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Common Coupled Fixed Point Theorems for w-Compatible Mappings in Partial Metric Spaces

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Abstract

In this paper, we prove some coupled coincidence point theorems for mappings satisfying some contractive conditions on complete partial metric spaces. Moreover, if the mappings F and g are w-compatible, we obtain a unique common coupled fixed point of the form (gx, gy). Our results unify, extend and generalize the results of Aydi [7] and Sabetghadam [28].

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1. Introduction

The notion of coupled fixed point for a partially ordered set X was introduced by Bhaskar and Lakshmikantham [9]. Several other authors such as Ciric and Lakshmikantham [12], Sabetghadam *et al.* [28] and Olaleru *et al.* [22] have proved some coupled fixed point theorems in metric spaces. In 1992, the notion of partial metric space (PMS) was introduced by Matthews [18] as part of the study of denotational semantics of data flow networks. It is well known that partial metric spaces play an important role in constructing models in the theory of computation (see, e.g. [14], [22], [24]). The PMS is a generalization of the usual metric spaces in which $d(x, x)$ need not be zero.

Several famous mathematicians have contributed to the development of this research fields. Masiha *et al.* [17] proved some fixed point results for weakly contractive type mappings in ordered partial metric spaces. They applied their results to nonlinear fractional boundary value problem. Altun *et al.* [5] established some fixed point theorems for generalized contractive type mappings on partial metric spaces. They also proved a homotopy result. Aydi *et al.* [8] introduced the concept of a partial Hausdorff metric and they initiated the study of fixed point theory for multi-valued mappings on partial metric spaces using the partial Hausdorff metric and proved an analogous to the well-known Nadler's fixed point theorem. Abbas *et al.* [2] proved a Suzuki type fixed point theorem for a generalized multi-valued mapping on a partial Hausdorff metric space. As a consequence of their results, they discussed the existence and uniqueness of the bounded solution of a functional equation arising in dynamic programming. Altun and Acar [6] introduced the notion of (δ, L)-weak contraction and (φ, L)-weak contraction in the sense of Berinde in partial metric spaces. They proved some fixed point theorems in partial metric spaces using these new concepts. Ćirić *et al.* [13] established some common fixed point theorems for four mappings satisfying a generalized nonlinear contraction type condition on partial metric spaces. Chi *et al.* [10] proved some fixed point theorems for generally contractive mappings in complete partial metric spaces. Their results generalized the results of Ilić *et al.* [15]. Chi *et al.* [11] proved a fixed point theorem for a pair of generalized weakly contractive mappings in complete partial metric spaces. Romaguera [27] established two fixed point theorems in a complete partial metric space, which generalizes in several directions the celebrated Boyd and Wong fixed point theorem and Matkowski fixed point theorem.

Recently, Shahzad and Valero [30] established a Nemytskii-Edelstein type fixed point theorem for self-mappings in partial metric spaces in such a way that the classical one can be retrieved as a particular case of their new result. They provided examples to show that the assumed hypothesis in their result cannot be weakened. Several other interesting results in this research area abound in literature (see, e.g. [1], [3],[4] [5], [18], [19],[20], [23], [24], [25], [26], [27], [14], [29]). Aydi [7] proved some coupled fixed point results on PMS. Our new results is a unification, an extension and a generalization of [7] and [28].

In the sequel, we give some definitions of some applicable concepts.

Definite 1.1. [7] An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.2. [7] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(p1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$,

(p2) $p(x, y) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, we call that the ordered pair (X, p) is a partial metric space.

Remark 1.3. Observe that if $p(x, y) = 0$, then by (p1), (p2) and (p3) we have $x = y$. But the converse is not true.

If p is a partial metric on a nonempty set X, then the function $p^s : X \times X \to \mathbb{R}_+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, is a metric on X.

Example 1.4. [30] Consider the function $p_{\max} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ given by $p_{\max}(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. It is not hard to see that $(\mathbb{R}^+, p_{\max})$ is a partial metric space. Moreover, it is clear that $p^s_{\max}(x, y) = |y - x|$ and it implies that $(\mathbb{R}^+, p^s_{\max})$ is a complete metric space.

Example 1.5. [18] If $X := \{[a, b] |\ a, b \in \mathbb{R}, a \leq b\}$, then $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, b\}$ defines a partial metric p on X.

Definition 1.6. [7] Let (X, p) be a partial metric space. Then

(i) a sequence $\{x_n\}$ in (X, p) is said to be convergent to a point $x \in X$ if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$;

(ii) a sequence $\{x_n\}$ in (X, P) is said to be Cauchy if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$;

(iii) a space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is, $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.7. [7] Let (X, p) be a partial metric space. Then

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(b) a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete, furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n,\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 1.8. ([1],[16]) An element $(x, y) \in X \times X$ is called

(g1) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called a coupled point of coincidence,

(g2) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Definition 1.9. [1] The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$, whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Aydi [7] proved the following coupled fixed point results on a partial metric space.

Theorem 1.10. [7] Let (X, p) be a complete partial metric space. Suppose that the mapping $F : X \times X \to X$ satisfies one of the following contractive conditions (i),(ii),(iii):

(i) for all $x, y, u, v \in X$ and nonnegative constants k, l with $k + l < 1$,

$$p(F(x, y), F(u, v)) \leq kp(x, u) + lp(y, v).$$

(ii) for all $x, y, u, v \in X$ and nonnegative constants k, l with $k + l < 1$,

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), x) + lp(F(u, v), u).$$

(iii) for all $x, y, u, v \in X$ and nonnegative constants k, l with $k + 2l < 1$,

$$p(F(x, y), F(u, v)) \leq kp(F(x, y), u) + lp(F(u, v), x).$$

Then, F has a unique coupled fixed point.

2. Main Results

Now we introduce and prove the main theorems.

Theorem 2.1. Let (X, p) be a complete partial metric space. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,

$$\begin{aligned} p(F(x, y), F(u, v)) \leq\ & a_1 p(gx, gu) + a_2 p(gy, gv) + a_3 p(F(x, y), gx) \\ & + a_4 p(F(u, v), gu) + a_5 p(F(x, y), gu) \\ & + a_6 p(F(u, v), gx) \end{aligned} \tag{2.1}$$

where $a_1, a_2, \cdots, a_6$ are nonnegative constants with $a_1 + a_2 + a_3 + a_4 + a_5 + 2a_6 < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then F and g have a coupled coincidence point in X. Moreover, if F and g are w-compatible, then F and g have unique common coupled fixed point.

Proof. Since $F(X \times X) \subseteq g(X)$, for $x_0, y_0 \in X$, we can define $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Continuing this process we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$. Then by (2.1), we obtain

$$\begin{aligned} p(x_n, x_{n+1}) &= p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq a_1 p(gx_{n-1}, gx_n) + a_2 p(gy_{n-1}, gy_n) + a_3 p(F(x_{n-1}, y_{n-1}), gx_{n-1}) \\ &\quad + a_4 p(F(x_n, y_n), gx_n) + a_5 p(F(x_{n-1}, y_{n-1}), gx_n) \\ &\quad + a_6 p(F(x_n, y_n), gx_{n-1}) \\ &= a_1 p(gx_{n-1}, gx_n) + a_2 p(gy_{n-1}, gy_n) + a_3 p(gx_n, gx_{n-1}) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_5 p(gx_n, gx_n) + a_6 p(gx_{n+1}, gx_{n-1}) \\ &\leq a_1 p(gx_{n-1}, gx_n) + a_2 p(gy_{n-1}, gy_n) + a_3 p(gx_n, gx_{n-1}) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_5 p(gx_n, gx_{n+1}) + a_6 p(gx_{n+1}, gx_{n-1}) \\ &\leq a_1 p(gx_{n-1}, gx_n) + a_2 p(gy_{n-1}, gy_n) + a_3 p(gx_n, gx_{n-1}) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_5 p(gx_n, gx_{n+1}) \\ &\quad + a_6 p(gx_{n+1}, gx_n) + a_6 p(gx_n, gx_{n-1}). \end{aligned} \tag{2.2}$$

Similarly, we have

$$\begin{aligned} p(gy_n, gy_{n+1}) &\leq a_1 p(gy_{n-1}, gy_n) + a_2 p(gx_{n-1}, gx_n) + a_3 p(gy_n, gy_{n-1}) \\ &\quad + a_4 p(gy_{n+1}, gy_n) + a_5 p(gy_n, gy_{n+1}) + a_6 p(gy_{n+1}, y_n) \\ &\quad + a_6 p(gy_n, gy_{n-1}). \end{aligned} \tag{2.3}$$

Set

$$\begin{aligned} d_n &= p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) \\ &\leq a_1 p(gx_{n-1}, gx_n) + a_2 p(gy_{n-1}, gy_n) + a_3 p(gx_n, gx_{n-1}) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_5 p(gx_n, gx_{n+1}) + a_6 p(gx_{n+1}, gx_n) \\ &\quad + a_6 p(gx_n, gx_{n-1}) + a_1 p(gy_{n-1}, gy_n) + a_2 p(gx_{n-1}, gx_n) \\ &\quad + a_3 p(gy_n, gy_{n-1}) + a_4 p(gy_{n+1}, gy_n) + a_5 p(gy_n, gy_{n+1}) \\ &\quad + a_6 p(gy_{n+1}, gy_n) + a_6 p(gy_n, gy_{n-1}) \\ &= (a_1 + a_2 + a_3 + a_6) p(gx_n, gx_{n-1}) \\ &\quad + (a_1 + a_2 + a_3 + a_6) p(gy_n, gy_{n-1}) \\ &\quad + (a_4 + a_5 + a_6) p(gx_n, gx_{n+1}) + (a_4 + a_5 + a_6) p(gy_n, gy_{n+1}). \end{aligned} \tag{2.4}$$

Hence we have

$$\begin{aligned} &(1 - a_4 - a_5 - a_6)[p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1})] \\ &\leq (a_1 + a_2 + a_3 + a_6) p(gx_n, gx_{n-1}) + (a_1 + a_2 + a_3 + a_6) p(gy_n, gy_{n-1}). \end{aligned} \tag{2.5}$$

This implies that

$$p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1}) \leq \frac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_5 - a_6}\big[p(gx_n, gx_{n-1}) + p(gy_n, gy_{n-1})\big]. \tag{2.6}$$

That is, $d_n \leq \lambda d_{n-1}$, where $\lambda = \dfrac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_5 - a_6} < 1$. Consequently, for all $n \in N$, we obtain

$$d_n \leq \lambda d_{n-1} + \lambda^2 d_{n-2} \leq \cdots \leq \lambda^n d_0. \tag{2.7}$$

If $d_0 = 0$, then $p(gx_0, gx_1) + p(gy_0, gy_1) = 0$. Hence, from Remark 1.3, we obtain $gx_0 = gx_1 = F(x_0, y_0)$ and $gy_0 = gy_1 = F(y_0, x_0)$. This means that (gx_0, gy_0) is a coupled coincidence point of F and g. If $d_0 > 0$, then for all $n \geq m$, we obtain,in view of (p4),

$$\begin{aligned} p(gx_n, gx_m) \leq\ & p(gx_n, gx_{n-1}) + p(gx_{n-1}, gx_{n-2}) - p(gx_{n-1}, gx_{n-1}) \\ & + p(gx_{n-2}, gx_{n-3}) + p(gx_{n-3}, gx_{n-4}) - p(gx_{n-3}, gx_{n-3}) \\ & \vdots \\ & + p(gx_{m+2}, gx_{m+1}) + p(gx_{m+1}, gx_m) - p(gx_{m+1}, gx_{m+1}) \\ \leq\ & p(gx_n, gx_{n-1}) + p(gx_{n-1}, gx_{n-2}) + \cdots + p(gx_{m+1}, gx_m). \end{aligned} \tag{2.8}$$

Similarly, we have

$$p(gy_n, gy_m) \leq p(gy_n, gy_{n-1}) + p(gy_{n-1}, gy_{n-2}) + \cdots + p(gy_{m+1}, gy_m). \tag{2.9}$$

Hence, we obtain that

$$\begin{aligned} p(gx_n, gx_m) + p(gy_n, gy_m) &\leq d_{n-1} + d_{n-2} + \cdots + d_m \\ &\leq (\lambda^{n-2} + \lambda^{n-2} + \cdots + \lambda^m) d_0 \\ &\leq \frac{\lambda^m}{1 - \lambda} d_0. \end{aligned} \tag{2.10}$$

Using the definition of p^s, we obtain $p^s(gx, gy) \leq 2p(gx, gy)$. Hence for each $n \geq m$,

$$\begin{aligned} p^s(gx_n, gx_m) + p^s(gy_n, gy_m) &\leq 2p(gx_n, gx_m) + 2p(gy_n, gy_m) \\ &\leq \frac{2\lambda^m}{1 - \lambda} d_0. \end{aligned} \tag{2.11}$$

Since $\lambda = \dfrac{a_1 + a_2 + a_3 + a_6}{1 - a_4 - a_5 - a_6} < 1$, (2.11) implies that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X, p^s). Since (X, p) is complete partial metric space, by Lemma 1.7, the space (X, p^s) is complete, so there exists $u^*, v^* \in X$ such that

$$\lim_{n\to\infty} p^s(gx_n, gu^*) = \lim_{n\to\infty} p^s(gy_n, gv^*) = 0. \tag{2.12}$$

From Lemma 1.7, we obtain

$$p(gu^*, gu^*) = \lim_{n\to\infty} p(gx_n, gu^*) = \lim_{n\to\infty} p(gx_n, gx_n) \tag{2.13}$$

and

$$p(gv^*, gv^*) = \lim_{n\to\infty} p(gy_n, gv^*) = \lim_{n\to\infty} p(gy_n, gy_n). \tag{2.14}$$

Using condition (p2) and (2.7), we have

$$p(gx_n, gx_n) \le p(gx_n, gx_{n+1}) \le d_n \le \lambda^n d_0. \tag{2.15}$$

But $\lambda \in [0, 1)$, by letting $n \to \infty$, we obtain $\lim_{n\to\infty} p(gx_n, gx_n) = 0$. Hence,

$$p(gu^*, gu^*) = \lim_{n\to\infty} p(gx_n, gu^*) = \lim_{n\to\infty} p(gx_n, gx_n) = 0. \tag{2.16}$$

Similarly, we have

$$p(gv^*, gv^*) = \lim_{n\to\infty} p(gy_n, gv^*) = \lim_{n\to\infty} p(gy_n, gy_n) = 0. \tag{2.17}$$

Consequently, by (p4) and (2.1), we have

$$\begin{aligned} p(F(u^*, v^*), gu^*) &\le p(F(u^*, v^*), gx_{n+1}) + p(gx_{n+1}, gu^*) - p(gx_{n+1}, gx_{n+1}) \\ &\le p(F(u^*, v^*), F(x_n, y_n)) + p(gx_{n+1}, gu^*) \\ &\le a_1 p(gu^*, gx_n) + a_2 p(gv^*, gy_n) + a_3 p(F(u^*, v^*), gu^*) \\ &\quad + a_4 p(F(x_n, y_n), gx_n) + a_5 p(F(u^*, v^*), gx_n) \\ &\quad + a_6 p(F(x_n, y_n), gu^*) + p(gx_{n+1}, gu^*) \\ &= a_1 p(gu^*, gx_n) + a_2 p(gv^*, gy_n) + a_3 p(F(u^*, v^*), gu^*) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_5 p(F(u^*, v^*), gx_n) \\ &\quad + a_6 p(gx_{n+1}, gu^*) + p(gx_{n+1}, gu^*) \\ &\le a_1 p(gu^*, gx_n) + a_2 p(gv^*, gy_n) + a_3 p(F(u^*, v^*), gu^*) \\ &\quad + a_4 p(gx_{n+1}, gx_n) + a_4 p(gu^*, gx_n) + a_5 p(F(u^*, v^*), gu^*) \\ &\quad + a_5 p(gu^*, gx_n) + a_6 p(gx_{n+1}, gu^*) + p(gx_{n+1}, gu^*). \end{aligned} \tag{2.18}$$

Now letting $n \to \infty$, and using (2.13) and (2.14), we have

$$\begin{aligned} p(F(u^*, v^*), gu^*) &\le a_3 p(F(u^*, v^*), gu^*) + a_5 p(F(u^*, v^*), gu^*) \\ &= (a_3 + a_5) p(F(u^*, v^*), gu^*). \end{aligned} \tag{2.19}$$

Suppose that $p(F(u^*, v^*), gu^*) \neq 0$. From (2.19), we can conclude that $1 \le (a_3 + a_5)$ which is a contradiction. Hence $p(F(u^*, v^*), gu^*) = 0$, i.e $F(v^*, u^*) = gv^*$. By w-compatibility of F and g, we have

$$g(F(u^*, v^*)) = F(gu^*, gv^*).$$

Hence (gu^*, gv^*) is a common coupled fixed point of F and g.

Next, we prove the uniqueness of the common coupled fixed point of F and g. Let (gu', gv') is an another common coupled fixed point of F and g. Then by using (2.1),

$$
\begin{aligned}
p(gu', gu^*) &= p(F(u', v'), F(u^*, v^*)) \\
&\le a_1 p(gu', gu^*) + a_2 p(gv', gv^*) + a_3 p(F(u', v'), gu') \\
&\quad + a_4 p(F(u^*, v^*), gu^*) + a_5 p(F(u', v'), gu^*) \\
&\quad + a_6 p(F(u^*, v^*), gu') \\
&= a_1 p(gu', gu^*) + a_2 p(gv', gv^*) + a_3 p(gu', gu') \\
&\quad + a_4 p(gu^*, gu^*) + a_5 p(gu', gu^*) + a_6 p(gu^*, gu') \\
&\le a_1 p(gu', gu^*) + a_2 p(gv', gv^*) + a_3 p(gu', gu^*) \\
&\quad + a_4 p(gu', gu^*) + a_5 p(gu', gu^*) + a_6 p(gu^*, gu').
\end{aligned}
\tag{2.20}
$$

Similarly, we have

$$
\begin{aligned}
p(gv', gv^*) \le\ & a_1 p(gv', gv^*) + a_2 p(gu', gu^*) + a_3 p(gv', gv^*) \\
& + a_4 p(gv', gv^*) + a_5 p(gv', gv^*) + a_6 p(gv^*, gv').
\end{aligned}
\tag{2.21}
$$

Hence, we obtain that

$$
\begin{aligned}
& p(gu', gu^*) + p(gv', gv^*) \\
& \le (a_1 + a_2 + a_3 + a_4 + a_5 + a_6)[p(gu', gu^*) + p(gv', gv^*)].
\end{aligned}
\tag{2.22}
$$

Since $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 < 1$, this implies that $p(gu', gu^*) + p(gv', gv^*) = 0$, and so $gu^* = gu'$ and $gv^* = gv'$. Hence, F and g have a unique common coupled fixed point. This completes the proof. ∎

We can get the following corollary directly from Theorem 2.1.

Corollary 2.2. Let (X, p) be a complete partial metric space. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition:

$$
\begin{aligned}
p(F(x, y), F(u, v)) \le \frac{a_1}{6}\big[& p(gx, gu) + p(gy, gv) + p(F(x, y), gx) \\
& + p(F(u, v), gu) + p(F(x, y), gu) + p(F(u, v), gx)\big],
\end{aligned}
\tag{2.23}
$$

for all $x, y, u, v \in X$, where $0 \le a_1 < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then F and g have a coupled coincidence point in X. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point.

Example 2.3. Let $X = [0, +\infty)$ endowed with the usual partial metric ρ defined by $p : X \times X \to [0, +\infty)$ with $p(x, y) = \max\{x, y\}$. The partial metric space (X, p) is complete because (X, p^s) is complete. Indeed, for any $x, y \in X$,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) = 2\max\{x, y\} - (x + y) = |x - y|. \tag{2.24}$$

Thus, (X, p^s) is an Euclidean metric space which is complete. Consider the mappings $F : X \times X \to X$ and $g : X \to X$ defined by

$$g(x) = \begin{cases} 6x, & \text{if } x \in [0, 1), \\ 4x, & \text{if } x \in (1, \infty) \end{cases}$$

and

$$F(x, y) = \begin{cases} \dfrac{x}{6} + \dfrac{y}{6}, & \text{if } x \in [0, 1] \text{ and } y \in \mathbb{R}, \\ \dfrac{x}{8} + \dfrac{y}{8}, & \text{if } x \in (1, \infty) \text{ and } y \in \mathbb{R}. \end{cases}$$

Clearly, F and g satisfies the conditions of Theorem 2.1. If we take $a_1 = a_3 = \dfrac{1}{5}$, $a_2 = a_4 = \dfrac{1}{12}$, $a_5 = a_6 = 0$. We observe that $(0, 0)$ is common coupled coincidence point of F and g.

Remark 2.4. Theorem 2.1 is a unification, an extension and a generalization of Theorem 2.1, Theorem 2.4 and Theorem 2.5 in [7]. If $a_3 = a_4 = a_5 = a_6 = 0$, then we obtain the result of Theorem 2.1. If $a_1 = a_2 = a_5 = a_6 = 0$, then we obtain Theorem 2.4. If $a_1 = a_2 = a_3 = a_4 = 0$, then we obtain Theorem 2.5. Similarly, Corollary 2.2 extends, unifies and generalizes Corollary 2.2, Corollary 2.6 and Corollary 2.7 in [7].

Theorem 2.5. Let (X, p) be a complete partial metric space. Suppose that the mappings $F : X \times X \to X$ and $g : X \to X$ satisfy the following contractive condition:

$$p(F(x, y), F(u, v)) \leq k\ell_{x,y,u,v}, \tag{2.25}$$

for all $x, y, u, v \in X$, where

$$\ell_{x,y,u,v} = \max\Big\{p(gx, gu), p(gy, gv), p(F(x, y), gx), p(F(u, v), gu), \frac{p(F(x, y), gu) + p(F(u, v), gx)}{2}\Big\} \tag{2.26}$$

and $k \in [0, 1)$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then F and g have a coupled coincidence point in X. Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point.

Proof. Since $F(X \times X) \subseteq g(X)$, for $x_0, y_0 \in X$ we can define the function g such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Continuing this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$. From (2.25) and (p2), we obtain that

$$p(gx_n, gx_{n+1}) = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq k\ell_{x,y,u,v}, \tag{2.27}$$

where

$$\begin{aligned}\ell_{x,y,u,v} &= \max\Big\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(F(x_{n-1}, y_{n-1}), gx_{n-1}),\\ &\quad p(F(x_n, y_n), gx_n), \frac{p(F(x_{n-1}, y_{n-1}), gx_n) + p(F(x_n, y_n), gx_{n-1})}{2}\Big\}\\ &= \max\Big\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n-1}), p(gx_{n+1}, gx_n),\\ &\quad \frac{p(gx_n, gx_n) + p(gx_{n+1}, gx_{n-1})}{2}\Big\}\\ &\leq \max\Big\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n-1}), p(gx_{n+1}, gx_n),\\ &\quad \frac{p(gx_n, gx_{n+1}) + p(gx_{n+1}, gx_n) + p(gx_{n+1}, gx_{n-1})}{2}\Big\}\\ &= \max\Big\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n-1}), p(gx_{n+1}, gx_n),\\ &\quad p(gx_{n+1}, gx_n) + \frac{p(gx_{n+1}, gx_{n-1})}{2}\Big\}\\ &\leq \max\Big\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n-1}), p(gx_{n+1}, gx_n),\\ &\quad p(gx_{n+1}, gx_n) + \frac{p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})}{2}\Big\}.\end{aligned} \tag{2.28}$$

Similarly, we have

$$p(gy_n, gy_{n+1}) = p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \leq k\ell_{x,y,u,v}, \tag{2.29}$$

where

$$\begin{aligned}\ell x, y, u, v = \max\Big\{&p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n), p(gy_n, gy_{n-1}), p(gy_{n+1}, gy_n),\\ &\frac{p(gy_n, gy_{n+1}) + p(gy_{n+1}, gy_n) + p(gy_n, gy_{n-1})}{2}\Big\}\end{aligned} \tag{2.30}$$

Set $d_n = p(gx_n, gx_{n+1}) + p(gy_n, gy_{n+1})$.

Case 1. If $\ell_{x,y,u,v} = p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n)$, then we have

$$d_n \leq p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) = d_{n-1} \leq kd_{n-1}.$$

Hence, (2.25) is satisfied.

Case 2. If $\ell_{x,y,u,v} = p(gx_{n+1}, gx_n) + p(gy_{n+1}, gy_n)$, then we have

$$d_n = p(gx_{n+1}, gx_n) + p(gy_{n+1}, gy_n) \le kd_n.$$

Hence, (2.25) is satisfied.

Case 3. If

$$\begin{aligned}\ell_{x,y,u,v} = p(gx_{n+1}, gx_n) + \frac{p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})}{2} \\ + p(gy_{n+1}, gy_n) + \frac{p(gy_{n+1}, gy_n) + p(gy_n, gy_{n-1})}{2},\end{aligned}$$

then we have

$$\begin{aligned}d_n &\le p(gx_{n+1}, gx_n) + \frac{p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})}{2} \\ &\quad + p(gy_{n+1}, gy_n) + \frac{p(gy_{n+1}, gy_n) + p(gy_n, gy_{n-1})}{2} \\ &= \frac{3}{2}p(gx_{n+1}, gx_n) + \frac{p(gx_n, gx_{n-1})}{2} + \frac{3}{2}p(gy_{n+1}, gy_n) + \frac{p(gy_n, gy_{n-1})}{2}.\end{aligned} \tag{2.31}$$

Hence, from (2.31) we obtain $0 \le \frac{p(gx_n, gx_{n-1})}{2} + \frac{p(gy_n, gy_{n-1})}{2}$, this implies that

$$0 \le p(gx_n, gx_{n-1}) + p(gy_n, gy_{n-1}) = d_{n-1}.$$

Hence (2.25) is satisfied in all cases. Consequently, for all $n \in N$, we obtain that

$$d_n \le kd_{n-1} + k^2 d_{n-2} \le \cdots \le k^n d_0. \tag{2.32}$$

If $d_0 = 0$, then $p(gx_0, gx_1) + p(gy_0, gy_1) = 0$. Hence, from Remark 1.3, we obtain $g(x_0) = g(x_1) = F(x_0, y_0)$ and $g(y_0) = g(y_1) = F(y_0, x_0)$, it means that (gx_0, gy_0) is a common coupled coincidence point of F and g. If $d_0 > 0$, for all $n \ge m$, we obtain, in view of (p4)

$$\begin{aligned}p(gx_n, gx_m) &\le p(gx_n, gx_{n-1}) + p(gx_{n-1}, gx_{n-2}) - p(gx_{n-1}, gx_{n-1}) \\ &\quad + p(gx_{n-2}, gx_{n-3}) + p(gx_{n-3}, gx_{n-4}) - p(gx_{n-3}, gx_{n-3}) \\ &\quad + \cdots + p(gx_{m+2}, gx_{m+1}) + p(gx_{m+1}, gx_m) \\ &\quad - p(gx_{m+1}, gx_{m+1}) \\ &\le p(gx_n, gx_{n-1}) + p(gx_{n-1}, gx_{n-2}) \\ &\quad + \cdots + p(gx_{m+1}, gx_m).\end{aligned} \tag{2.33}$$

Similarly, we obtain:

$$\begin{aligned}p(gy_n, gy_m) &\le p(gy_n, gy_{n-1}) + p(gy_{n-1}, gy_{n-2}) \\ &\quad + \cdots + p(gy_{m+1}, gy_m).\end{aligned} \tag{2.34}$$

Hence, we have

$$\begin{aligned} p(gx_n, gx_m) + p(gy_n, gy_m) &\le d_{n-1} + d_{n-2} + \cdots + d_m \\ &\le (k^{n-1} + k^{n-2} + \cdots + k^m) d_0 \\ &\le \frac{k^m}{1-k} d_0. \end{aligned} \tag{2.35}$$

By the definition of p^s, we obtain $p^s(gx, gy) \le 2p(gx, gy)$, and so, for each $n \ge m$

$$p^s(gx_n, gx_m) + p^s(gy_n, gy_m) \le 2p(gx_n, gx_m) + 2p(gy_n, gy_m) \le 2\frac{k^m}{1-k} d_0. \tag{2.36}$$

Since $k \in [0, 1)$, (2.36) implies that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (X, p^s). From Lemma 1.7, the metric space (X, p^s) is complete, there exist $u^*, v^* \in X$ such that

$$\lim_{n\to\infty} p^s(gx_n, gu^*) = \lim_{n\to\infty} p^s(gy_n, gv^*) = 0, \tag{2.37}$$

Hence, from Lemma 1.7, we have

$$p(gu^*, gu^*) = \lim_{n\to\infty} p(gx_n, gu^*) = \lim_{n\to\infty} p(gx_n, gx_n) \tag{2.38}$$

and

$$p(gv^*, gv^*) = \lim_{n\to\infty} p(gy_n, gv^*) = \lim_{n\to\infty} p(gy_n, gy_n). \tag{2.39}$$

Using condition (p2) and (2.32), we obtain that

$$p(gx_n, gx_n) \le p(gx_n, gx_{n+1}) \le d_n \le \lambda^n d_0. \tag{2.40}$$

But $\lambda \in [0, 1)$, by letting $n \to \infty$, we obtain $\lim_{n\to\infty} p(x_n, x_n) = 0$. Hence,

$$p(gu^*, gu^*) = \lim_{n\to\infty} p(gx_n, gu^*) = \lim_{n\to\infty} p(gx_n, gx_n) = 0. \tag{2.41}$$

Similarly, we obtain

$$p(gv^*, gv^*) = \lim_{n\to\infty} p(gy_n, gv^*) = \lim_{n\to\infty} p(gy_n, gy_n) = 0. \tag{2.42}$$

Hence, by using (2.25) and (p4), we obtain:

$$\begin{aligned} p(F(u^*, v^*), gu^*) &\le p(F(u^*, v^*), gx_{n+1}) + p(gx_{n+1}, gu^*) - p(gx_{n+1}, gx_{n+1}) \\ &\le p(F(u^*, v^*), F(x_n, y_n)) + p(gx_{n+1}, gu^*) \\ &\le k\ell_{x_n, y_n, u^*, v^*} + p(gx_{n+1}, gu^*), \end{aligned} \tag{2.43}$$

where

$$\ell_{x_n,y_n,u^*,v^*} = \max\Big\{p(gu^*, gx_n), p(gv^*, gy_n), p(F(u^*, v^*), gu^*), p(F(x_n, y_n), gx_n), \frac{p(F(u^*, v^*), gx_n) + p(F(x_n, y_n), gu^*)}{2}\Big\}$$
$$= \max\Big\{p(gu^*, gx_n), p(gv^*, gy_n), p(F(u^*, v^*), gu^*), p(gx_{n+1}, gx_n), \frac{p(F(u^*, v^*), gx_n) + p(gx_{n+1}, gu^*)}{2}\Big\}. \tag{2.44}$$

We now consider the following cases:

Case 1^0. If $\ell_{x_n,y_n,u^*,v^*} = p(gu^*, gx_n)$, then from (2.43) we obtain

$$p(F(u^*, v^*), gu^*) \le kp(gu^*, gx_n) + p(gx_{n+1}, gu^*). \tag{2.45}$$

By letting $n \to \infty$ and using (2.41) and (2.42), we obtain $p(F(u^*, v^*), gu^*) = 0$.

Case 2^0. If $\ell_{x_n,y_n,u^*,v^*} = p(gv^*, gy_n)$, then from (2.43), we obtain:

$$p(F(u^*, v^*), gu^*) \le kp(gv^*, gy_n) + p(gx_{n+1}, gu^*). \tag{2.46}$$

By letting $n \to \infty$ and using (2.41) and (2.42), we obtain $p(F(u^*, v^*), gu^*) = 0$.

Case 3^0. If $\ell_{x_n,y_n,u^*,v^*} = p(F(u^*, v^*), gu^*)$, then from (2.43), we obtain

$$p(F(u^*, v^*), gu^*) \le kp(F(u^*, v^*), gu^*) + p(gx_{n+1}, gu^*). \tag{2.47}$$

By letting $n \to \infty$ and using (2.41), we have

$$p(F(u^*, v^*), gu^*) \le kp(F(u^*, v^*), gu^*). \tag{2.48}$$

From (2.48), we can obtain a contradiction if we assume that $p(F(u^*, v^*), gu^*) \neq 0$, this implies that $1 \le k$ which ultimately gives a contradiction. Hence we have

$$p(F(u^*, v^*), gu^*) = 0.$$

Case 4^0. If $\ell_{x_n,y_n,u^*,v^*} = p(gx_{n+1}, gx_n)$, then from (2.43), we obtain

$$\begin{aligned} p(F(u^*, v^*), gu^*) &\le kp(gx_{n+1}, gx_n) + p(gx_{n+1}, gu^*) \\ &\le kp(gx_{n+1}, gu^*) + kp(gu^*, gx_n) + p(gx_{n+1}, gu^*). \end{aligned} \tag{2.49}$$

By letting $n \to \infty$ and using (2.41) and (2.42), we obtain $p(F(u^*, v^*), gu^*) = 0$.

Case 5^0. If $\ell_{x_n,y_n,u^*,v^*} = \dfrac{p(F(u^*, v^*), gx_n) + p(gx_{n+1}, gu^*)}{2}$, then from (2.43), we obtain

$$\begin{aligned}
p(F(u^*, v^*), gu^*) &\leq k\left[\frac{p(F(u^*, v^*), gx_n) + p(gx_{n+1}, gu^*)}{2}\right] + p(gx_{n+1}, gu^*) \\
&\leq k\left(\frac{p(F(u^*, v^*), gu^*)}{2}\right) + k\left(\frac{p(gu^*, gx_n)}{2}\right) \\
&\quad + k\left(\frac{p(gx_{n+1}, gu^*)}{2}\right) + p(gx_{n+1}, gu^*).
\end{aligned} \tag{2.50}$$

By letting $n \to \infty$ and using (2.41) and (2.42), we obtain

$$p(F(u^*, v^*), gu^*) \leq k\left(\frac{p(F(u^*, v^*), gu^*)}{2}\right). \tag{2.51}$$

Thus, by Case 3^0, we have $p(F(u^*, v^*), gu^*) = 0$ since $k < 1$ implies that $\dfrac{k}{2} < 1$. Hence in all cases, we have

$$p(F(u^*, v^*), gu^*) = 0,$$

that is $F(u^*, v^*) = gu^*$.

Similarly, we obtain $F(v^*, u^*) = gv^*$. By w-compatibility of F and g, we obtain

$$g(F(u^*, v^*)) = F(gu^*, gv^*).$$

This means that (gu^*, gv^*) is a common coupled fixed point of F and g.

Next, we prove that the common coupled fixed point of F and g is unique. Suppose that (gu', gv') is another common coupled fixed point of F and g, then in view of (2.25), we have

$$\begin{aligned}
p(gu', gu^*) &= p(F(u', v'), F(u^*, v^*)) \\
&\leq k\ell_{u',u^*},
\end{aligned} \tag{2.52}$$

where

$$\begin{aligned}
\ell_{u',u^*} = \max\Big\{&p(gu', gu^*), p(gv', gv^*), p(F(u', v'), gu'), p(F(u^*, v^*), gu^*), \\
&\frac{p(F(u', v'), gu^*) + p(F(u^*, v^*), gu')}{2}\Big\}.
\end{aligned} \tag{2.53}$$

Similarly, we have

$$\begin{aligned}
p(gv', gv^*) &= p(F(v', u'), F(v^*, u^*)) \\
&\leq k\ell_{v',v^*},
\end{aligned} \tag{2.54}$$

where

$$\ell_{v',v^*} = \max\Big\{p(gv', gv^*), p(gu', gu^*), p(F(v', u'), gv'), p(F(v^*, u^*), gv^*), \frac{p(F(v', u'), gv^*) + p(F(v^*, u^*), gv')}{2}\Big\}. \tag{2.55}$$

Combining (2.52) and (2.54), we obtain $p(gu', gu^*) + p(gv', gv^*)$.

We now consider the following cases:

Case 1[1]. If $\ell_{u',u^*} = p(gu', gu^*)$, $\ell_{v',v^*} = p(gv', gv^*)$, then from (2.52) and (2.54), we obtain

$$p(gu', gu^*) + p(gv', gv^*) \leq k[p(gu', gu^*) + p(gv', gv^*)]. \tag{2.56}$$

Since $k \in [0, 1)$, (2.56) implies that $p(gu', gu^*) + p(gv', gv^*) = 0$. Hence, $gu^* = gu'$ and $gv^* = gv'$.

Case 2[1]. If $\ell_{u',u^*} = p(gv', gv^*)$, $\ell_{v',v^*} = p(gu', gu^*)$, then from (2.52) and (2.54), we obtain

$$p(gu', gu^*) + p(gv', gv^*) \leq k[p(gv', gv^*) + p(gu', gu^*)]. \tag{2.57}$$

Since $k \in [0, 1)$, (2.57) implies that $p(gu', gu^*) + p(gv', gv^*) = 0$. Hence, $gu^* = gu'$ and $gv^* = gv'$.

Case 3[1]. If
$\ell_{u',u^*} = p(F(u', v'), gu')$, $\ell_{v',v^*} = p(F(v', u'), gv')$, then we have

$$\begin{aligned} p(gu', gu^*) + p(gv', gv^*) &\leq k[p(F(u', v'), gu') + p(F(v', u'), gv')] \\ &= k[p(gu', gu') + p(gv', gv')] \\ &\leq k[p(gu', gu^*) + p(gv', gv^*)]. \end{aligned} \tag{2.58}$$

Since $k \in [0, 1)$, (2.58) implies that $p(gu', gu^*) + p(gv', gv^*) = 0$. Hence, $gu^* = gu'$ and $gv^* = gv'$.

Case 4[1]. If $\ell_{u',u^*} = p(F(u^*, v^*), gu^*)$, $\ell_{v',v^*} = p(F(v^*, u^*), gv^*)$, then we have

$$\begin{aligned} p(gu', gu^*) + p(gv', gv^*) &\leq k[p(F(u^*, v^*), gu^*) + p(F(v^*, u^*), gv^*)] \\ &= k[p(gu^*, gu^*) + p(gv^*, gv^*)] \\ &\leq k[p(gu', gu^*) + p(gv', gv^*)]. \end{aligned} \tag{2.59}$$

Since $k \in [0, 1)$, (2.59) implies that $p(gu', gu^*) + p(gv', gv^*) = 0$. Hence, $gu^* = gu'$ and $gv^* = gv'$.

Case 5[1]. If $\ell_{u',u^*} = \dfrac{p(F(u',v'),gu^*) + p(F(u^*,v^*),gu')}{2}$,
$\ell_{v',v^*} = \dfrac{p(F(v',u'),gv^*) + p(F(v^*,u^*),gv')}{2}$, then we obtain

$$\begin{aligned}
p(gu',gu^*) + p(gv',gv^*) &\le k\Bigg[\frac{p(F(u',v'),gu^*) + p(F(u^*,v^*),gu')}{2} \\
&\quad + \frac{p(F(v',u'),gv^*) + p(F(v^*,u^*),gv')}{2}\Bigg] \\
&= k\Bigg[\frac{p(gu',gu^*) + p(gu^*,gu')}{2} + \frac{p(gv',gv^*) + p(gv^*,gv')}{2}\Bigg] \\
&= k[p(gu',gu^*) + p(gv',gv^*)]
\end{aligned} \tag{2.60}$$

Since $k \in [0,1)$, (2.60) implies that $p(gu',gu^*) + p(gv',gv^*) = 0$. Hence, $gu^* = gu'$ and $gv^* = gv'$. Hence, in all cases, we have established that $gu^* = gu'$ and $gv^* = gv'$. Hence (gu^*, gv^*) is a unique common coupled fixed point of F and g. This completes the proof. ■

Remark 2.6. Theorem 2.1 is a unification, an extension and a generalization of the results of [7] and [28].

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