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Some characterisations of rectifying curves in four dimensional Galilean space $\mathbb{G}^4$

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Abstract

In this paper, we define rectifying curves in Galilean $4-$spaces. A rectifying curve is a curve whose position vector lies in the orthogonal complement of normal vector. We obtain some characterisations of these curves. In particular, we prove that there are no rectifying curves in $\mathbb{G}^4$ with nonzero constant curvatures.

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1. Introduction

The notion of rectifying curves was given by B. Y. Chen in [3]. In Euclidean 3-spaces, he defined a rectifying curve as a curve whose position vector always lies in its rectifying plane spanned by the tangent and the binormal vector fields of the curve. Thus, the position vector α of the rectifying curve in $\mathbb{E}^3$ satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

for some differentiable functions $\lambda(s)$ and $\mu(s)$ in arclength parameter $s \in I \subset \mathbb{R}$. Rectifying curves in Euclidean-3 are studied in [3, 4] and have some interesting properties, e.g., the ratio of its torsion and curvature vector is non-constant linear function of s. In [4], it is shown that there is a simple relationship between the rectifying curves and the centrodes, which is of greater importance in mechanics, kinematics and also in differential geometry while defining the curves of constant precession. In Mikowski 3-space $\mathbb{E}^3_1$, rectifying curves are studied in [5]. In $\mathbb{E}^3_1$, the rectifying curves have the

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similar geometric properties as in Euclidean 3-spaces. In [8], it is proved that rectifying curves are geodesics on a cone. Therefore, the authors of that paper called rectifying curves conical geodesics. Then, in [6], authors generalised the concept of rectifying curves to Euclidean 4-space and obtained some characterisations theorems there. The analogous form of [6] in Minkowski space-time is obtained in [9, 1], where the authors in [9] obtained explicit equations of null, psuedo null and partially null rectifying curves in $\mathbb{E}_1^4$.

Extending the study of rectifying curves to other ambient spaces the authors in [9] and [10] studied the rectifying curves in three dimensional sphere and psuedo-Galilean spaces, respectively. Recently the authors in [12] generalised the definition of rectifying curves to extended rectifying curves and obtained the relations between nonhelical rectifying curves and their Darboux vectors. Meanwhile, the authors in [2] generalised the definition of rectifying curves to n-dimensional Euclidean spaces.

2. Preliminaries

The geometry of Galilean Relativity acts like a connector from Euclidean geometry to special Relativity. Galilean space is the space of Galilean Relativity. The Galilean space is a 3D complex projective space P_3 in which the absolute figure $\{\omega, f_1, I_1, I_2\}$ consists of a real plane ω(the absolute plane), a real line $f \subset \omega$(the absolute line) and two complex points $I_1, I_2 \in f$.

The group of motions in $\mathbb{G}^3$ is a six parameter group given in affine coordinates by

$$\begin{aligned}\bar{x} &= (\cos\alpha)x + (\sin\alpha)y + (\nu\cos\beta)t + a,\\ \bar{y} &= -(\sin\alpha)x + (\cos\alpha)y + (\nu\sin\beta)t + b,\\ \bar{t} &= t + d.\end{aligned}$$

Yaglom [11] stressed that four dimensional Galilean Geometry, which studies all properties invariant under motions of objects in space. Yaglom also stated this geometry could be described more precisely as the study of those properties of four dimensional space with coordinates that are invariant under the general Galilean transformations.

$$\begin{aligned}\bar{x} &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)x + (\sin\beta\cos\alpha - \cos\gamma\cos\beta\sin\alpha)y + (\sin\gamma\sin\alpha)z\\ &\quad +(\nu\cos\delta_1)t + a,\\ \bar{y} &= -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)x + (-\sin\beta\sin\alpha + \cos\gamma\cos\beta\cos\alpha)y + (\sin\gamma\cos\alpha)z\\ &\quad +(\nu\cos\delta_2)t + b,\\ \bar{z} &= (\sin\gamma\sin\beta)x - (\sin\gamma\cos\beta)y + (\cos\gamma)z + (\nu\cos\delta_3)t + c,\\ \bar{t} &= t + d,\end{aligned}$$

with $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$. In affine coordinates, the inner product of two vectors $p = (p_1, p_2, p_3, p_4)$ and $q = (q_1, q_2, q_3, q_4)$ is defined by

$$\langle p, q\rangle = \begin{cases} p_1q_1, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0,\\ p_2q_2 + p_3q_3 + p_4q_4, & \text{if } a_1 = 0 \text{ and } b_1 = 0.\end{cases}$$

For vectors $p = (p_1, p_2, p_3, p_4)$, $q = (q_1, q_2, q_3, q_4)$ and $r = (r_1, r_2, r_3, r_4)$, the Galilean cross product in $\mathbb{G}^4$ is defined as

$$p \times q \times r = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{vmatrix},$$

where e_i are the standard basis vectors.
Let $\alpha : I \subset \mathbb{R} \to \mathbb{G}^4, \alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arc length s in $\mathbb{G}^4$. For α, the Frenet formulas are

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.1}$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying

$$\begin{cases} \langle T, T \rangle = \langle N, N \rangle = \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = 1, \\ \langle T, N \rangle = \langle T, B_1 \rangle = \langle T, B_2 \rangle = \langle N, B_1 \rangle = \langle N, B_2 \rangle = \langle B_1, B_2 \rangle = 0. \end{cases} \tag{2.2}$$

We recall some important results of rectifying curves.

Theorem 2.1. [3] Let $\mathbf{x} : I \to \mathbb{E}^3$ be a curve $\mathbb{E}^3$ with $\kappa > 0$. Then $\mathbf{x}$ is a rectifying curve if and only if, up to parameterization, it is given by

$$\mathbf{x}(t) = a \sec t \mathbf{y}(t),$$

where a is a positive number and $\mathbf{y} = \mathbf{y}(t)$ is a unit speed curve in S^2.

Theorem 2.2. [6] Let $\alpha(s)$ be a unit speed curve in $\mathbb{E}^4$, with non-zero curvatures $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$. Then $\alpha(s)$ is congruent to a rectifying curve if and only if

$$\frac{\kappa_1(s)\kappa_3(s)(s+c)}{\kappa_2(s)} + \left(\frac{\kappa_1(s)\kappa_2(s) + (s+c)\kappa_1'(s)\kappa_2(s) - \kappa_1(s)\kappa_2'(s)}{\kappa_2^2(s)\kappa_3(s)} \right)' = 0, \quad c \in \mathbb{R}.$$

Theorem 2.3. [6] There exists no rectifying curve in $\mathbb{E}^4$ with nonzero constant curvatures.

Theorem 2.4. [4] The centrode of a unit speed curve in $\mathbb{E}^3$ with nonzero constant curvature κ and nonconstant torsion τ is a rectifying curve.

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with nonzero constant curvatures and nonconstant torsion.

Theorem 2.5. **[2]** Let α be an arc length parameterised curve in $\mathbb{E}^n$ with nonzero curvatures. Then α is congruent to a rectifying curve if and only if

$$\kappa_{n-1}(s)\sum_{k=0}^{n-4}\mu_{n-3,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{\kappa_1(s)}{\kappa_s(s)}\right)+\sum_{k=0}^{n-3}\left(\mu_{n-2,k}(s)\frac{\partial^k}{\partial s^k}\left(\frac{\kappa_1(s)}{\kappa_2(s)}\right)\right)'=0,$$

where

$$\mu_{1,0}(s)=s+c,\quad c\in\mathbb{R},\quad \mu_{2,0}(s)=\frac{1}{\kappa_3(s)},\quad \mu_{2,1}(s)=\frac{s+c}{\kappa_3(s)},$$

and for $i=\{3,4,\cdots,n-2\}$,

$$\mu_{i,0}(s)=\frac{\kappa_i(s)\mu_{i-2,0}(s)+\mu'_{i-1,0}(s)}{\kappa_{i+1}(s)},$$

$$\mu_{i,k}(s)=\frac{\kappa_i(s)\mu_{i-2,k}(s)+\mu'_{i-1,k}(s)+\mu_{i-1,k-1}(s)}{\kappa_{i+1}(s)},\quad k\in\{1,2,\cdots,i-3\},$$

$$\mu_{i,i-2}(s)=\frac{\mu_{i-1,i-3}(s)+\mu'_{i-1,i-2}(s)}{\kappa_{i+1}(s)},$$

$$\mu_{i,i-1}(s)=\frac{\mu_{i-1,i-2}(s)}{\kappa_{i+1}(s)}.$$

3. Rectifying curves in $\mathbb{G}^4$

Definition 3.1. Let α be an admissible curve in $\mathbb{G}^4$. We say that α is a rectifying curve if the position vector of α always lies in the orthogonal complement of N, i.e., in

$$N^{\perp}=\{W\in\mathbb{G}^4;\langle N,W\rangle\}=0. \tag{3.3}$$

Therefore, the position vector of α satisfies the equation

$$\alpha(s)=\lambda(s)T(s)+\mu(s)B_1(s)+\nu(s)B_2(s), \tag{3.4}$$

where $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are arbitrary differentiable functions in arc length parameter s.

Theorem 3.2. Let $\alpha(s)$ be a unit speed curve in $\mathbb{G}^4$ with κ, τ and σ non-vanishing, then $\alpha(s)$ is congruent to a rectifying curve if and only if

$$\frac{\kappa(s)(s+c)}{\tau}=A_1\cosh\int_0^s\sigma(s)ds+A_2\sinh\int_0^s\sigma(s)ds,\quad \text{where } A_1,A_2\in\mathbb{R}.$$

Proof. Differentiating (3.4), we get

$$T=\lambda'T+\lambda T'+\mu'B_1+\mu B_1'+\nu'B_2+\nu B_2'. \tag{3.5}$$

Using the Frenet equations (2.1), we obtain

$$\begin{cases} \lambda' = 1, \\ \lambda\kappa - \mu\tau = 0, \\ \mu' - \nu\sigma = 0, \\ \mu\sigma + \nu' = 0. \end{cases} \tag{3.6}$$

From above, we obtain

$$\begin{cases} \lambda(s) = s + c, \\ \mu(s) = \dfrac{(s+c)\kappa(s)}{\tau(s)}, \\ \nu(s) = \dfrac{\kappa(s)\tau(s) + (s+c)(\kappa'(s)\tau(s) - \kappa(s)\tau'(s))}{\tau^2(s)\sigma(s)}, \end{cases} \tag{3.7}$$

where $c \in \mathbb{R}$. Using the last equation of (3.6) and (3.7), we obtain

$$\frac{\kappa(s)\sigma(s)(s+c)}{\tau} + \left(\frac{\kappa\tau + (s+c)(\kappa'(s)\tau(s) - \kappa(s)\tau'(s))}{\tau^2(s)\sigma(s)}\right)' = 0, \quad c \in \mathbb{R}. \tag{3.8}$$

(3.8) can be written as

$$\frac{\kappa(s)(s+c)}{\tau(s)} - \frac{1}{\sigma}\left[\frac{1}{\sigma}\left(\frac{\kappa(s)(s+c)}{\tau}\right)'\right]' = 0. \tag{3.9}$$

Using change of variable method such that

$$t = \int_0^s \sigma(s)ds.$$

Then (3.9) is equivalent to

$$\frac{\kappa(s)(s+c)}{\tau(s)} - \left(\frac{\kappa(s)(s+c)}{\tau(s)}\right)'' = 0. \tag{3.10}$$

From (3.10), we derive

$$\frac{\kappa(s)(s+c)}{\tau(s)} = A_1 \cosh \int_0^s \sigma(s)ds + A_2 \sinh \int_0^s \sigma(s)ds. \tag{3.11}$$

Conversely, assume that $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ of an arbitrary unit speed curve satisfy (3.8). Let us consider $X \in \mathbb{G}^4$ be given by

$$\begin{aligned} X(s) \;&=\; \alpha(s) - (s+c)T(s) - \left(A_1 \cosh \int_0^s \sigma(s)ds + A_2 \sinh \int_0^s \sigma(s)ds\right) B_1(s) \\ &\quad + \left(A_1 \sinh \int_0^s \sigma(s)ds + A_2 \cosh \int_0^s \sigma(s)ds\right) B_2(s). \end{aligned}$$

Using (2.1) and (3.11), we get $X' = 0$. This implies that α is congruent to a rectifying curve. ∎

Remark 3.3.

1. From (3.8), we see that there are no rectifying curves lying in $\mathbb{G}^4$ with κ, τ and σ all non-zero constants.

2. From (3.7) and (3.11), we have

$$\begin{cases} \lambda(s) = s + c, \\ \mu(s) = A_1 \cosh \int_0^s \sigma(s)ds + A_2 \sinh \int_0^s \sigma(s)ds, \\ \nu(s) = -\left(A_1 \sinh \int_0^s \sigma(s)ds + A_2 \cosh \int_0^s \sigma(s)ds\right). \end{cases} \tag{3.12}$$

Theorem 3.4. Let α be a unit speed curve in $\mathbb{G}^4$, then α is a rectifying curve if

1. $\kappa(s)$, $\tau(s)$ are constants and $\sigma(s) = \pm\frac{1}{\sqrt{-s^2 - 2c_1 s + c_2}}$, $c_1, c_2 \in \mathbb{R}$ and $-s^2 - 2c_1 s + c_2 > 0$.

2. $\tau(s)$ and $\sigma(s)$ be constants and $\kappa(s) = \frac{1}{(s+c)}(A_1 \sin(s\sigma) + A_2 \cos(s\sigma))$, where $A_1, A_2 \in \mathbb{R}$.

3. $\kappa(s)$ and $\sigma(s)$ be constants and $\tau(s) = \frac{(s+c)}{A_1 e^{-s\sigma} + A_2 e^{s\sigma}}$, where $A_1, A_2 \in \mathbb{R}$ not both zero.

Proof. Suppose $\kappa(s) =$ constant, $\tau(s) =$constant and $\sigma(s)$ is a non-constant function, then from (3.9), we obtain

$$\sigma'(s) - \sigma^3(s)(s + c) = 0, \quad c \in \mathbb{R}.$$

This implies that

$$\sigma(s) = \pm\frac{1}{\sqrt{-s^2 - 2c_1 s + c_2}}, \quad c_1, c_2 \in \mathbb{R} \text{ and } -s^2 - 2c_1 s + c_2 > 0.$$

Now, suppose $\tau(s)$ and $\sigma(s)$ be constants and $\kappa(s)$ be a non-constant function, then from (3.9), we derive

$$(s + c)\kappa(s)\sigma^2 + (\kappa(s)(s + c))'' = 0.$$

Solving the above differential equation we obtain

$$\kappa(s) = \frac{1}{(s+c)}(A_1 \sin(s\sigma) + A_2 \cos(s\sigma)), \text{ where} A_1, A_2 \in \mathbb{R}.$$

Now, let us suppose that $\kappa(s) = \kappa$ and $\sigma(s) = \sigma$ be constants and $\tau(s)$ be a non-constant function, then from (3.9), we have

$$\frac{\kappa(s+c)}{\tau(s)} - \frac{1}{\sigma}\left[\frac{1}{\sigma}\left(\frac{\kappa(s+c)}{\tau(s)}\right)'\right]' = 0.$$

Set $\frac{\kappa(s+c)}{\tau(s)} = Z$, from above equation, we get

$$Z'' - \sigma^2 Z = 0. \tag{3.13}$$

Solving (3.13), we get

$$\tau(s) = \frac{(s+c)}{A_1 e^{-s\sigma} + A_2 e^{s\sigma}}, \quad \text{where } A_1 \text{ and } A_2 \text{ are constants not both zero.}$$

■

Theorem 3.5. Let $\alpha(s)$ be a unit speed curve in Galilean 4-spaces with κ, τ and σ being nonzero. Then the following statements hold.

1. The tangential function $\rho(s) = \|\alpha(s)\|$ satisfies $\rho^2(s) = s^2 + c_1 s + c_2$, $c_1 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$.
2. The tangential component of α is given by $\langle \alpha(s), T(s) \rangle = s + c$, $c \in \mathbb{R}$.
3. The normal component $\alpha^N(s)$ of the position vector of α has constant length and the distance function $\rho(s)$ is non-constant.
4. The first and second binormal components α are given by

$$\begin{aligned} \langle \alpha(s), B_1(s) \rangle &= A_1 \cosh \int_0^s \sigma(s) ds + A_2 \sinh \int_0^s \sigma(s) ds, \\ \langle \alpha(s), B_2(s) \rangle &= A_1 \sinh \int_0^s \sigma(s) ds + A_2 \cosh \int_0^s \sigma(s) ds, \end{aligned} \tag{3.14}$$

respectively.

Proof. Multiplying the third equation of (3.6) by μ and the last equation of (3.6) by ν and on adding, we get

$$\mu^2(s) + \nu^2(s) = t^2, \quad \text{for some constant t} \in \mathbb{R}_0^+. \tag{3.15}$$

Using (3.15) and (3.4), we obtain $\rho^2(s) = \langle \alpha(s), \alpha(s) \rangle = s^2 + c_1 s + c_2$, $c_1 \in \mathbb{R}_0$ and $c_2 \in \mathbb{R}$, which proves the statement (i).
Using (3.4) and (3.7), we get $\langle \alpha(s), T \rangle = s + c$, $c \in \mathbb{R}$. This proves the statement (ii).
We can write (3.4) as $\alpha(s) = \gamma(s)T(s) + \alpha^N(s)$, where $\gamma(s)$ is arbitrary differentiable function and α^N is the normal component of α. From (3.4), we see that $\alpha^N = \mu(s)B_1(s) + \nu(s)B_2(s)$, this implies that $\langle \alpha^N(s), \alpha^N(s) \rangle = \mu^2(s) + \nu^2(s)$. Using (3.15), we find $\|\alpha^N\| = t$, $t \in \mathbb{R}_0^+$. Further using statement (i), we get statement (iii). Using (3.4) and (3.12), we get the results of statement (iv).
Conversely, suppose statement (i) holds i.e., $\langle \alpha(s), \alpha(s) \rangle = s^2 + c_1 s + c_2$. Differentiate the previous equation two times, we get $\langle \alpha(s), N(s) \rangle = 0$, which proves that α

is a rectifying curve. Secondly, suppose statement (2) holds. Differentiating the equation in statement (2), we get $\langle\alpha(s), N(s)\rangle = 0$. If the statement (3) holds, writing $\alpha(s) = \gamma(s)T(s) + \alpha^N(s)$, where $\gamma(s)$ is arbitrary differentiable function. Thus

$$\langle\alpha^N(s), \alpha^N(s)\rangle = \langle\alpha(s), \alpha(s)\rangle + \gamma^2(s) - 2\langle\alpha(s), T(s)\rangle.$$

From (3.4), $\langle\alpha(s), T(s)\rangle = \gamma(s)$, it follows that

$$\langle\alpha^N(s), \alpha^N(s)\rangle = \langle\alpha(s), \alpha(s)\rangle - \langle\alpha(s), T(s)\rangle^2,$$

where $\langle\alpha(s), \alpha(s)\rangle = \rho^2(s) \neq 0$. Differentiating the previous equation with respect to s and using (2.1), we can easily find that $\langle\alpha(s), N(s)\rangle = 0$. This implies that α is a rectifying curve.
If the statement (4) holds, taking derivative of first equation of (3.12), we obtain

$$-\tau(s)\langle\alpha(s), N(s)\rangle + \sigma(s)\langle\alpha(s), B_2(a)\rangle = A_1 \sinh \int_0^s \sigma(s)ds + A_2 \cosh \int_0^s \sigma(s)ds.$$

Using second equation of (3.12), the above equation reduces to $\langle\alpha(s), N(s)\rangle = 0$, this implies that α is a rectifying curve. ■

4. Conclusion

Galilean spaces are of greater importance while looking at their importance in physics and other fields. We derived some necessary and sufficient conditions for a curves to be a rectifying curve in Galilean 4-space. The generalisation of rectifying curves in Galilean n-spaces can be a problem of discussion. Also, the classification of rectifying curves in other ambient spaces like isotropic, semi-isotropic, Lie groups can be problems of great interest.

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