Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 2 (2017), pp. 475-492

http://www.ripublication.com

Applied Mathematics and Demonstrations to the Theory of Optimal Filters

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Abstract

In this review article, we introduce notions of the mathematical applications, including some demonstrations of the optimal filters, where we make a complete study of its properties like the filters of Kalman-Bucy. In addition, we establish a comparison of the Winer and Kalman filters and some of their solution techniques that allow solving the filtering of common signals that is model with the mentioned equations. It shows an application of mathematical models of the optimum filters with a physical application of basic engineering that demonstrates their effectiveness for the engineering calculations made.

Keyword: filter, optimal filter, Kalman-Bucy filter

INTRODUCTION

Most of the signals that appear in the fields of science and engineering are analogous in nature, that is, signals are functions of a continuous variable such as time or space and normally take values in a continuous range. Such signals are process directly by analog systems for the purpose of changing their characteristics or extracting any desired information. In this case, the signal is process directly in analog form.

Digital signal processing provides an alternative method for processing a signal. To perform the processing, an interface between the analog signal and the digital processor is required. After this signal is process through filters for further interpretation. To understand the development of the system it becomes necessary to know the theoretical aspects, which are develop bellow.

1. THE OPTIMAL FILTER THEORY.

1.1. Wiener Filter

The Wiener filter probably represents the first presentation of terminology in which two important ideas are rescue: dynamical systems and estimation optimal in the presence of noise. It is consider a signal $y(\dot{})$, which contains a noise, $v(\dot{})$ and a measure $z(\dot{}). y(\dot{}), v(\dot{}), y\ z(\dot{})$ may give rise to a problem of the continuous or discrete type in time depending on the nature of the same. Time signals are considered as continuous scalars defined in the interval $(-\infty, \infty)$ only. It is assumed that $y(\dot{}), and\ v(\dot{})$, are simple functions of stationary random processes[1]. Usually they are independent and have zero mean. Subsequently they are considered for the obtaining of $\varphi yy(jw)\ y\ \varphi vv(jw), w\ \in\ R - spectra$. The Wiener filter task is to use the measures $z(\dot{})$ to estimate $y(\dot{})$. More precisely, the estimation is required to be causal, online and optimal. Causal means that $y(t)$ is to be estimated using $z(s)$ to some $s\ <\ t$; On-line means that at time t the estimate of $y(t)$ should be optimally. Optimal means that $y(t)$, should have a minimum square error, $i.e.$ $\mathrm{E}[\mathrm{y(t)}\ -\ \hat{y}(\mathrm{t})]^2$ Which must be minimized. If $y(\dot{}), and\ v(\dot{})$ are Gaussian, this Means that $\hat{y}(t)$ is the conditional estimate, $\mathrm{E}\ [\frac{\mathrm{y(t)}}{\mathrm{z(s)}}\ , \mathrm{s}\ \leq\ \mathrm{t}]$.

Solution

The solution to this problem is given in the following explanation: The Wiener filter Is a linear, time-invariant, causal, stable system whose input-output relationship is given by a transfer function $h(\dot{})$:

$$\hat{y}(t) = \int_{-\infty}^{t} \mathrm{h(t\ -\ s)z(s)ds} \tag{1}$$

The signal $y(\dot{})$ and noise $v(\dot{})$ is represent as the output of a linear system excited by white noise. If $\varepsilon y(\dot{}), \varepsilon v(\dot{})$ are white noises with zero mean and intensity of variance 1, then

$$\mathrm{E}[\varepsilon \mathrm{y(t)} \varepsilon \mathrm{y(s)}] =\ \mathrm{E}[\varepsilon \mathrm{v(t)} \varepsilon \mathrm{v(s)}] =\ \delta(\mathrm{t}\ -\ \mathrm{s}), \tag{2}$$

and therefore

$$\varphi yy(jw) = |Wy(jw)|^2, \varphi vv(jw) = |Wv(jw)|^2 \tag{3}$$

The key of the problem is the obtainment of $\varphi yy(jw), \varphi vv(jw)$ for the response function to the impulse $h(t)$ or its transfer function $H(jw)$. The crucial step is the spectral factorization.

The spectrum of $z(\dot{})$ when $y(\dot{})$ and $v(\dot{})$ are independent is given by:

$$\varphi zz(jw) = \varphi yy(jw) + \varphi vv(jw) \tag{4}$$

Spectral factorization requires the determination of a transfer function $W_z(jw)$ such that $W_z(s)$ and (s) are analytic at $\mathbb{R}$, $s \geq 0$ and such that:

$$\varphi zz(jw) = |Wz(jw)|^2, \tag{5}$$

In [4] this spectral factorization operation is presented as a crucial step in the obtaining of $H(\dot{})$, which in [2] is the key to the determination of the optimum filter. Then proceed as follows. A signal $\varepsilon z(\dot{})$ is defined as the output of a linear system of a $W_z{}^{-1}(jw)$ transfer function driven by $z(\dot{})$. If it exists $W_z{}^{-1}(\dot{})$ Then $\varepsilon z(\dot{})$ is equivalent to $z(\dot{})$, that is, the estimate of $y(t)$ using $\varepsilon z(s)$ for $s < t$ should give the same result as the estimate of $y(t)$ using $z(s)\, for\, s < t$ and also $\varepsilon z(\dot{})$ is a white noise. This simplification is very important and is used to obtaining the optimum filter in [11 and 12].

In addition, it is notable that the construction of $W_z(\dot{})$ satisfies the conditions of stability and (5) and is an important step for the construction of H($\dot{}$). The question is how can Be done if φzz($\dot{}$) is rational? The key is polynomial factorization. In another case, use:

$$Wz(jw0) = \min_{\varepsilon \to 0} \exp\{ \{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log\varphi zz(jw)}{-j(w - w0) - \varepsilon} dw\} \tag{6}$$

Another way to solve the filtering problem, in the time domain, is to use the impulse response function h(t), which corresponds to the inverse transform of Laplace of the function H(jw), by the equation:

$$h(t) + \int_0^t h(\tau)K(\tau - s)ds = K(t), t \geq 0, \tag{7}$$

Where K(τ) is the covariance function of y(t). This equation is known as the equation Wiener-Hopf.

1.2 Kalman-Bucy Filter (Discrete Case)

Practically, everything established for the Kalman filter in continuous time, is to the case of the filter with discrete time. The theory in the continuous case is more transparent than in the discrete case, since it presents applicability to more problems[2].

The model is given by

$$x_k + 1 = F_k x_x + G_k w_k \tag{8}$$
$$\text{zk} = \text{H}^{\text{T}}\text{k}x_k + \text{vk}$$

with

$$\text{E}\begin{bmatrix} w_k \\ v_k \end{bmatrix} \begin{bmatrix} w_l^T & v_l^T \end{bmatrix} = \begin{bmatrix} Q_k & 0 \\ 0 & R_k \end{bmatrix} \delta_{\text{kl}}$$

And $\{w_k\}$, $\{v_k\}$ are sequences with zero mean. By convention, time is considered Initial k = 0. Adding that the mean $\hat{x}_0$ and the variance P_0 of x_0 are independent of $\{w_k\}$, $\{v_k\}$. All variables are Gaussian. The main idea is to distinguish the effect of dynamics and measurements on the filter. More precisely, be $\hat{x}_{\frac{k}{k}}$ the estimated optimal estimated average of x_k given z_l, l ≤ k, and let $\hat{x}_{\frac{k+1}{k}}$ given by $E\left[\frac{\text{kk+1}}{z_l}\right], \text{l} \leq \text{k}],$ the first step in the prediction of the estimate (*see in* [27]).

Solution

Since w_k is independent of z_l for $l \leq k$, we have

$$\hat{x}_{\frac{k+1}{k}} = F_k \hat{x}_{\frac{k}{k}} \tag{9}$$

This demonstrates how to update an estimate because of dynamical systems, when No extra measurements appear. (9) relies on

$$\mathsf{V}_{\frac{k+1}{k}} = F_k \mathsf{V}_{\frac{k}{k}} F_k^T + G_k Q_k G_k^T \tag{10}$$

Here $\mathsf{V}_{\frac{k+1}{k}}$ *and* $\mathsf{V}_{\frac{k}{k}}$ Are the error covariance's associated with $\hat{x}_{\frac{k}{k}}$ and $\hat{x}_{\frac{k+1}{k}}$ update the Equations of the estimates is equivalent to $\hat{x}_{\frac{k+1}{k}}$ and $\mathsf{V}_{\frac{k+1}{k}}$ *to* $\hat{x}_{\frac{k+1}{k+1}}$ *and* $\mathsf{V}_{\frac{k+1}{k+1}}$ and this is Show below:

$$\hat{x}_{\frac{k+1}{k}} = \hat{x}_{\frac{k+1}{k}} + \mathrm{V}_{\frac{k+1}{k}} \mathrm{H_k} + 1 \left[H_{k+1}^{T} \mathrm{V}_{\frac{k+1}{k}} H_k + R_k + 1\right]^{-1} \mathrm{X} \quad \left[z_{k+1} - H_{k+1}^{T} \hat{x}_{\frac{k+1}{k}}\right] \tag{11}$$

$$\bigvee_{\frac{k+1}{k+1}} = \bigvee_{\frac{k+1}{k}} - \bigvee_{\frac{k+1}{k}} \mathrm{H_{k+1}} \left[H_{k+1}^{T} \bigvee_{\frac{k+1}{k}} H_{k+1} + R_{k+1}\right]^{-1} H_{k+1}^{T} \bigvee_{\frac{k+1}{k}}$$

1.3 Kalman-Bucy Filter (Continuous Case)

The representation of the model is given by

$$\frac{dx}{dt} = \mathrm{F(t)x(t)} + \mathrm{G(t)w(t)} \tag{12}$$

$$\mathrm{z(t)} = H^{T}\,\mathrm{(t)x(t)} + \mathrm{v(t)} \tag{13}$$

In which F, G, H are matrices n × n, n × m, and n × p respectively. The process $w(\dot{})$ y $V(\dot{})$ are Gaussian white noises with zero mean such that

$$E \begin{bmatrix} \mathrm{w(t)} \\ \mathrm{v(t)} \end{bmatrix} \begin{bmatrix} w^{T}(s) & v^{T}(s) \end{bmatrix} = \begin{bmatrix} \mathrm{Q(t)} & \mathrm{S(t)} \\ S^{T}(s) & \mathrm{R(t)} \end{bmatrix} \delta(\mathrm{t} - \mathrm{s})$$

With $R(t) = R'(t) > 0$ for all t. Very often, $S(t) \equiv 0$, i.e. $W(\dot{})\, and\, v(\dot{})$ are Independent. Which is supposed. Then $Q(t) = Q^{T}(t) \geq 0$. A time is assumed Finite initial t_0. On the other hand $x(t_0)$ will be assumed as a Gaussian random variable with Mean x_0 and variance P_0. The task of estimation is to use $z(s)$ measurements for $s < t$ To estimate $x(t)$, this estimate is called $\hat{x}(t)$, which minimizes $E\ [||x(t) - \hat{x}(t)||^2]$. This Means that $\hat{x}(t)$ is necessarily an estimate of the conditional mean, with respect to the observations [1], [2].

Solution

The solution is obtained as follows. Define $P(t) = P^{T}(t) \geq 0$ as the Solution of

$$P^{\dot{}} = PF^{T} + \mathrm{FP} - \mathrm{PHR^{-1}}H^{T}P + \mathrm{GQ}G^{T}, P(t_0) = \mathrm{P0} \tag{14}$$

And $\hat{x}(t)$ is the solution of

$$\frac{d\hat{x}}{dt} = F(t)\hat{x}(t)\mathrm{P(t)H(t)R^{-1}(t)[z(t)} - H^{T}(t)\hat{x}(t)] \tag{15}$$

Where $P(t)\, H(t)\, R^{-1}(t)$ denotes the Kalman gain.

$E\ [x\ (t) - \hat{x}(t)] -\ [x(t) - \hat{x}(t)]^T\ = P\ (t)$. The effectiveness of the optimum estimator is measure by the error covariance, which are gives by the solution of the equation (14), and the existence of the solution to this equation in (t_0, ∞) is guaranteed.

Some differences of the Kalman filter with respect to the Wiener filter are gives in following table, see [31].

Wiener Filter	Kalman Filter
$t0\ =\ -\infty$	$t0\ \geq\ -\infty$
Stationary	Accepts non-stationary.
Infinite dimensional	finite dimensional
Noise not necessarily white	White noise
spectral factorization	Solution of the Riccati equation
Signal estimation	Estimating status

The problem of predictions solved by filter theory. This is to compute $x(t\ + \triangle)$ for some $\triangle$ positive, given $z(s)$ for $s\ <\ t$, that is:

$$\hat{x}(t\ + \triangle) =\ \Phi(\mathrm{t}\ + \triangle)\hat{x}(t) \tag{16}$$

1.4 Polynomial Optimum Filter

Be (Ω, F, P) a complete probability space with a growing and continuous family by the right of σ-algebra $F_t, t\ \geq\ t_0$, and are $(W_1(t), Ft, t\ \geq\ t_0)\ y\ (W_2(t), F_t, t\ \geq\ t_0)$ Wiener independent processes. The Ft measurable random process $(x(t), y(t))$ is describe by a nonlinear differential equation with a polynomial drift term for the state of systems [28]:

$$\mathrm{dx(t)}\ =\ \mathrm{f(x,t)dt}\ +\ \mathrm{b(t)d}W_1\mathrm{(t)},\ \mathrm{x(t0)}\ =\ \mathrm{x0}, \tag{17}$$

and a linear differential equation for the observation process

$$\mathrm{dy(t)}\ =\ (A_0\mathrm{(t)}\ +\ \mathrm{A(t)x(t))dt}\ +\ \mathrm{B(t)d}W_2\mathrm{(t)} \tag{18}$$

Here, $x(t)\ \in\ \mathbb{R}^n$ is the state vector and $y(t)\ \in\ \mathbb{R}^m$ is the vector of linear observation, $m\ \leq\ n$. The initial condition $x_0 \in\ \mathbb{R}^n$ is a Gaussian vector such that x_0, W_1 (t) $\in\ \mathbb{R}^p$, And $W_2(t) \in\ \mathbb{R}^q$ are independent. Where the observation matrix A(t) $\in\ \mathbb{R}^{m\times n}$ is not supposes that it is invertible or even square. It is assumed that $B(t)\ B^T(t)$ is a matrix defined positive, therefore, $m \leq q$. All coefficients in (17) - (18) are functions deterministic dimensions.

The optimal filter equations is obtain using the formula for differential of Ito of the conditional expectation

$$m(t) = E\ (x(t)|F_t^Y) \times (\mathrm{B(t)}\ B^T\mathrm{(t)})^{-1}(\mathrm{dy(t)} - (A_0\mathrm{(t)} + \mathrm{A(t)m(t)}), \tag{19}$$

Where $f(x, t)$ is the term of the polynomial drift in the state equation.

$$dm(t) = (f(x,t)F_t^Y)\mathrm{dt} + \mathrm{P(t)}\ A^T\mathrm{(t)}\big(B\mathrm{(t)}\ B^T(t)\big)^{-1} \times \big(\mathrm{dy(t)} - \big(A_0\mathrm{(t)} + \mathrm{A(t)m(t)}\big)\mathrm{dt}\big) \tag{20}$$

$$dP(t) = (E((x(t) - m(t))\big(f(x,t)\big)^T \mid F_t^Y) + E(f(x,t)(x(t) - m(t))^T) \mid F_t^Y) + \mathrm{b(t)}\ b^T\mathrm{(t)} - \mathrm{P(t)}\ A^T\ \mathrm{(t)(B(t)}\ B^T\mathrm{(t))}^{-1}\mathrm{A(t)P(t))dt}, \tag{21}$$

With the initial conditions

$$m(t_0) = E(x(t_0)|F_{t_0}^Y)\ and\ P(t_0) = E\ [(x\ (t_0) - m(t_0))(x(t_0) - M(t_0))T|F_{t_0}^Y].$$

In the particular case where $(f\ (x,t))$ takes the form

$$\mathrm{f(x,t)} = a_0\mathrm{(t)} + a_1\mathrm{(t)x} + a_2\mathrm{(t)x}x^T + a_3\mathrm{(t)xx}x^T$$

The following optimal closed-loop filtering equations are obtain in [11]:

$$dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + 3a_3(t)m(t)P(t) + a_3(t)m(t)m(t)m^T\ (t))dt + P(t)A^T\ (t)(B(t)B^T\ (t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))dt],$$

$$m(t0) = E(x(t_0) \mid F_t^Y)),$$

$$dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + (2a_2(t)m(t)P(t))T + 3(a_3(t)[P(t)P(t) + m(t)m^T\ (t)P(t)) + 3(a_3(t)[P(t)P(t) + m(t)m^T\ (t)P(t))T + b(t)b^T\ (t))dt - P(t)A^T\ (t)(B(t)BT\ (t))^{-1}A(t)P(t)dt.$$

$$P(t0) = E((x(t0) - m(t0))(x(t0) - m(t0))^T\ |F_t^Y)).$$

1.5 General Equation of Optimal Filtration

Consider the stochastic continuous process described by equation

$$\dot{X} = \phi(X,t) + \psi(X,t)V \tag{22}$$

Where X the n-dimensional state vector of the system, V is r-dimensional vector, which represents the Gaussian white noise, and $\varphi(X,t), \psi(X,t)$ are knowns functions of the state of system and time. The values of the function $\varphi(X,t)$ are n-dimensional vectors and the values of the function $\psi(X,t)$ are $n \times r$ matrices. If the state vector of the system X is measured continuously, then the n-dimensional random process $Y(t) = X(t) + U(t)$ would be the result of the measurements, where $U(t)$ is the error of the Measurement, which usually represents a random function of time [30]. On the other hand, if this is not true of the state vector, but if some of the components of the observation vector measure some functions of the state vector, the result of the Measurements is generally determine by the formula

$$Y = Y(t) = \phi_0(X,U,t), \tag{23}$$

Where Y is a n1-dimensional vector, U is the error of the measurement, representing a Random vector time function of dimension $r \geq n1$ and $\varphi_0(x,u,t)$ is a function Known state of the system, error and time measurement. The general model of measurements, which are carries out in a system, is describe by the equation differential:

$$\dot{Y} = \phi 1(Y,X,U,t) \tag{24}$$

The result of the measurements represents the random process Y. The filtering problem Is proposed for the state vector of the system X at each instant $t > t_0$, using the results of continuous measurements of the process Y determined by equation (24) in the time interval [t0, t]. Let be a random vector of a process $[Y^T X^T]^T$ determined by differential equations Ito warehouse

$$dY = \phi 1(Y,X,t)dt + \psi 1(Y,X,t)dW, \tag{25}$$
$$dX = \phi(Y,X,t)dt + \psi(Y,X,t)dW,$$

Where Y is a n1-dimensional random process, X is an n-dimensional process, W is a n-dimensional process, $\varphi 1(y,x,t)$ and $\varphi(y,x,t)$ are mapping vector functions the space $\mathbb{R}^{n1} \times \mathbb{R}^{n} \times \mathbb{R}$ in the spaces $\mathbb{R}^{n1}$ and $\mathbb{R}^{n}$ respectively and $\psi 1(y,x,t)$ and $\psi\ (y,x,t)$

are Matrices of known functions that map $\mathbb{R}^{n1} \times \mathbb{R}^{n} \times \mathbb{R}$ in $\mathbb{R}^{n1r}$ and $\mathbb{R}^{nr}$ respectively. This is the approach of the filtering problem for the state vector of the system at some time $t > t_0$ using the results of continuous measurements of the process Y in the time interval $[t_0, t]$.

The general solution to the optimal filtering problem is obtained from the following property for second-order moments: the lowest of all second-order moments of A scalar random variable is its variance. From this it follows that the best approximation of A random variable by a non-random variable using the square mean criterion Is given by its conditional expectation regarding observations [30], [31]. Let Y_{to}^{t} the set of Values of the process measured in the time interval $[t_0, t]$, $Y_{to}^{t} = \{Y(\tau): \tau \in [t_0, t]\}$. Then the optimal estimate of the vector $X_u = X(u)$, which gives the solution of the problem for $u = t$ is determined by the formula

$$\hat{X}_u = E\frac{X_u}{Y_{to}^{t}} \tag{26}$$

This formula determines the optimal estimator of the X_u value for some random function $X(u)$ using the results of the measurements of another random function $Y(t)$ in the interval $[t_0, t]$. It is also valid for the case of a vector with argument t and the measurement of the function $Y(t)$ in some set T of values of t. The application of the formula (26) is necessary to find the conditional distribution of X_u. This can be a problem, which at times is resolve. In the particular case where $Y(t)$ and $X(t)$ are determine by equations (25), this are solves under some constraints additional. The general formula for the stochastic differential of the optimal estimate of a function of the given state vector is the basis of optimum filtering theory. Let $f(Xt, t)$ some scalar function of the n-dimensional state vector of a system and time. Its estimated optimum using the observation results Y_{to}^{t} According to (26) is determine by the formula:

$$\hat{f}(t) = E[\frac{f(Xt,t)}{Y_{to}^{t}}] \tag{27}$$

This estimate represents a functional of the random process $Y(t)$ in the time interval

$[t_0, t]$, and consequently is itself a function of t. A mathematical problem which is helpful is to find the Ito stochastic differential of this random process. This problem is solve under the condition that $W(t)$ in equations (25) represents the Wiener process whose dimension r is not less than n1 which is the dimension of the measurement process $Y(t)$, and that the function φ1 in equations (25) not depend on X. The system (25) takes the form

$$dY = \phi 1(Y,X,t)dt + \psi 1(Y,t)dW, \quad (28)$$
$$dX = \phi(Y,X,t)dt + \psi(Y,X,t)dW,$$

Ito Differential for an Optimal Estimator Function

The stochastic differential equation of the optimum estimator of the random variable $f(Xt,t)$ for equations (25) is given by the formula, see [26], [30], [31]

$$d\hat{f} = E[ft(X,t) + fx(X,t)^T\phi(Y,X,t) \quad (29)$$
$$+ \frac{1}{2}\ tr\frac{\{fxx(X,t)(\psi v\psi\text{^}T\ )(Y,X,t)\}}{Y_{to}^{t}}]dt + E[f(X,t)\{\phi 1(Y,X,t)^T - \hat{\phi}_1^T\}$$
$$+fx(X,t)^T\ (\psi v\psi_1^T\ )(Y,X,t)/Y_{to}^{t}\,]\psi 1v\psi_1^T\ )^{-1}(Y,t)(dY - \hat{\phi}1dt),$$

where

$$(\psi v\psi^T\ )(x,y,t) = \psi(y,x,t)v(t)\psi(y,x,t)^T\ , \quad (30)$$
$$(\psi v\psi_1^T\ )(x,y,t) = \psi(y,x,t)v(t)\psi(y,t)^T\ ,$$
$$\left(\psi v\psi_1^T\ \right)^{-1}(\mathrm{y,t}) = [\psi 1(\mathrm{y,t})\nu(\mathrm{t})\psi 1(y,t)^T\ )]^{-1},$$
$$\phi 1 = \int_{-\infty}^{\infty}\phi 1\mathrm{pt(x)dx} = \mathrm{E}[\phi 1(\mathrm{Xt,Yt},\frac{t}{Y_{to}^{t}}\ ],$$

$Pt(x)$ is the conditional density of Xt relative to Y_{to}^{t} ; The derivatives ft, fx, fxx and all Conditional hopes of the right side exist.

Equation for the Characteristic Function

Substituting in equation (29) $f(x,t) = e^{\mathrm{i}\lambda^{\mathrm{T}}X_t}$ we obtain the stochastic equation for the conditional function characteristic of the random vector Xt:

$$gt(\lambda) = E\left[\frac{e^{\mathrm{i}\lambda^{\mathrm{T}}X_t}}{Y_{to}^{t}}\right] \quad (31)$$

Making substitutions

$$ft = 0, fx = i\lambda e^{\mathrm{i}\lambda^{\mathrm{T}}X_t}\, fxx = -\lambda\lambda^T\ e^{\mathrm{i}\lambda^{\mathrm{T}}X_t} \quad (32)$$
$$tr\{\lambda\lambda^T\ \ (\psi v\psi^T\ )(y,x,t)\} = \lambda^T\ (\psi v\psi^T\ )(y,x,t)\lambda,$$

Of the equation (29) is obtain

$$dgt(\lambda) = E[i\lambda^T\phi(Y,X,t) - \tfrac{1}{2}\ (\psi\nu\psi^T\ )(Y,X,t)\lambda\}e^{i\lambda^T X_t}/Y_{to}^t\ ]dt \qquad (33)$$

$$+E[\{\phi 1(Y,X,t)T\ -\ \hat{\phi}_1^T\ +\ i\lambda^T\ (\psi\nu\psi_1^T\ )(Y,X,t)\}$$

$$\times\, e^{i\lambda^T X_t}/Y_{to}^t\ ](\psi 1\nu\psi_1^T\ )^{-1}(Y,t)(dY\ -\ \hat{\phi}1dt).$$

The right side represents a function of λ. The conditional distribution of the random vector X is completely and solely determined by its characteristic function. Solving the equation (33) it is possible to evaluate the optimal estimate $\widehat{Xt}$ of the determined state vector Xt by the formula (26). By means of these formulas it is possible to obtain the expression for the hope in terms of the characteristic function [31].

$$\hat{X}t\ =\ E\left[\frac{Xt}{Y_{to}^t}\right] = \left[\frac{\partial gt(\lambda)}{\partial\lambda}\right], \qquad \lambda = 0 \qquad (34)$$

Equation for Conditional

Density the stockastic equation for the conditional density $pt(x)$ of the random vector Xt is derive below

$$dpt(x)\ =\ -\frac{\partial T}{\partial x}[\phi(Y,x,t)pt(x)]dt \qquad (35)$$

$$+\frac{1}{2}\ tr\ \frac{\partial}{\partial x}\ \frac{\partial^T}{\partial x}\ [(\psi\nu\psi^T\ )(Y,X,t)pt(x)]\}\ (\psi 1\nu\psi_1^T\ )^{-1}(Y,t)(dY\ -\ \hat{\phi}1dt)$$

or

$$dpt(x)\ =\ L*pt(x)dt\ +\ \{[\psi 1(Y,x,t)T\ -\ \phi_1^T]pt(x) \qquad (36)$$

$$-\frac{\partial^T}{\partial x}[(\psi\nu\psi^T\ )\ )pt(x)]\}(\psi 1\nu\psi_1^T\ )^{-1}(Y,t)(dY\ -\ \hat{\phi}1dt)$$

Where L * is the operator

$$L = \ \phi(Y,x,t)^T\frac{\partial}{\partial x} + \frac{1}{2}tr[(\psi\nu\psi^T\ )(Y,x,t)\frac{\partial}{\partial x}\ \frac{\partial^T}{\partial x} \qquad (37)$$

Looking at the last equation of (30), we conclude that the equation (35) represents an integral-differential equation relative to the conditional density $pt(x)$. As the moment Initial to the function $pt_0(x)$ serves as the initial condition for the equation (35). After

t_0 solve the equation (35), can be found in accordance with formula (2.76) Estimated 'optimal $\hat{X}t$ of the state vector Xt of the system

$$\hat{X}t = E\left[\frac{Xt}{Y_{to}^{t}}\right] = \int_{-\infty}^{\infty} \text{xpt(x)dx} \tag{38}$$

As the formula (25) determines the stochastic differential of Ito of the random process $\hat{f}(t)$, equations (33) and (35) represent stochastic equations of Ito. The equation (35) are originally obtain in another form and under rigid constraints in Stratonovich. It's referred to as the Stratonovich stochastic equation. The same time, the equation for pt in the form of Ito is obtain in (Kushner1964, 1967) also under stricter restrictions. Hence it is usually called the equation of Stratonovich-Krushner, see [32].

Stochastic Differential of Mathematical Hope

The formula (26) determined the optimal estimate as the conditional expectation of $\hat{X}$ of the corresponding random variable X. The optimal estimate obtained as a result of measurements is characterized by the conditional covariance matrix R. These formulas can be obtained from the general formula (29). As the formula (29) determines the Differential of a scalar function of the state of the system, it is necessary to apply it For each element of the $\hat{X}$ and R matrices separately. Substituting in (29) $f(X,t) = Xl$, $ft = 0$, $fx = [0, \ldots, 1, \ldots]\, T, fxx = 0$, and formula (29) takes the form

$$d\hat{X}l = \hat{\phi}_1^T dt + E[Xl(\phi_l^T - \hat{\phi}_l^T) \tag{39}$$

$$+(\psi v \psi_l^T)l/Y_{to}^t](\psi l v \psi_l^T)^{-1}(dY - \hat{\phi}ldt)(l = 1, \ldots, n)$$

Where, according to the latter equation of (29) $\hat{\varphi}l = E\,[\varphi l\,(Y, X, t)\,/\,Y_{to}^t], (\psi v \psi_l^T)\,l$, where The lera column of the matrix $\psi v \psi^T$ L and the arguments of the functions $\varphi l, \psi v \psi_l^T$ and $\psi l v \psi_l^T)^{-1}$ Are omitted for brevity. Then the matrix for the stochastic differential of the estimated The optimal $\hat{X}$ of the state vector of system X is given by

$$d\hat{X} = \hat{\phi}dt + E[X\{(\phi l(Y, X, t)^T - \phi_l^T)\} \tag{40}$$

$$+(\psi v \psi_l^T)(Y, X, t)/Y_{to}^t](\psi l v \psi_l^T)^T(Y, t)(dY - \hat{\phi}ldt)$$

Stochastic Differential of Conditional Moment of Second Order

Substituting in (29) $f(X,t) = X_k X_l$ with $k < 1, ft = 0, fx = [0, \ldots Xl \ldots Xk \ldots, 0]^T$,

$$f_{xx} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ . & .. & . & .. & . & .. & . \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ . & .. & . & .. & . & .. & . \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ . & .. & . & .. & . & .. & . \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Since the two columns and central rows contain some, the corresponding to k, and l respectively, of the formula (29) has

$$d\Gamma kl = E[Xk\phi l + Xl\phi k + (\psi\nu\psi^T)kl/Y_{to}^t]dt \tag{41}$$
$$+E[XkXl(\phi_l^T - \hat{\phi}_l^T) + Xk(\psi\nu\psi_l^T)l$$
$$+Xl(\psi\nu\psi_l^T)k/Y_{to}^t](\psi l\nu\psi_l^T)^{-1}(dY - \hat{\phi}1dt)(k,l = 1,\ldots,n),$$

Where $d\Gamma kl = E[Xk\phi l \,/\, Y_{to}^t]$, and $(\psi\nu\psi^T)kl$ is the corresponding element of the matrix $(\Psi\nu\psi^2)$. Re-writing the formula (41) as:

$$dT_{kl} = E\left[\frac{X_{k\partial l} + X_{l\partial k} + (\theta\vartheta\theta^T)_{kl}}{Y_{t0}^t}\right]dt \tag{42}$$
$$+\sum_{P=1}^{r} E\left[\frac{X_k X_{lpa} + X_k b_{lk} + X_l b_{kp}}{Y_{t0}^t}\right](dY_P - \partial_{1p}dt)$$

Where a_p is the p-th Matrix element $(\phi_1^T - \hat{\phi}_1^T)(\psi_1 \nu\psi_1^T)^{-1}$ *and* b_{kp} is the element of the k-th row and the ρ-th column of the matrix $\psi\nu\psi_1^T\ (\psi 1\nu\psi_1^T)^{-1}$.

Denoting by B the ρ-'most column of the matrix $\psi v \psi_1^T\ (\psi 1 v \psi_1^T)^{-1}, b_p = [b_{p1}, \ldots, b_{pp}]^{-T}\ (\rho = 1, \ldots, r)$, if Obtains the following stochastic differential formula of the conditional moment of second Order Γ of the vector system status:

$$dT = E\left[(\emptyset v \emptyset^T)(Y, X, t) + \varphi(Y, X, t)X^T + \frac{(\emptyset v \emptyset^T)(Y,X,t)}{Y_{t0}^t}\right] dt + \sum_{p=1}^{r} E\left[XX^T ap(Y, X, t) + Xb_p(Y, X, t)^T + \frac{b_p X^T}{Y_{t0}^t}\right](dY_P - \partial_{1p} dt) \tag{43}$$

Covariance Matrix Differential

To find the stochastic differential of the conditional covariance matrix R of the state vector of the system will be used to the known formula that relates the hope, the second-order moment, and the covariance matrix of the random vector $R = \Gamma - \hat{X}\hat{X}^T$, or in the scalar form $R_{Kl} = \Gamma_{Kl} - \hat{X}_k\ \hat{X}_l$. Deriving on both sides of the latter form, gets the expression $dR_{Kl} = d\Gamma_{Kl} - d\ (\hat{X}_k\ \hat{X}_l)$. To find $d(\hat{X}_k\ \hat{X}_l)$ the formula is used

$$d(Z_1 Z_2) = Z_1 dZ_2 + Z_2 dZ_1 + Y_1 v Y_2^t dt \tag{44}$$

$Z(t) = [Z1, Z2]$ is a Ito process which is given by

$$dZ(t) = x(t)dt + Y(t)dW(t) \tag{45}$$

Here $t_0 > 0, W(t)$ is a Wiener process, where Y_1 and Y_2 represent the first and second columns of the matrix $Y = [Y_1, Y_2]$ respectively. $X(t), Y_1(t), Y_2(t)$ are functions random conditions that satisfy the conditions of existence. Z_1, Z_2 are the components of the random vector $Z(t)$. according to (40)

$$E\left[X_k(\varphi_1^T - \partial^\wedge{}_1^T) + \frac{(\partial v \partial^T)_k}{Y_{t0}^t}\right](\partial v_1 \partial_1^t)^{-1}\, \partial_1, \tag{46}$$

$$E\left[X_l(\varphi_1^T - \partial^\wedge{}_1^T) + \frac{(\partial v \partial^T)_l}{Y_{t0}^t}\right](\partial v_1 \partial_1^t)^{-1}\, \partial_1$$

Play the role of the lines Y_1, Y_2 of the matrix, in this case we get to

$$d(X_k X_l) = X_k dX_l + dX_k X_l \tag{47}$$

$$E\left[X_k\left(\varphi_1^T - \varphi\ {}_1^T\right) + \frac{(\partial v\partial_1^T)_k}{Y_{t0}^t}\right] (\partial_1 v\partial_1^T)^{-1}\partial_1 v\partial_1^T(\partial_1 v\partial_1^T)^{-1}. E\left[X_l(\varphi_1^T - \varphi^\wedge{}_1^T) + \frac{(\partial v\partial_1^T)_l}{Y_{t0}^t}\right] dt$$

Substituting here the expressions for dX_K and $dX_l$of equation (39), we have

$$d(X_kX_l) = \{X_{kp1} + X_{lpk} + E\left[X_k(\varphi_1^T - \varphi^{\wedge T}{}_1) + \frac{(\partial v\partial_1^T)_k}{Y_{t0}^t}\right](\partial_1 v\partial_1^T)^{-1}E\left[X_l(\varphi_1^T - \varphi^\wedge{}_1^T) + \frac{(\partial v\partial_1^T)_l}{Y_{t0}^t}\right]dt + E\left[(X_k{}^\wedge X_l + X_l{}^\wedge X_k)(\varphi_1^T - \varphi^{\wedge T}{}_1) + X_k(\partial v\partial_1^T)_l + \frac{X_l(\partial v\partial_1^T)_k}{Y_{t0}^t}\right](\partial_1 v\partial_1^T)^{-1}(dY - {}^\wedge\varphi l dt). \quad (48)$$

By subtracting this formula from (42) and adding the term

$$E\left[\frac{X_kX_l(\varphi_1^T - \varphi^{\wedge T}{}_1)}{Y_{t0}^t}\right] = X_kX_l(\varphi^{\wedge T}{}_1 - \varphi^{\wedge T}{}_1) = 0 \quad (49)$$

We obtain

$$dR_{kl} = \left\{E\left[({}^\wedge X_k - X_k)_{\varphi l} + ({}^\wedge X_k - X_k)_{\varphi k} + \frac{(\partial v\partial_1^T)_{kl}}{Y_{t0}^t}\right] - E\left[X^\wedge{}_k(\varphi_1^T - \varphi^{\wedge T}{}_1) + \frac{(\partial v\partial_1^T)_k}{Y_{t0}^t}\right](\partial_1 v\partial_1^T)^{-1}E\left[(\varphi_1 - \varphi^\wedge{}_1) + \frac{(\partial v\partial_1^T)_l}{Y_{t0}^t}\right]\right\}dt + E\left[({}^\wedge X_k - X_k)({}^\wedge X_l - X_l)(\varphi_1^T - \varphi^\wedge{}_1^T) + \frac{(X_l - {}^\wedge X_l)(\partial v\partial_1^T)_k}{Y_{t0}^t}\right](\partial_1 v\partial_1^T)^{-1}.(dY - {}^\wedge\varphi l dt)(k, l = 1, ..n) \quad (50)$$

By making some transformations in the previous formula (50), we obtain the formula of the stochastic differential matrix for the covariance matrix as the solution of

$$dR = \left\{E\left[(X - {}^X)_\varphi(Y, X, t)^T + \varphi(Y, X, t)\left(X^T - \hat{X}^T\right) - E\left[X\{(\varphi(Y, X, t)^T - {}^\wedge\varphi_1^T) + \frac{(\partial v\partial_1^T)(Y, {}^X, t)}{Y_{t0}^t}\right]\right]\right\}dt + \sum_{p=1}^r E\left[({}^X - X({}^\wedge X^T - X^T)ap(Y, {}^X, t) + ({}^X - Xbp(Y, {}^X, t)^T + \frac{(X - {}^X)^T}{Y_{t0}^t}\right]\left(dY_p - \varphi_{lp}dt\right). \quad (51)$$

So far, the problem and solution approach are establish for the case of a system represented by linear state equations, and linear observations, both with the presence of disturbances that behave like Gaussian white noises, which was developed by Kalman-Bucy.

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