differential equations. Also, Haar wavelet method is applied for different kind of integral equations, which among Lepik et al. [10-13] presented the solution for differential and integral equations. Babolian et al. [14] and Shiralashetti et al. [15] applied for solving nonlinear Fredholm integral equations. Aziz et al. [16] have introduced a new algorithm for the numerical solution of nonlinear Fredholm and Volterra integral equations. Some of the author's have approached for the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations from various methods. Such as Legendre collocation method [17], Legendre approximation [18], CAS wavelet [19]. In this paper, we applied the Haar wavelet collocation method for the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations.

The article is organized as follows: In Section 2, properties of Haar wavelets and its operational matrix is given. In Section 3, the method of solution is discussed. In section 4, we report our numerical results and demonstrated the accuracy of the proposed scheme. Lastly, the conclusion of the proposed work is given in section 5.

2. PROPERTIES OF HAAR WAVELETS

2.1. Haar wavelets

The scaling function $h_1(x)$ for the family of the Haar wavelet is defined as

$$h_1(x)=\begin{cases}1 & for\ x\in\left[0,1\right)\\ 0 & otherwise\end{cases} \tag{2.1}$$

The Haar Wavelet family for $x\in[0,1)$ is defined as,

$$h_i(x)=\begin{cases}1 & for\, x\in[\alpha,\beta),\\ -1 & for\, x\in[\beta,\gamma),\\ 0 & elsewhere,\end{cases} \tag{2.2}$$

where $\alpha=\dfrac{k}{m},\quad \beta=\dfrac{k+0.5}{m},\quad \gamma=\dfrac{k+1}{m},$

where $m=2^l$, $l=0,1,...,J,$ J is the level of resolution; and $k=0,1,...,m-1$ is the translation parameter. Maximum level of resolution is J. The index i in (2.2) is calculated using $i=m+k+1$. In case of minimal values $m=1,\ k=0$ then $i=2$. The maximal value of i is $N=2^{J+1}$.

Let us define the collocation points $x_j=\dfrac{j-0.5}{N}, j=1,2,...,N$, Haar coefficient matrix $H\left(i,j\right)=h_i(x_j)$ which has the dimension $N\times N$.

For instance, $J=3\Rightarrow N=16$, then we have

$$\mathrm{H}(16,16)=\begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&-1&-1&-1&-1&-1&-1&-1&-1\\
1&1&1&1&-1&-1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&-1&-1&-1&-1\\
1&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&-1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&-1&-1\\
1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1
\end{pmatrix}$$

Any function $f(x)$ which is square integrable in the interval (0, 1) can be expressed as an infinite sum of Haar wavelets as,

$$f(x)=\sum_{i=1}^{\infty} a_i h_i(x) \tag{2.3}$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or it can be approximated as piecewise constant during each subinterval. Given a function $f(x)\in L^2(R)$ a multi-resolution analysis (MRA) of $L^2(R)$ produces a sequence of subspaces $V_j, V_{j+1},\dots$ such that the projections of $f(x)$ onto these spaces gives finer approximation of the function $f(x)$ as $j\to\infty$.

2.2. Operational Matrix of Haar Wavelet

The operational matrix P which is an N square matrix is defined by

$$P_{1,i}(x)=\int_0^x h_i(t)\,dt \tag{2.4}$$

often, we need the integrals

$$P_{r,i}(x)=\underbrace{\int_A^x\int_A^x \dots\int_A^x}_{r-times} h_i(t)\ dt^r = \frac{1}{(r-1)!}\int_A^x (x-t)^{r-1} h_i(t)\,dt \tag{2.5}$$

$r=1,2,\dots,n$ *and* $i=1,2,\dots,N.$

For $r=1,$ corresponds to the function $P_{1,i}(x)$, with the help of (2.2) these integrals can be calculated analytically; we get,

$$P_{1,i}(x)=\begin{cases} x-\alpha & for\ \ x\in[\alpha,\beta) \\ \gamma-x & for\ \ x\in[\beta,\gamma) \\ 0 & Otherwise \end{cases} \tag{2.6}$$

$$P_{2,i}(x)=\begin{cases} \frac{1}{2}(x-\alpha)^2 & for\ \ x\in[\alpha,\beta) \\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-x)^2 & for\ \ x\in[\beta,\gamma) \\ \frac{1}{4m^2} & for\ \ x\in[\gamma,1) \\ 0 & Otherwise \end{cases} \tag{2.7}$$

In general, the operational matrix of integration of r^{th} order is given as

$$P_{r,i}(x)=\begin{cases} \frac{1}{r!}(x-\alpha)^r & for \quad x\in[\alpha,\beta) \\ \frac{1}{r!}\{(x-\alpha)^r-2(x-\beta)^r\} & for \quad x\in[\beta,\gamma) \\ \frac{1}{r!}\{(x-\alpha)^r-2(x-\beta)^r+(x-\gamma)^r\} & for \quad x\in[\gamma,1) \\ 0 & Otherwise \end{cases} \tag{2.8}$$

For instance, J=3 ⇒ N = 16, then we have

$$P_{1,i}(16,16)=\frac{1}{32}\begin{pmatrix}
1&3&5&7&9&11&13&15&17&19&21&23&25&27&29&31\\
1&3&5&7&9&11&13&15&15&13&11&9&7&5&3&1\\
1&3&5&7&7&5&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&5&7&7&5&3&1\\
1&3&3&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&3&3&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&3&3&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&3&3&1\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1
\end{pmatrix}$$

and

$$P_{2,i}(16,16)=\frac{1}{2048}\begin{pmatrix}
1&9&25&49&81&121&169&225&289&361&441&529&625&729&841&961\\
1&9&25&49&81&121&169&225&287&343&391&431&463&487&503&511\\
1&9&25&49&79&103&119&127&128&128&128&128&128&128&128&128\\
0&0&0&0&0&0&0&0&1&9&25&49&79&103&119&127\\
1&9&23&31&32&32&32&32&32&32&32&32&32&32&32&32\\
0&0&0&0&1&9&23&31&32&32&32&32&32&32&32&32\\
0&0&0&0&0&0&0&0&1&9&23&31&32&32&32&32\\
0&0&0&0&0&0&0&0&0&0&0&0&1&9&23&31\\
1&7&8&8&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&1&7&8&8&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&1&7&8&8&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&1&7&8&8&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&1&7&8&8&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&1&7&8&8&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&1&7&8&8\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&7
\end{pmatrix}$$

3. METHOD OF SOLUTION

In this section, we present a Haar wavelet collocation method (HWCM) based on Leibnitz rule for the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equation of the form,

$$u(x) = f(x) + \int_0^x K_1(x,t)F(t,u(t))dt + \int_0^1 K_2(x,t)G(t,u(t))\,dt\ , \tag{3.1}$$

where $K_1(x, t)$ and $K_2(x, t)$ are known functions which are called kernels of the integral equation and $f(x)$ is also a known function, while the unknown function $u(x)$ represents the approximate solution of the integral equation. Basic principle is that for conversion of the integral equation into equivalent differential equation with initial conditions. The conversion is achieved by the well-known Leibnitz rule [20].

Numerical computational Procedure is as follows,

Step 1: Differentiating (3.1) twice w.r.t *x*, using Leibnitz rule, we get differential equations with subject to initial conditions $u(0) = \beta,\ u'(0) = \gamma$.

Step 2: Applying Haar wavelet collocation method,

Let us assume that,
$$u''(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{3.2}$$

Step 3: By integrating (3.2) twice and substituting the initial conditions, we get,

$$u'(x) = \gamma + \sum_{i=1}^{N} a_i p_{1,i}(x) \tag{3.3}$$

$$u(x) = \beta + \gamma x + \sum_{i=1}^{N} a_i p_{2,i}(x) \tag{3.4}$$

Step 4: Substituting (3.2) - (3.4) in the differential equation, which reduces to the nonlinear system of *N* equations with *N* unknowns and then the Newton's method is used to obtain the Haar coefficients a_i, i = 1,2,...., *N*. Substituting Haar coefficients in (3.4) to obtain the required approximate solution of equation (3.1).

4. ILLUSTRATIVE EXAMPLES

In this section, we consider the some of the examples to demonstrate the capability of the present method and error function is presented to verify the accuracy and efficiency of the following numerical results,

$$Error = E_{\max} = \left\| u_e(x_i) - u_a(x_i) \right\|_{\max} = \sqrt{\sum_{i=1}^{n} \left(u_e(x_i) - u_a(x_i) \right)^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 4.1 Consider the Nonlinear Volterra-Fredholm-Hammerstein integral equation [17],

$$u(x) = \frac{x}{2} - \frac{x^4}{12} - \frac{1}{3} + \int_0^1 (x+t)u(t)\,dt + \int_0^x (x-t)u^2(t)\,dt,\ 0 \le x, t \le 1 \tag{4.1}$$

Initial condition's: $u(0) = 0, u'(0) = 1$. Which has the exact solution $u(x) = x$.

Differentiating Eq. (4.1) twice w.r.t *x* and using Leibnitz rule which reduces to the differential equation,

$$u'(x) = \frac{1}{2} - \frac{1}{3}x^3 + \int_0^1 u(t)\,dt + \int_0^x u^2(t)\,dt \tag{4.2}$$

$$u''(x) - [u(x)]^2 + x^2 = 0 \tag{4.3}$$

Assume that,
$$u''(x) = \sum_{i=1}^{N} a_i h_i(x) \tag{4.4}$$

Integrating Eq. (4.4) twice,

$$u'(x) = \sum_{i=1}^{N} a_i p_{1,i}(x) + 1 \tag{4.5}$$

$$u(x) = \sum_{i=1}^{N} a_i p_{2,i}(x) + x \tag{4.6}$$

Substituting Eq. (4.4) – Eq. (4.6) in Eq. (4.3), we get the system of *N* equations with *N* unknowns,

$$\sum_{i=1}^{N} a_i h_i(x) - \left[\sum_{i=1}^{N} a_i p_{2,i}(x) + x \right]^2 + x^2 = 0. \tag{4.7}$$

Solving Eq. (4.7) using Newton's method to find Haar wavelet coefficients a_i's for $N = 16$, i.e., [-1.93e-11 1.93e-11 8.48e-19 3.70e-11 -2.56e-18 1.08e-12 6.99e-11 -5.24e-20 -1.59e-18 -7.87e-18 8.86e-18 4.92e-13 1.53e-12 2.60e-12 9.36e-11]. Substituting a_i's, in Eq. (4.6) and obtained the required HWCM solution with exact solution is presented in table 2. Error analysis is shown in table 1. Hence, justifies the efficiency of the HWCM.

Table 1. Error analysis of example 4.1.

N	E_{max} (HWCM)
4	1.93e-13
8	1.68e-13
16	5.29e-13
32	6.66e-13
64	6.55e-13
128	9.38e-13

Table 2. Comparison of exact and approximate solution of example 4.1.

x(/32)	Exact	(HWCM)	Error (HWCM)
1	0.03125	0.03125	0
3	0.09375	0.09375	0
5	0.15625	0.15625	0
7	0.21875	0.21875	0
9	0.28125	0.28125	0
11	0.34375	0.34375	0
13	0.40625	0.40625	0
15	0.46875	0.46875	0
17	0.53125	0.53125	0
19	0.59375	0.59375	4.44e-16
21	0.65625	0.65625	4.44e-15
23	0.71875	0.71875	1.41e-14
25	0.78125	0.78125	3.82e-14
27	0.84375	0.84375	7.70e-14
29	0.90625	0.90625	1.67e-13
31	0.96875	0.96875	5.29e-13

Example 4.2. Next, consider the Nonlinear Volterra-Hammerstein Integral equation [18],

$$u(x)=\frac{3}{2}-\frac{1}{2}e^{-2x}-\int_0^x[(u(t))^2+u(t)]\,dt, \qquad 0\le x\le 1, \tag{4.8}$$

with initial conditions $u(0)=1$. Which has the exact solution $u(x)=e^{-x}$.

Successively differentiating Eq. (4.8) w.r.t x and using Leibnitz rule reduces to the differential equation,

$$u'(x)=e^{-2x}+(u(x)^2+u(x)) \tag{4.9}$$

$$u'(x)-(u(x)^2+u(x))-e^{-2x}=0 \tag{4.10}$$

Assume that, $$u'(x)=\sum_{i=1}^{2M}a_i h_i(x) \tag{4.11}$$

Integrating Eq. (4.11),

$$u(x)=\sum_{i=1}^{2M}a_i p_{1,i}(x)+1 \tag{4.12}$$

Substituting Eqs. (4.11) and (4.12) in Eq. (4.9), we get the system of N equations with N unknowns.

$$\sum_{i=1}^{2M}a_i h_i(x)-\left(\left(\sum_{i=1}^{2M}a_i p_{1,i}(x)+1\right)^2+\left(\sum_{i=1}^{2M}a_i p_{1,i}(x)+1\right)\right)-e^{-2x}=0 \tag{4.13}$$

solving (4.13) using Matlab to find Haar wavelet coefficients a_i's, for N = 16 i.e, [-0.63 -0.16 -0.10 -0.06 -0.06 -0.04 -0.03 -0.03 -0.03 -0.03 -0.02 -0.02 -0.02 -0.02 -0.01 -0.01]. Substituting a_i's, in Eq. (4.12) and obtained the required HWCM solution compared with exact solutions is shown in table 3. Error analysis is given in table 4, which justifies the efficiency of the HWCM.

Table 3. Comparison of Exact and HWCM for N=16 of example of 4.2.

x=(/32)	HWCM	Exact
1	0.9697	0.9692
3	0.9109	0.9105
5	0.8556	0.8553
7	0.8037	0.8035
9	0.7550	0.7548
11	0.7092	0.7091
13	0.6662	0.6661
15	0.6259	0.6258
17	0.5879	0.5879
19	0.5523	0.5523
21	0.5188	0.5188
23	0.4874	0.4874
25	0.4578	0.4578
27	0.4301	0.4301
29	0.4040	0.4040
31	0.3795	0.3796

Table 4. Error analysis of the example 4.2.

N	E_{max} (HWCM)
4	5.3e-3
8	1.6e-3
16	4.38e-4
32	1.15e-4
64	2.96e-5
128	7.52e-6

5. CONCLUSION

In the present work, Haar wavelet collocation method based on Leibnitz rule is applied to obtain the numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equation of the second kind. The Haar wavelet function and its operational matrix were employed to solve the resultant integral equations. The numerical results are obtained by the proposed method have been demonstrated in tables and figures. Illustrative examples are tested with error analysis to justify the efficiency and possibility of the proposed technique.

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