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Existence and stability of neutral–type BAM neural networks with time delay in the neutral and leakage terms on time scales[1]

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Abstract

By employing the theory of exponential dichotomy on time scales, Banach fixed point theorem and the method of Lyapunov functional, new sufficient conditions are established to ensure the existence and exponential stability of almost periodic solutions for neutral–type BAM neural networks. Unlike previously considered networks, the addressed system involves time delays in both the neutral derivative as well as the leakage terms. For numerical treatment, we construct a concrete example to illustrate the effectiveness of the proposed results. Our theorems are essentially new and extend existing results in the literature.

AMS subject classification:
Keywords: Neutral type BAM neural networks, Almost periodic solutions, Time scales, Leakage delays, Exponential stability.

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1. Introduction

In [1], Kosko proposed Bidirectional Associative Memories (BAM) as faithful models of neurodynamics. As such, they are able to develop attractors that correspond to desired associations. This allows them to be noise tolerant and able to perform pattern completion. Over the next years, many researchers have found BAM undoubtedly important in performing multi–step patterns recognition, learning real–valued correlated patterns, having increased storage capacities and extracting features from given inputs [2–8]. Due to the finite switching speed of neurons and amplifiers, it is well known that time delay has major impact that is unavoidable in the hardware implementation of neural networks. Indeed, the time delay often tends to cause instability or oscillation in the process describing neural networks. In practical applications, there exist many types of time delays such as discrete delays, time–varying delays, distributed delays and leakage delays (or forgetting delays). The incorporation of delays in BAM networks have been reported by many authors who studied the dynamics behavior of their solutions by employing several methods and techniques such as the fixed point theorems, theory of set–valued maps, functional differential inclusions, the matrix theory, the Lyapunov functional and certain mathematical inequalities [9–19].

The leakage delay, in particular, which exists in the negative feedback term of BAM, has recently emerged as a research topic of great significance. In [20], Gopalsamy was the first who investigated the stability of the BAM neural networks with constant leakage delays. Later on, a substantial achievements have been reached regarding the dynamics of BAM with delay in the leakage term [21–25]. On the other hand, it is natural that the BAM contains certain information about the derivative of the past state. This is expressed via the encompass of delay in the neutral derivative of BAM which often appears in the study of automatic control, population dynamics and vibrating masses attached to an elastic bar; the reader may consult the papers [26–31] for more details.

The theory of time scales, which has recently received a lot of considerable attention, was introduced by Stefan Hilger in his Ph.D. dissertation in 1988. The two main objectives of this theory are the unification and extension. Indeed, it allows simultaneous treatment for both differential and difference equations and extends those equations to the so called dynamic equations [32]. This theory has been recently assimilated into BAM networks [34–43]. To the best of our observation, however, the progress in this direction is heavily passing and thus it needs additional investigations. Inspired by the above discussion, we consider the problem of existence and stability for BAM neural networks on time scales. Unlike previously proposed and investigated network models, we precisely carry out our investigations for neutral–type BAM neural networks involving delays in the neutral derivative and leakage terms on time scales of the form

$$\begin{cases} x_i^{\nabla}(t) = -a_i(t)x_i(t-\alpha_i(t)) + \sum_{j=1}^{m} b_{ij}(t)g_j\big(y_j(t-\tau_j(t))\big) \\ \qquad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(t)p_j\big(y_j^{\nabla}(t-\tau_j(t))\big)q_l\big(y_l(t-\tau_l(t))\big) + I_i(t),\ i=1,2,\ldots,n, \\ y_j^{\nabla}(t) = -d_j(t)y_j(t-\beta_j(t)) + \sum_{i=1}^{n} c_{ji}(t)f_i\big(x_i(t-\delta_i(t))\big) \\ \qquad + \sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}(t)u_i\big(x_i^{\nabla}(t-\delta_i(t))\big)v_l\big(x_l(t-\delta_l(t))\big) + J_j(t),\ j=1,2,\ldots,m. \end{cases} \tag{1.1}$$

To see the significant differences between system (1.1) and those obtained in [39–41], the reader is recommended to check the concluding remark at the end of this paper. In system (1.1), $x_i(t)$ and $y_j(t)$ denote the potential (or voltage) of the cell at time t where n and m correspond to the number of units in a neural network; $a_i > 0$ and $d_j > 0$ denote respectively the strengths at which the ith and the jth cell reset their potential to the resting state when isolate from the other cells and inputs; g_j, p_j, f_i, u_i are the activation functions; q_l and v_l are transmission functions which are used to regulate the rate of activation functions; $\alpha_i, \beta_j, \tau_j$ and δ_i are nonnegative and correspond to finite speed of axonal signal transmission delays satisfying $t-\alpha_i(t)$, $t-\beta_j(t)$, $t-\tau_j(t)$, $t-\delta_i(t) \in \mathbb{T}$; b_{ij}, c_{ji}, e_{ijl} and s_{jil} are the first and second–order connection weights of the neural network, respectively and I_i and J_j denote the ith and the jth component of an external input.

The main features of this paper that make it distinctive and innovative are stated as follows: The general structure of system (1.1) which allows covering many particular cases, the incorporation of regulator functions in the system which helps measuring the rate of activation functions, the insertion of delays in the neutral derivative and leakage terms which provide better description for the dynamics of neural reactions and the establishment od a unified framework to handle both continuous and discrete BAM cases through the implementation of time scales calculus.

Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = (0, +\infty)$, and $\mathbb{T}$ be an almost periodic time scale. The definitions and properties of $\mathbb{T}$ and almost periodic functions on time scale will be stated in Section 2. Denote $h^+ = \sup_{t\in\mathbb{T}} |h(t)|$ and $h^- = \inf_{t\in\mathbb{T}} |h(t)|$, where $h : \mathbb{T} \to \mathbb{R}$ is an almost periodic function.

Set $X = \Big\{\phi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_n)^T : \varphi_i, \psi_j \in C^1(\mathbb{T}, \mathbb{R}), \varphi_i, \psi_j$ are almost periodic functions on $\mathbb{T}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\Big\}$ with the norm $\|\phi\| = \max\{|\varphi|_1, |\psi|_1\}$, where $|\varphi|_1 = \max\{|\varphi|_0, |\varphi^{\nabla}|_0\}$, $|\psi|_1 = \max\{|\psi|_0, |\psi^{\nabla}|_0\}$, $|\varphi|_0 = \max_{1\le i\le n} \varphi_i^+$, $|\varphi^{\nabla}|_0 = \max_{1\le i\le n} (\varphi_i^{\nabla})^+$, $|\psi|_0 = \max_{1\le i\le n} \psi_i^+$, $|\psi^{\nabla}|_0 = \max_{1\le i\le n} (\psi_i^{\nabla})^+$ and $C^1(\mathbb{T}, \mathbb{R})$ is the set of continuous functions with continuous nabla derivatives on $\mathbb{T}$. Clearly X is a

Banach space.

The initial values associated with system (1.1) are given by

$$x_i(s) = \varphi_i(s),\ y_j(s) = \psi_j(s),\ s \in [-\sigma, 0]_{\mathbb{T}} = \{t :\ t \in [-\sigma, 0] \cap \mathbb{T}\}, \tag{1.2}$$

where $\varphi_i, \psi_j \in C^1([-\sigma, 0]_{\mathbb{T}}, \mathbb{R})$ and $\sigma = \max\Big\{ \max\limits_{1\le i\le n, 1\le j\le m} \{\tau_j^+, \delta_i^+\}\Big\}$. Throughout the remaining part of the paper, the following conditions are satisfied:

H.1 $b_{ij}, e_{ijl}, I_i, c_{ji}, s_{jil}, J_j \in C(\mathbb{T}, \mathbb{R})$ are all almost periodic functions for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $l = 1, 2, \ldots, m$.

H.2 $a_i, \alpha_i, \tau_i, d_j, \beta_j, \delta_i \in C(\mathbb{T}, \mathbb{R}^+)$ are all almost periodic functions for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

H.3 $f_i, g_j \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants $L_j^f, L_i^g, L_j^p, L_j^q, L_i^u, L_i^v, P_j, Q_j, U_i$ and V_i such that

$$|f_j(s_1) - f_j(s_2)| \le L_j^f |s_1 - s_2|,\ \ |g_i(s_1) - g_i(s_2)| \le L_i^g |s_1 - s_2|,$$

$$|p_j(s_1) - p_j(s_2)| \le L_j^p |s_1 - s_2|,\ \ |q_j(s_1) - q_j(s_2)| \le L_j^q |s_1 - s_2|,$$

$$|u_i(s_1) - u_i(s_2)| \le L_i^u |s_1 - s_2|,\ \ |v_i(s_1) - v_i(s_2)| \le L_i^v |s_1 - s_2|,$$

and

$$|p_j(s)| \le P_j,\ |q_j(s)| \le Q_j,\ |u_i(s)| \le U_i,\ |v_i(s)| \le V_i,$$

for all $s_1, s_2, s \in \mathbb{R}$ and $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

It is obvious that condition (H.3) is of Lipschitz type that will be used to ensure the existence and uniqueness od solutions for system (1.1).

Our approach is based on the Banach fixed point theorem and the method of Lyapunov functional to establish new sufficient conditions for the existence and exponential stability of almost periodic solutions for system (1.1). We will employ the theory of exponential dichotomy on time scales and apply certain inequality techniques to prove the main results.

The contents of the paper is outlined as follows. Section 2 is devoted to some basic notations, lemmas and definitions on $\mathbb{T}$ that will be used in the sequel. Section 3 provides the existence and uniqueness of the almost periodic solution of system (1.1). The almost periodic solution of system (1.1) which is exponentially stable is proved in Section 4. In Section 5, the proposed results are verified by demonstrating a numerical example. We finish the paper by a concluding remark.

2. Preliminaries

For the sake of convenience, we recall some basic notations, definitions and lemmas of time scales that are needed to prove the main results.

Let $\mathbb{T}$ be a time scale which is a closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, defined by,

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The point $t \in \mathbb{T}$ is called left–dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left–scattered if $\rho(t) < t$, right–dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right–scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a right–scattered minimum m, set $\mathbb{T}_k := \mathbb{T} \backslash \{m\}$, otherwise $\mathbb{T}_k := \mathbb{T}$. The backwards graininess $\nu : \mathbb{T}_k \to [0, +\infty)$ is defined by $\nu(t) = t - \rho(t)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called left–dense continuous (*ld*-continuous) provided it is continuous at left–dense point in $\mathbb{T}$ and its right–side limits exist at right–dense points in $\mathbb{T}$.

Definition 2.1. [32] Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}_k$. Then define $f^{\nabla}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there exits a neighborhood U of t such that

$$\Big| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \Big| \leq \varepsilon |\rho(t) - s|,$$

for all $s \in U$, f^{∇} is called to be the nabla derivative of f at t.

If f is *ld*-continuous, then there exists a function F such that $F^{\nabla}(t) = f(t)$, and we define

$$\int_a^b f(t) \nabla t = F(b) - F(a).$$

The function p is $\nu-$regressive if $1 - \nu(t)p(t) \neq 0$ for all $t \in \mathbb{T}_k$. Let us denote $R_\nu = \{p : \mathbb{T} \to \mathbb{R} | p$ is *ld*-continous and ν–regressive$\}$ and $R_\nu^+ = R_\nu^+(\mathbb{T} \to \mathbb{R}) = \{p \in R_\nu : 1 - \nu(t)p(t) > 0$, for all $t \in \mathbb{T}\}$.

The nabla exponential function is defined by

$$\hat{e}_p(t, s) = \exp\Big\{ \int_s^t \hat{\xi}_\nu(\tau)(p(\tau)) \nabla \tau \Big\},$$

for $s, t \in \mathbb{T}$, where $p \in R_\nu$, the ν-cylinder transformation is expressed by

$$\hat{\xi}_h(z) = \begin{cases} -\dfrac{\log(1 - hz)}{h}, & h \neq 0, \\ z, \quad h = 0. \end{cases}$$

Definition 2.2. [32,33] If $p, q \in R_\nu$, then a circle plus addition is defined by $(p \oplus_\nu q)(t) := p(t) + q(t) - p(t)q(t)\nu(t)$, for all $t \in \mathbb{T}_k$. For $p \in R_\nu$, we defined a circle minus p by $\ominus_\nu p := -\dfrac{p}{1 - \nu p}$.

Lemma 2.3. [32,33] If $p, q \in R_\nu$, and $s, t, r \in \mathbb{T}$. Then

I. $\hat{e}_0(t,s) \equiv 1$ and $\hat{e}_p(t,t) \equiv 1$;

II. $\hat{e}_p(\rho(t),s) = (1-\nu(t)p(t))\hat{e}_p(t,s)$;

III. $\hat{e}_p(t,s) = 1/\hat{e}_p(s,t)$;

IV. $\hat{e}_p(t,r)\hat{e}_p(r,s) = \hat{e}_p(t,s)$;

V. $(\hat{e}_p(t,s))^\nabla = p(t)\hat{e}_p(t,s)$.

Lemma 2.4. [33] Let f, g be nabla differential functions on $\mathbb{T}$. Then

I. $(v_1 f + v_2 g)^\nabla = v_1 f^\nabla + v_2 g^\nabla$, for any constants v_1, v_2;

II. $(fg)^\nabla = f^\nabla(t)g(t) + f(\rho(t))g^\nabla(t) = f(t)g^\nabla(t) + f^\nabla(t)g(\rho(t))$;

III. If f and f^∇ are continuous, then $\left(\int_a^t f(t,s)\nabla s\right)^\nabla = f(\rho(t),t) + \int_a^t f(t,s)\nabla s$.

Definition 2.5. [33] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\prod := \left\{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\right\} \neq \{0\}.$$

Definition 2.6. [33] Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T},\mathbb{R})$ is called an almost periodic function if the $\varepsilon-$translation set of f

$$E\{\varepsilon, f\} = \left\{\tau \in \prod : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\right\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$\left|f(t+\tau) - f(t)\right| < \varepsilon, \forall t \in \mathbb{T},$$

where τ is called the $\varepsilon-$translation number of and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

Definition 2.7. [32,34] Let $A(t)$ be an $n \times n$ matrix–valued function on $\mathbb{T}$. Then the linear system

$$x^\nabla(t) = A(t)x(t), t \in \mathbb{T}, \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist constants $k_i, \alpha_i, i = 1, 2$, projection P and the fundamental solution matrix $X(t)$ of Eq. (2.1) satisfying

$$\|X(t)PX^{-1}(s)\| \leq k_1\hat{e}_{\ominus_\nu \alpha_1}(t,s), s, t \in \mathbb{T}, t \geq s,$$

$$\|X(t)(I-P)X^{-1}(s)\| \leq k_2\hat{e}_{\ominus_\nu \alpha_2}(t,s), s, t \in \mathbb{T}, t \leq s.$$

Lemma 2.8. [34] If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system

$$x^{\nabla}(t) = A(t)x(t) + g(t), \quad t \in \mathbb{T}, \tag{2.2}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\rho(s))g(s)\nabla s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}\rho(s)g(s)\nabla s,$$

where $X(t)$ is the fundamental solution matrix of Eq. (2.1).

Lemma 2.9. [33, 34] Assume $c_i(t) \in R_{\nu}^{+} (i = 1, 2, \ldots, n)$ are almost periodic on $\mathbb{T}$, and $\min\limits_{1\le i\le n}\{\inf\limits_{t\in\mathbb{T}} c_i(t)\} = \bar{m} > 0$. Then the linear system

$$x^{\nabla}(t) = diag\Big(-c_1(t), -c_2(t), \ldots, -c_n(t)\Big)x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

Definition 2.10. [33, 34] Let $z^* = (x_1^*, x_2^*, \ldots, x_n^*, y_1^*, y_2^*, \ldots, y_m^*)^T$ be a continuously differentiable almost periodic solution of system (1.1) with initial value $\phi^*(s) = (\varphi_1^*(s), \varphi_2^*(s), \ldots, \varphi_n^*(s), \psi_1^*(s), \psi_2^*(s), \ldots, \psi_m^*(s))^T$. If there exists a constant $\lambda > 0$ such that for every solution $z(t) = (x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t))^T$ of system (1.1) with initial value $\phi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s), \psi_1(s), \psi_2(s), \ldots, \psi_m(s))^T$ satisfying

$$\begin{aligned}
&x_i(t) - x_i^*(t) = O(\hat{e}_{\ominus_\nu \lambda}(t, 0)),\\
&y_j(t) - y_j^*(t) = O(\hat{e}_{\ominus_\nu \lambda}(t, 0)),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.
\end{aligned}$$

Then, the solution $z^*(t)$ is said to be exponential stable.

3. The main results

This section is devoted to the main results of this paper in which we will establish sufficient conditions for the existence and exponential stability of almost periodic solutions of system (1.1). As per expected, the impenetrable structure of system (1.1) will lead to quite lengthy computations. The analysis will be presented in two separate divisions.

3.1. Existence of almost periodic solutions

Define $\phi^0(t) = \big(\varphi_1^0(t), \varphi_2^0(t), \ldots, \varphi_n^0(t), \psi_1^0(t), \psi_2^0(t), \ldots, \psi_m^0(t)\big)^T$, where

$$\varphi_i^0(t) = \int_{-\infty}^{t} \hat{e}_{-a_i}(t, \rho(s))I_i(s)\nabla s, i = 1, 2, \ldots, n,$$

and

$$\psi_j^0(t) = \int_{-\infty}^{t} \hat{e}_{-b_j}(t, \rho(s)) J_j(s) \nabla s,\ j = 1, 2, \ldots, m.$$

Let L be a constant satisfying

$$\max \left\{ \begin{array}{l} \|\phi^0\|,\ \max\limits_{1\le j\le m} |g_j(0)|,\ \max\limits_{1\le j\le m} |f_i(0)|,\ \max\limits_{1\le j\le m} |p_j(0)|, \\ \max\limits_{1\le j\le m} |q_j(0),\ \max\limits_{1\le j\le m} |u_i(0)||,\ \max\limits_{1\le j\le m} |v_i(0)| \end{array} \right\} \le L.$$

and assume the following conditions

H.4 There exists a constant $\rho > 0$ such that $\chi_k(t) \ge \rho, t \in \mathbb{R}, k = 1, 2, \ldots, n+m$, where

$$\chi_k(t) = \begin{cases} a_i(t),\ k = i, i = 1, 2, \ldots, n, \\ d_j(t),\ k = n + j, j = 1, 2, \ldots, m. \end{cases}$$

H.5 For $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$,

$$\max \left\{ \max_{1\le i\le n} \left\{ \Theta_i, \left(1 + \frac{a_i^+}{a_i^-}\right) \Theta_i,\ \max_{1\le j\le m} \left\{ \Pi_j, \left(1 + \frac{d_i^+}{d_i^-}\right) \Pi_j \right\} \right\} \right\} \le 1,$$

and

$$\max \left\{ \max_{1\le i\le n} \{\kappa_{i1}, \kappa_{i2}\},\ \max_{1\le j\le m} \{\epsilon_{j1}, \epsilon_{j2}\} \right\} < 1,$$

where

$$\Theta_i = 2a_i^+\alpha_i^+ + 2\sum_{j=1}^{m} b_{ij}^+ L_j^g + \sum_{j=1}^{m} b_{ij}^+ + L\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ (2L_j^p + 1)(2L_l^q + 1),$$

$$\Pi_j = 2d_j^+\beta_j^+ + 2\sum_{i=1}^{n} c_{ji}^+ L_i^f + \sum_{i=1}^{n} c_{ji}^+ + L\sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}^+ (2L_i^u + 1)(2L_l^v + 1),$$

$$\kappa_{i1} = \frac{1}{a_i^-}\left[a_i^+\alpha_i^+ + \sum_{j=1}^{m} b_{ij}^+ L_j^g + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ (Q_l L_j^p + P_j L_l^q) \right],$$

$$\kappa_{i2} = \left(1 + \frac{a_i^+}{\alpha_i^-}\right)\left[a_i^+\alpha_i^+ + \sum_{j=1}^{m} b_{ij}^+ L_j^g + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ (Q_l L_j^p + P_j L_l^q) \right],$$

$$\epsilon_{j1} = \frac{1}{d_j^-}\left[d_j^+\beta_j^+ + \sum_{j=1}^{m} c_{ji}^+ L_i^f + \sum_{j=1}^{m}\sum_{l=1}^{m} s_{jil}^+ (V_l L_i^u + U_i L_l^v) \right]$$

and

$$\epsilon_{j2} = \left(1 + \frac{d_j^+}{d_j^-}\right)\left[d_j^+\beta_j^+ + \sum_{i=1}^{n} c_{ji}^+ L_i^f + \sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}^+ (V_l L_i^u + U_i L_l^v) \right].$$

Clearly, condition (H.5) is used to dominate certain quantities in the inequalities in the proofs of the main results.

Theorem 3.1. Suppose that the assumptions H.1–H.5 are satisfied. Then there exists a unique almost periodic solution of system (1.1) in $X_0 = \{\phi \in X : \|\phi - \phi^0\| \le L\}$, where

$$\phi(t) = \Big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \psi_1(t), \psi_2(t), \ldots, \psi_m(t)\Big)^T.$$

Proof. First, we rewrite the following form of system (1.1)

$$\begin{cases} x_i^{\nabla}(t) = -a_i(t)x_i(t) + a_i(t)\displaystyle\int_{t-\alpha(t)}^{t} x_i^{\nabla}(s)\nabla s + \sum_{j=1}^{m} b_{ij}(t)g_j(y_j(t-\tau_j(t))) \\ \qquad + \displaystyle\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(t)p_j(y_j^{\nabla}(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_i(t),\ i = 1, 2, \ldots, n, \\ y_j^{\nabla}(t) = -d_j(t)y_j(t) + d_j(t)\displaystyle\int_{t-\alpha(t)}^{t} y_j^{\nabla}(s)\nabla s + \sum_{i=1}^{n} c_{ji}(t)f_i(x_i(t-\delta_i(t))) \\ \qquad + \displaystyle\sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}(t)u_i(x_i^{\nabla}(t-\delta_i(t)))v_l(x_l(t-\delta_l(t))) + J_j(t),\ j = 1, 2, \ldots, m, \end{cases} \tag{3.3}$$

For any given $\phi \in X$, we consider the following auxiliary almost periodic system on time scale:

$$\begin{cases} x_i^{\nabla}(t) = -a_i(t)x_i(t) + F_i(t, \varphi, \psi) + I_i(t),\ i = 1, 2, \ldots, n, \\ y_j^{\nabla}(t) = -d_j(t)y_j(t) + G_j(t, \varphi, \psi) + J_j(t),\ j = 1, 2, \ldots, m, \end{cases} \tag{3.4}$$

where

$$\begin{cases} F_i(t, \varphi, \psi) = a_i(t)\displaystyle\int_{t-\alpha(t)}^{t} \varphi_i^{\nabla}(s)\nabla s + \sum_{j=1}^{m} b_{ij}(t)g_j(\psi_j(t-\tau_j(t))) \\ \qquad + \displaystyle\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(t)p_j(\psi_j^{\nabla}(t-\tau_j(t)))q_l(\psi_l(t-\tau_l(t))), \\ G_j(t, \varphi, \psi) = d_j(t)\displaystyle\int_{t-\alpha(t)}^{t} \psi_j^{\nabla}(s)\nabla s + \sum_{i=1}^{n} c_{ji}(t)f_i(\varphi_i(t-\delta_i(t))) \\ \qquad + \displaystyle\sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}(t)u_i(\varphi_i^{\nabla}(t-\delta_i(t)))v_l(\varphi_l(t-\delta_l(t))). \end{cases} \tag{3.5}$$

From (H.4) and Lemma 2.4, it follows that the linear system

$$\begin{cases} x_i^{\nabla}(t) = -a_i(t)x_i(t),\ i = 1, 2, \ldots, n, \\ y_j^{\nabla}(t) = -d_j(t)y_j(t),\ j = 1, 2, \ldots, m, \end{cases}$$

admits an exponential dichotomy on $\mathbb{T}$. Therefore, by Lemma 2.3, we have that system (3.4) has a unique almost periodic solution, which takes the following form:

$$\begin{cases} x_i^{\varphi}(t) = \displaystyle\int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t, \rho(s))[F_i(s, \varphi, \psi) + I_i(s)]\nabla s, \; i = 1, 2, \ldots, n, \\ y_j^{\psi}(t) = \displaystyle\int_{-\infty}^{t} \hat{e}_{-\int_s^t d_j(u)\nabla u}(t, \rho(s))[G_j(s, \varphi, \psi) + J_j(s)]\nabla s, \; j = 1, 2, \ldots, m. \end{cases} \tag{3.6}$$

For $\phi \in X_0$, we have $\|\phi\| \le \|\phi - \phi^0\| + \|\phi^0\| \le 2L$. Define the following nonlinear operator Φ by

$$\begin{aligned} \Phi : X_0 \; &\to X_0, \; \left(\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \cdots, \psi_m\right)^T \\ &\to \left(x_1^{\varphi}, x_2^{\varphi}, \ldots, x_n^{\varphi}, y_1^{\psi}, y_2^{\psi}, \ldots, y_m^{\psi}\right)^T, \end{aligned} \tag{3.7}$$

where $x_i^{\varphi}(i = 1, 2, \ldots, n)$ and $y_j^{\psi}(j = 1, 2, \ldots, m)$ are defined by (3.4). In what follows, we will show that Φ is a contraction mapping.

First, we prove that $\Phi\phi \in X_0$, for any $\phi \in X_0$. It follows from (3.5) that

$$\begin{aligned} \left|F_i(t, \varphi, \psi)\right| \; &= \; \Big|a_i(t) \int_{t-\alpha(t)}^{t} \varphi_i^{\nabla}(s)\nabla s + \sum_{j=1}^{m} b_{ij}(t) g_j(\psi_j(t - \tau_j(t))) \\ &\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(t) p_j(\psi_j^{\nabla}(t - \tau_j(t))) q_l(\psi_l(t - \tau_l(t)))\Big| \\ &\le \; a_i(t) \int_{t-\alpha_i(t)}^{t} |\varphi_i^{\nabla}(u)|\nabla u \\ &\quad + \sum_{j=1}^{m} b_{ij}(t)\big(|g_j(\psi_j(t - \tau_j(t))) - g_j(0)| + |g_j(0)|\big) \\ &\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(t)\big(|p_j(\psi_j^{\nabla}(t - \tau_j(t))) \\ &\quad - p_j(0)|\big)\big(|q_l(\psi_l(t - \tau_l(t))) - q_l(0)| + |q_l(0)|\big) \\ &\le \; a_i^+ \alpha_i^+ |\varphi^{\nabla}|_0 + \sum_{j=1}^{m} b_{ij}^+ L_j^g |\psi_j(t - \tau_j(t))| + \sum_{j=1}^{m} b_{ij}^+ |g_j(0)| \\ &\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+\big(L_j^p |\psi_j^{\nabla}(t - \tau_j(t))| \\ &\quad + |p_j(0)|\big)\big(L_l^q |\psi_l(t - \tau_l(t))| + |q_l(0)|\big) \end{aligned}$$

$$
\begin{aligned}
&\leq a_i^+\alpha_i^+|\varphi^\nabla|_0+\sum_{j=1}^m b_{ij}^+L_j^g|\psi|_0+\sum_{j=1}^m b_{ij}^+|g_j(0)| \\
&\quad+\sum_{j=1}^m\sum_{l=1}^m e_{ijl}^+\big(L_j^p|\psi^\nabla|_0+|p_j(0)|\big)\big(L_l^q|\psi|_0+|q_l(0)|\big) \\
&\leq a_i^+\alpha_i^+|\phi|+\sum_{j=1}^m b_{ij}^+L_j^g|\phi|+\sum_{j=1}^m b_{ij}^+|g_j(0)| \\
&\quad+\sum_{j=1}^m\sum_{l=1}^m e_{ijl}^+\big(L_j^p|\phi|+|p_j(0)|\big)\big(L_l^q|\phi|+|q_l(0)|\big) \\
&\leq 2a_i^+\alpha_i^+L+2\sum_{j=1}^m b_{ij}^+L_j^gL+\sum_{j=1}^m b_{ij}^+L+\sum_{j=1}^m\sum_{l=1}^m e_{ijl}^+(2L_j^pL+L)(2L_l^qL+L) \\
&\leq \left[2a_i^+\alpha_i^++2\sum_{j=1}^m b_{ij}^+L_j^g+\sum_{j=1}^m b_{ij}^++L\sum_{j=1}^m\sum_{l=1}^m e_{ijl}^+(2L_j^p+1)(2L_l^q+1)\right]L \\
&:= \Theta_iL,\ \ i=1,2,\ldots,n,
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\big|G_j(t,\varphi,\psi)\big| &= \Big|d_j(t)\int_{t-\beta(t)}^t\psi_j^\nabla(s)\nabla s+\sum_{i=1}^n c_{ji}(t)f_i(\varphi_i(t-\delta_i(t))) \\
&\quad+\sum_{i=1}^n\sum_{l=1}^n s_{jil}(t)u_i(\varphi_i^\nabla(t-\delta_i(t)))v_l(\varphi_l(t-\delta_l(t)))\Big| \\
&\leq d_j(t)\int_{t-\beta_j(t)}^t|\psi_j^\nabla(u)|\nabla u \\
&\quad+\sum_{i=1}^n c_{ji}(t)(|g_i(\varphi_i(t-\delta_i(t)))-f_i(0)|+|f_i(0)|) \\
&\quad+\sum_{i=1}^n\sum_{l=1}^n s_{jil}(t)(|u_i(\varphi_i^\nabla(t\delta_i(t))) \\
&\quad-u_i(0)|)(|v_l(\varphi_l(t-\delta_l(t)))-v_l(0)|+|v_l(0)|) \\
&\leq d_j^+\beta_j^+|\psi^\nabla|_0+\sum_{i=1}^n c_{ji}^+L_i^f|\varphi_i(t-\delta_i(t))|+\sum_{i=1}^n c_{ji}^+|f_i(0)| \\
&\quad+\sum_{i=1}^n\sum_{l=1}^n s_{jil}^+(L_i^u|\varphi_i^\nabla(t-\delta_i(t))| \\
&\quad+|v_i(0)|)(L_l^v|\varphi_l(t-\delta_l(t))|+|v_l(0)|)
\end{aligned}
$$

$$
\begin{aligned}
&\leq d_j^+\beta_j^+|\psi^\nabla|_0+\sum_{i=1}^n c_{ji}^+L_i^f|\varphi|_0+\sum_{i=1}^n c_{ji}^+|g_i(0)|\\
&\quad+\sum_{i=1}^n\sum_{l=1}^n s_{jil}^+(L_i^u|\varphi^\nabla|_0+|u_i(0)|)(L_l^v|\varphi|_0+|v_l(0)|)\\
&\leq d_j^+\beta_j^+|\phi|+\sum_{i=1}^n c_{ji}^+L_i^f|\phi|+\sum_{i=1}^n c_{ji}^+|f_i(0)|\\
&\quad+\sum_{i=1}^n\sum_{l=1}^n s_{jil}^+(L_i^u|\phi|+|u_i(0)|)(L_l^v|\phi|+|v_l(0)|)\\
&\leq 2d_j^+\beta_j^+L+2\sum_{i=1}^n c_{ji}^+L_i^fL+\sum_{i=1}^n c_{ji}^+L+\sum_{i=1}^n\sum_{l=1}^n s_{jil}^+(2L_i^uL+L)(2L_l^vL+L)\\
&\leq \left[2d_j^+\beta_j^++2\sum_{i=1}^n c_{ji}^+L_i^f+\sum_{i=1}^n c_{ji}^++L\sum_{i=1}^n\sum_{l=1}^n s_{jil}^+(2L_i^u+1)(2L_l^v+1)\right]L\\
&:= \Pi_jL,\quad j=1,2,\dots,m, \qquad (3.9)
\end{aligned}
$$

where Θ_i and Π_j are defined in (H.5).

By virtue of (3.5)–(3.7), we get

$$
\begin{aligned}
\left|(\Phi\phi-\phi^0)_i(t)\right| &= \left|\int_{-\infty}^t \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))F_i(s,\varphi,\psi)\nabla s\right|\\
&\leq \int_{-\infty}^t \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))|F_i(s,\varphi,\psi)|\nabla s\\
&\leq \int_{-\infty}^t \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Theta_iL\nabla s\\
&\leq \frac{\Theta_iL}{a_i^-},\quad i=1,2,\dots,n, \qquad (3.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(\Phi\phi-\phi^0)_{n+j}(t)\right| &= \left|\int_{-\infty}^t \hat{e}_{-\int_s^t d_j(u)\nabla u}(t,\rho(s))G_j(s,\varphi,\psi)\nabla s\right|\\
&\leq \int_{-\infty}^t \hat{e}_{-\int_s^t d_j(u)\nabla u}(t,\rho(s))|G_j(s,\varphi,\psi)|\nabla s\\
&\leq \int_{-\infty}^t \hat{e}_{-\int_s^t d_j(u)\nabla u}(t,\rho(s))\Pi_jL\nabla s\\
&\leq \frac{\Pi_jL}{d_j^-},\quad j=1,2,\dots,m. \qquad (3.11)
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
\left|(\Phi\phi - \phi^0)_i^{\nabla}(t)\right| &= \left|\left(\int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s)) F_i(s,\varphi,\psi)\nabla s\right)^{\nabla}\right| \\
&= \left|F_i(t,\varphi,\psi) - a_i(t)\int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s)) F_i(s,\varphi,\psi)\nabla s\right| \\
&\le |F_i(t,\varphi,\psi)| + a_i(t)\int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s)) |F_i(s,\varphi,\psi)|\nabla s \\
&\le \Theta_i L + \Theta_i L \frac{a_i^+}{a_i^-} \\
&= (1 + \frac{a_i^+}{a_i^-})\Theta_i L, \ i = 1,2,\ldots,n, \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|(\Phi\phi - \phi^0)_{n+j}^{\nabla}(t)\right| &= \left|\left(\int_{-\infty}^{t} \hat{e}_{-\int_s^t d_j(u)\nabla u}(t,\rho(s)) G_j(s,\varphi,\psi)\nabla s\right)^{\nabla}\right| \\
&= \left|G_j(t,\varphi,\psi) - b_j(t)\int_{-\infty}^{t} \hat{e}_{-b_j\int_0^{\infty} \tilde{k}_j(u)\nabla u}(t,\rho(s)) G_j(s,\varphi,\psi)\nabla s\right| \\
&\le \left|G_j(t,\varphi,\psi)| + d_j(t)\int_{-\infty}^{t} \hat{e}_{-\int_s^t d_j(u)\nabla u}(t,\rho(s)) |G_j(s,\varphi,\psi)\right| \nabla s \\
&\le \Pi_j L + \frac{d_i^+}{d_i^-}\Pi_j L \\
&= \left(1 + \frac{d_j^+}{d_j^-}\right)\Pi_j L, \ j = 1,2,\ldots,m. \qquad (3.13)
\end{aligned}
$$

In view of (3.8)–(3.11), we have

$$
\|\Phi\phi - \phi^0\| = \max\left\{\max_{1\le i\le n}\left\{\Theta_i, \left(1 + \frac{a_i^+}{a_i^-}\right)\Theta_i\right\}, \max_{1\le j\le m}\left\{\Pi_j, \left(1 + \frac{d_i^+}{d_i^-}\right)\Pi_j\right\}\right\} \le L,
$$

which yields that $\Phi\phi \in X_0$.

Hereafter, we show that Φ is a contraction.
For $\phi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \psi_1, \psi_2, \ldots, \psi_m)^T$, $\bar{\phi} = (\bar{\varphi}_1, \bar{\varphi}_2, \ldots, \bar{\varphi}_n, \bar{\psi}_1, \bar{\psi}_2, \ldots, \bar{\psi}_m)^T \in$

X_0, we have

$$
\begin{aligned}
|(\Phi\phi - \Phi\bar{\phi})_i(t)| &= \Big|\int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Big[a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u) - \bar{\varphi}_i^{\nabla}(u))\nabla u \\
&\quad + \sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s))) - g_j(\bar{\psi}_j(s-\tau_i(s)))) \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&\quad - p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big]\nabla s\Big| \\
&\le \int_{-\infty}^{t} \hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Big[|a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u) - \bar{\varphi}_i^{\nabla}(u))\nabla u| \\
&\quad + \left|\sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s))) - g_j(\bar{\psi}_j(s-\tau_i(s))))\right| \\
&\quad + \Big|\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&\quad - p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big|\Big]\nabla s \\
&\le \int_{-\infty}^{t} \hat{e}_{-a_i^-(t-s)}(t,\rho(s))\Big[a_i^+\alpha_i^+|\varphi - \bar{\phi}|_1 \\
&\quad + \left|\sum_{j=1}^{m} b_{ij}^+ L_j^g |\psi_j(s-\tau_j(s)) - \bar{\varphi}_j(s-\tau_i(s))\right| \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ Q_L L_j^p |\psi_j(s-\tau_i(s)) - \bar{\psi}_j(s-\tau_i(s))| \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ P_j L_l^q |\varphi_j(s-\tau_i(s)) - \bar{\varphi}_j(s-\tau_i(s))|\Big]\nabla s \\
&\le \frac{1}{a_i^-}\Big[a_i^+\alpha_i^+ + \sum_{j=1}^{m} b_{ij}^+ L_j^g \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ Q_l L_j^p + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ P_j L_l^q\Big]\|\phi - \bar{\phi}\|, \qquad (3.14)
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
|(\Phi\phi-\Phi\bar{\phi})_i^{\nabla}(t)| = {}& \Big|\Big\{\int_{-\infty}^{t}\hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Big[a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u)-\bar{\varphi}_i^{\nabla}(u))\nabla u \\
&+\sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s)))-g_j(\bar{\psi}_j(s-\tau_i(s)))) \\
&+\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&-p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big]\nabla s\Big\}^{\nabla}\Big| \\
\le{}& \Big|a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u)-\bar{\varphi}_i^{\nabla}(u))\nabla u \\
&+\sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s)))-g_j(\bar{\psi}_j(s-\tau_i(s)))) \\
&+\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&-p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s)))) \\
&-a_i(t)\int_{-\infty}^{t}\hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Big[a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u)-\bar{\varphi}_i^{\nabla}(u))\nabla u \\
&+\sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s)))-g_j(\bar{\psi}_j(s-\tau_i(s)))) \\
&+\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&-p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big]\nabla s\Big| \\
\le{}& \Big|a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u)-\bar{\varphi}_i^{\nabla}(u))\nabla u\Big| \\
&+\Big|\sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s)))-g_j(\bar{\psi}_j(s-\tau_i(s))))\Big| \\
&+\Big|\sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&-p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big| \\
&+a_i(t)\int_{-\infty}^{t}\hat{e}_{-\int_s^t a_i(u)\nabla u}(t,\rho(s))\Big[\Big|a_i(s)\int_{s-\alpha_i(s)}^{s}(\varphi_i^{\nabla}(u)-\bar{\varphi}_i^{\nabla}(u))\nabla u\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big| \sum_{j=1}^{m} b_{ij}(s)(g_j(\psi_j(s-\tau_j(s))) - g_j(\bar{\psi}_j(s-\tau_i(s))))\Big| \\
&+ \Big| \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}(s)(p_j(\psi_j(s-\tau_i(s)))q_l(\psi_l(s-\tau_l(s))) \\
&- p_j(\bar{\psi}_j(s-\tau_i(s)))q_l(\bar{\psi}_l(s-\tau_l(s))))\Big|\Big]\nabla s
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|(\Phi\phi - \Phi\bar{\phi})_i^{\nabla}(t)| &\leq a_i^+\alpha_i^+|\varphi-\bar{\varphi}|_1 + \sum_{j=1}^{m} b_{ij}^+ L_j^g|\psi-\bar{\psi}|_1 \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ Q_L L_j^p|\psi-\bar{\psi}|_1 + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ P_j L_l^q|\psi-\bar{\psi}|_1 \\
&\quad + \frac{a_i^+}{a_i^-}\big[a_i^+\alpha_i^+|\varphi-\bar{\varphi}|_1 + \sum_{j=1}^{m} b_{ij}^+ L_j^g|\psi-\bar{\psi}|_1 \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ Q_L L_j^p|\psi-\bar{\psi}|_1 + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ P_j L_l^q|\psi-\bar{\psi}|_1\big] \\
&= (1+\frac{a_i^+}{a_i^-})\big[a_i^+\alpha_i^+|\varphi-\bar{\varphi}|_1 + \sum_{j=1}^{m} b_{ij}^+ L_j^g|\psi-\bar{\psi}|_1 \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ Q_L L_j^p|\psi-\bar{\psi}|_1 + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ P_j L_l^q|\psi-\bar{\psi}|_1\big] \\
&\leq (1+\frac{a_i^+}{a_i^-})\big[a_i^+\alpha_i^+ + \sum_{j=1}^{m} b_{ij}^+ L_j^g \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m} e_{ijl}^+ (Q_l L_j^p + P_j L_l^q)\big]\|\phi-\bar{\phi}\|. \qquad (3.15)
\end{aligned}
$$

By applying the same arguments, one can deduce that

$$
\big|(\Phi\phi - \Phi\bar{\phi})_{n+j}(t)\big| \leq \frac{1}{d_j^-}\Big[d_j^+\beta_j^+ + \sum_{i=1}^{n} c_{ji}^+ L_i^f + \sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}^+(V_l L_i^u + U_i L_l^v)\Big]\|\phi-\bar{\phi}\|, \tag{3.16}
$$

and

$$
\big|(\Phi\phi - \Phi\bar{\phi})_{n+j}^{\nabla}(t)\big| \leq \left(1+\frac{d_j^+}{d_j^-}\right)\Big[d_j^+\beta_j^+ + \sum_{i=1}^{n} c_{ji}^+ L_i^f + \sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}^+(V_l L_i^u + U_i L_l^v)\Big]\|\phi-\bar{\phi}\|. \tag{3.17}
$$

From (H.4), however, we have

$$\max\left\{\max_{1\le i\le n}\{\kappa_{i1},\kappa_{i2}\},\ \max_{1\le j\le m}\{\epsilon_{j1},\epsilon_{j2}\}\right\}<1. \tag{3.18}$$

It follows from (3.12)–(3.16) that $\|\Phi\phi-\Phi\bar{\phi}\|<\|\phi-\tilde{\phi}\|$, which means that Φ is a contraction mapping. Based on Banch fixed point theorem, Φ has a fixed point $\phi^*\in X_0$ such that $\Phi\phi^*=\phi^*$. That is, (3.1) has a unique almost periodic solution $\phi^*\in X_0$. Therefore, there exists a unique almost periodic solution ϕ^* of system (1.1) in X_0. The proof of Theorem 3.1 is completed. ■

3.2. Exponential stability of almost periodic solution

Let

H.6 For $t\in(0,\infty)_{\mathbb{T}}$, there exist positive constants $\lambda\in R_\nu^+$, γ_i and χ_j such that

$$\begin{cases}-[a_i^- -(a_i^+)^2]\gamma_i+\sum_{j=1}^m\chi_j\big[b_{ij}^+L_j^g+\sum_{l=1}^m e_{ijl}^+(Q_lL_j^p+P_jL_l^q)\big]<0,\\ -[d_j^- -(d_j^+)^2]\chi_j+\sum_{i=1}^n\gamma_i\big[c_{ji}^+L_i^f+\sum_{l=1}^n s_{jil}^+(V_lL_i^u+U_iL_l^v)\big]<0,\end{cases}$$

and

$$\begin{cases}\big[(1-\nu\lambda)a_i^+ +(\lambda+(1-\nu\lambda))a_i^+\int_{t-\alpha_i(t)}^t\hat{e}_\lambda(t,\omega)\nabla\omega\big]\\ +\hat{e}_\lambda(\rho(t),0){\gamma_i}^{-1}\sum_{j=1}^m\chi_j\big[b_{ij}^+L_j^g\hat{e}_\lambda(0,t-\tau_{ij}(t))\\ +\sum_{l=1}^m e_{ijl}^+(Q_lL_j^p+P_jL_l^q)\hat{e}_\lambda(0,t-\tau_{ij}(t))\big]<1,\\ \big[(1-\nu\lambda)d_j^+ +(\lambda+(1-\nu\lambda))d_j^+\int_{t-\beta_j(t)}^t\hat{e}_\lambda(t,\omega)\nabla\omega\big]\\ +\hat{e}_\lambda(\rho(t),0){\chi_j}^{-1}\sum_{i=1}^n\gamma_i\big[c_{ji}^+L_i^f\hat{e}_\lambda(0,t-\delta_{ji}(t))\\ +\sum_{l=1}^n s_{jil}^+(V_lL_i^u+U_iL_l^v)\hat{e}_\lambda(0,t-\delta_{ji}(t))\big]<1.\end{cases}$$

Theorem 3.2. Let H.1–H.6 hold. Then the almost solution of system (1.1) is exponentially stable.

Proof. In view of Theorem 3.1, system (1.1) has an almost periodic solution

$$z^*(t)=\Big(x_1^*(t),x_2^*(t),\ldots,x_n^*(t),\ y_1^*(t),y_2^*(t),\ldots,y_m^*(t)\Big)^T$$

with initial condition

$$\phi^*(s) = \Big(\varphi_1^*(s), \varphi_2^*(s), \dots, \varphi_n^*(t), \psi_1^*(s), \psi_2^*(s), \dots, \psi_m^*(s)\Big)^T.$$

Suppose that $z(t) = \big(x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t)\big)^T$ is an arbitrary solution of system (1.1) with initial condition $\phi(s) = \big(\varphi_1(s), \varphi_2(s), \dots, \varphi_n(t), \psi_1(s), \psi_2(s), \dots, \psi_m(s)\big)^T$. For $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, let

$$\begin{cases} \mu_i(t) = x_i(t) - x_i^*(t) \\ \zeta_j(t) = y_j(t) - y_j^*(t) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \bar{G}_j(t) = g_j(\zeta_j(t - \tau_j(t))) - g_j(\zeta_j^*(t - \tau_j(t))), \\ \bar{F}_i(t) = f_i(\mu_i(t - \delta_i(t))) - f_i(\mu_i^*(t - \delta_i(t))), \\ \bar{H}_{jl}(t) = p_j(\zeta_j^\nabla(t - \tau_j(t))q_l(\zeta_j(t - \tau_l(t))) - p_j(\zeta_j^{*\nabla}(t - \tau_j(t))q_l(\zeta_j^*(t - \tau_l(t))), \\ \bar{N}_{il}(t) = u_i(\mu_i^\nabla(t - \delta_i(t))v_l(\mu_i(t - \delta_l(t))) - u_i(\mu_i^{*\nabla}(t - \delta_i(t))v_l(\mu_i^*(t - \delta_l(t))). \end{cases} \tag{4.2}$$

Then

$$\begin{cases} u_i^\nabla(t) = -a_i(t)u_i(t) + a_i(t)\displaystyle\int_{t-\alpha_i(t)}^t u_i^\nabla(\omega)\nabla\omega \\ +\displaystyle\sum_{j=1}^m b_{ij}(t)\bar{G}_j(t) + \sum_{j=1}^m\sum_{l=1}^m e_{ijl}(t)\bar{H}_{jl}(t), \\ \zeta_j^\nabla(t) = -d_i(t)\zeta_j(t) + d_i(t)\displaystyle\int_{t-\beta_j(t)}^t \zeta_j^\nabla(\omega)\nabla\omega \\ +\displaystyle\sum_{i=1}^n c_{ji}(t)\bar{F}_i(t) + \sum_{i=1}^n\sum_{l=1}^n s_{jil}(t)\bar{N}_{il}(t). \end{cases} \tag{4.3}$$

Define

$$X_i(t) = \hat{e}_\lambda(t, 0)u_i(t), \quad Y_j(t) = \hat{e}_\lambda(t, 0)\zeta_j(t), \quad i = 1, 2, \dots, n, \; j = 1, 2, \dots, m. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} X_i^\nabla(t) &= \lambda\hat{e}_\lambda(t, 0)u_i(t) + \hat{e}_\lambda(\rho(t), 0)u_i^\nabla(t) \\ &= \lambda X_i(t) + \hat{e}_\lambda(\rho(t), 0)\Big[-a_i(t)u_i(t) + a_i(t)\int_{t-\alpha_i(t)}^t u_i^\nabla(\omega)\nabla\omega \\ &\quad+\sum_{j=1}^m b_{ij}(t)\bar{G}_j(t) + \sum_{j=1}^m\sum_{l=1}^m e_{ijl}(t)\bar{H}_{jl}(t)\Big] \\ &= \lambda X_i(t) - (1 - \nu\lambda)a_i(t)X_i(t) + (1 - \nu\lambda)a_i(t)\int_{t-\alpha_i(t)}^t \hat{e}_\lambda(t, \omega)X_i^\nabla(\omega)\nabla\omega \\ &\quad+\hat{e}_\lambda(\rho(t), 0)\sum_{j=1}^m\Big[b_{ij}(t)\bar{G}_j(t) + \sum_{j=1}^m\sum_{l=1}^m e_{ijl}(t)\bar{H}_{jl}(t)\Big], \; i = 1, 2, \dots, n. \end{aligned} \tag{4.5}$$

In similar manner, we have

$$\begin{cases} Y_j^{\nabla}(t) &= \lambda Y_j(t) - (1-\nu\lambda)d_j(t)Y_j(t) + (1-\nu\lambda)d_j(t)\displaystyle\int_{t-\beta_j(t)}^{t} \hat{e}_{\lambda}(t,\omega)Y_j^{\nabla}(\omega)\nabla\omega \\ &+ \ \hat{e}_{\lambda}(\rho(t),0)\displaystyle\sum_{j=1}^{m}\big[c_{ji}(t)\bar{F}_i(t) + \sum_{i=1}^{n}\sum_{l=1}^{n} s_{jil}(t)\bar{N}_{il}(t)\big], \ j=1,2,\dots,m. \end{cases}$$

Define two continuous functions $\Gamma_i(\xi)$ and $\Gamma_j(\xi)$, $i=1,2,\dots,n,\ j=1,2,\dots,m$, by setting

$$\begin{cases} \Gamma_i(\xi) &= -\big[(1-\nu\xi)a_i^- - (\xi + (1-\nu\xi)a_i^+\displaystyle\int_{t-\alpha_i(t)}^{t}\hat{e}_{\xi}(t,\omega)\nabla\omega)\big]\gamma_i \\ &+\hat{e}_{\xi}(\rho(t),0)\displaystyle\sum_{j=1}^{m}\chi_j\big[b_{ij}^{+}L_j^{g}\hat{e}_{\lambda}(0,t-\tau_{ij}(t)) \\ &+\displaystyle\sum_{l=1}^{m} e_{ijl}^{+}(Q_lL_j^{p} + P_jL_l^{q})\hat{e}_{\lambda}(0,t-\tau_{ij}(t))\big], \\ \Gamma_j(\xi) &= -\big[(1-\nu\xi)d_j^- - (\xi + (1-\nu\xi)d_j^+\displaystyle\int_{t-\beta_j(t)}^{t}\hat{e}_{\xi}(t,\omega)\nabla\omega)\big]\chi_j \\ &+\hat{e}_{\xi}(\rho(t),0)\displaystyle\sum_{i=1}^{n}\gamma_i\big[c_{ji}^{+}L_i^{f}\hat{e}_{\lambda}(0,t-\delta_{ji}(t)) \\ &+\displaystyle\sum_{l=1}^{n} s_{jil}^{+}(V_lL_i^{u} + U_iL_l^{v})\hat{e}_{\lambda}(0,t-\delta_{ji}(t))\big]. \end{cases} \tag{4.6}$$

Then, for $i=1,2,\dots,n$, $j=1,2,\dots,m$,

$$\begin{cases} \Gamma_i(0) = -[a_i^- - (a_i^+)^2]\gamma_i + \displaystyle\sum_{j=1}^{m}\chi_j\big[b_{ij}^{+}L_j^{g} + \sum_{l=1}^{m} e_{ijl}^{+}(Q_lL_j^{p} + P_jL_l^{q})\big] < 0, \\ \Gamma_j(0) = -[d_j^- - (d_j^+)^2]\chi_j + \displaystyle\sum_{i=1}^{n}\gamma_i\big[c_{ji}^{+}L_i^{f} + \sum_{l=1}^{n} s_{jil}^{+}(V_lL_i^{u} + U_iL_l^{v})\big] < 0, \end{cases}$$

which means that there exists a constant $\lambda > 0$ such that

$$\begin{cases} -\big[(1-\nu\xi)a_i^- - (\xi + (1-\nu\xi)a_i^+\displaystyle\int_{t-\alpha_i(t)}^{t}\hat{e}_{\xi}(t,\omega)\nabla\omega)\big]\gamma_i \\ +\hat{e}_{\xi}(\rho(t),0)\displaystyle\sum_{j=1}^{m}\chi_j\big[b_{ij}^{+}L_j^{g}\hat{e}_{\lambda}(0,t-\tau_{ij}(t)) + \sum_{l=1}^{m} e_{ijl}^{+}(Q_lL_j^{p} + P_jL_l^{q})\hat{e}_{\lambda}(0,t-\tau_{ij}(t))\big] < 0, \\ -\big[(1-\nu\xi)d_j^- - (\xi + (1-\nu\xi)d_j^+\displaystyle\int_{t-\beta_j(t)}^{t}\hat{e}_{\xi}(t,\omega)\nabla\omega)\big]\chi_j \\ +\hat{e}_{\xi}(\rho(t),0)\displaystyle\sum_{i=1}^{n}\gamma_i\big[c_{ji}^{+}L_i^{f}\hat{e}_{\lambda}(0,t-\delta_{ji}(t)) + \sum_{l=1}^{n} s_{jil}^{+}(V_lL_i^{u} + U_iL_l^{v})\hat{e}_{\lambda}(0,t-\delta_{ji}(t))\big] < 0. \end{cases}$$

Let

$$M = \max\left\{ \max_{1\le i\le n}\{|X_i(s)|, |X_i^{\nabla}(s)|\}, \max_{1\le j\le m}\{|Y_j(s)|, |Y_j^{\nabla}(s)|\}, s \in [-\sigma, 0]_{\mathbb{T}} \right\}.$$

Therefore, for any $t \in [-\sigma, 0]_{\mathbb{T}}$, there exists $K > 0$ such that

$$\begin{cases} |X_i(t)| \le M < K\gamma_i, \ |X_i^{\nabla}(t)| \le M < K\gamma_i, \\ |Y_j(t)| \le M < K\chi_j, \ |Y_j^{\nabla}(t)| \le M < K\chi_j. \end{cases}$$

Next, we prove that

$$\begin{cases} |X_i(t)| < K\gamma_i, \ |X_i^{\nabla}(t)| < K\gamma_i, t \in \mathbb{T}, \ i = 1, 2, \ldots, n, \\ |Y_j(t)| < K\chi_j, \ |Y_j^{\nabla}(t)| < K\chi_j, t \in \mathbb{T}, \ j = 1, 2, \ldots, m. \end{cases}$$

For the sake of contradiction, one can find two indices $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$ and a constant $\theta \in (0, \infty)_{\mathbb{T}}$ such that one of the following cases:

(i) $|X_i(\theta)| = K\gamma_i$ and $|X_i(t)| < K\gamma_i$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i^{\nabla}(t)| < K\gamma_i$, $|Y_j(t)| < K\chi_j$, $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$;

(ii) $|X_i^{\nabla}(\theta)| = K\gamma_i$ and $|X_i^{\nabla}(t)| < K\gamma_i$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i(t)| < K\gamma_i$, $|Y_j(t)| < K\chi_j$, $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$;

(iii) $|Y_j(\theta)| = K\chi_j$ and $|Y_j(t)| < K\chi_j$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i(t)| < K\gamma_i$, $|X_i^{\nabla}(t)| < K\gamma_i$, $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$;

(iv) $|Y_j^{\nabla}(\theta)| = K\chi_j$ and $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i(t)| < K\gamma_i$, $|X_i^{\nabla}(t)| < K\gamma_i$, $|Y_j(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$.

Let (i) hold, then either
(i)$'$ $X_i(\theta) = K\gamma_i$, $X_i^{\nabla}(\theta) \ge 0$ and $|X_i(t)| < K\gamma_i$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i^{\nabla}(t)| < K\gamma_i$, $|Y_j(t)| < K\chi_j$, $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$; or
(i)$''$ $X_i(\theta) = -K\gamma_i$, $X_i^{\nabla}(\theta) \le 0$ and $|X_i(t)| < K\gamma_i$ for all $t \in [-\sigma, \theta)_{\mathbb{T}}$, and $|X_i^{\nabla}(t)| < K\gamma_i$, $|Y_j(t)| < K\chi_j$, $|Y_j^{\nabla}(t)| < K\chi_j$ for all $t \in [-\sigma, \theta]_{\mathbb{T}}$.

For (i)$'$, from H.6 and equation (4.5), we get

$$\begin{aligned}
0 \quad &\le X_i^{\nabla}(\theta) = \lambda X_i(\theta) - (1-\nu\lambda)a_i(\theta)X_i(\theta) \\
&\quad +(1-\nu\lambda)a_i(\theta)\int_{\theta-\alpha_i(\theta)}^{\theta}\hat{e}_\lambda(\theta,\omega)X_i^{\nabla}(\omega)\nabla\omega \\
&\quad +\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}(\theta)\bar{G}_j(\theta)+\sum_{l=1}^{m}e_{ijl}(\theta)\bar{H}_{jl}(\theta)\big] \\
&\le -[(1-\nu\lambda)a_i^- - \lambda]X_i(\theta) + a_i^+(1-\nu\lambda)\int_{\theta-\alpha_i(\theta)}^{\theta}\hat{e}_\lambda(\theta,\omega)|X_i^{\nabla}(\omega)|\nabla\omega \\
&\quad +\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}^+L_j^g\hat{e}_\lambda(0,\theta-\tau_j(\theta))|Y_j(\theta-\tau_j(\theta))| \\
&\quad +\sum_{l=1}^{m}e_{ijl}^+(Q_LL_j^p+P_jL_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))|Y_j^{\nabla}(\theta-\tau_l(\theta))|\big] \\
&< -\big[(1-\nu\lambda)a_i^- - \lambda\big]K\gamma_i + (1-\nu\lambda)a_i^+\int_{\theta-\alpha_i(\theta)}^{\theta}\hat{e}_\lambda(\theta,\omega)K\gamma_i\nabla\omega \\
&\quad +\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}^+L_j^g\hat{e}_\lambda(0,\theta-\tau_j(\theta))K\chi_j \\
&\quad +\sum_{l=1}^{m}e_{ijl}^+(Q_LL_j^p+P_jL_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))K\chi_j\big] \\
&= \big\{-\big[(1-\nu\lambda)a_i^- - (\lambda+(1-\nu\lambda)a_i^+\int_{\theta-\alpha_i(\theta)}^{\theta}\hat{e}_\lambda(\theta,\omega)\nabla\omega)\big]\gamma_i \\
&\quad +\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\chi_j\big[b_{ij}^+L_j^g\hat{e}_\lambda(0,\theta-\tau_j(\theta)) \\
&\quad +\sum_{l=1}^{m}e_{ijl}^+(Q_LL_j^p+P_jL_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))\big]\big\}K < 0,
\end{aligned}$$

which is a contraction.

For (i)$''$, from H.6 and equation (4.5), we get

$$\begin{aligned}
0 \quad &\ge X_i^{\nabla}(\theta) \ge -\big[(1-\nu\lambda)a_i(\theta)-\lambda\big]X_i(\theta) \\
&\quad -a_i(\theta)(1-\nu\lambda)\int_{\theta-\alpha_i(\theta)}^{\theta}\hat{e}_\lambda(\theta,\omega)|X_i^{\nabla}(\omega)|\nabla\omega \\
&\quad -\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}(\theta)L_j^g\hat{e}_\lambda(0,\theta-\tau_j(\theta))|Y_j(\theta-\tau_j(\theta))| \\
&\quad +\sum_{l=1}^{m}e_{ijl}(\theta)(Q_LL_j^p+P_jL_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))|Y_j^{\nabla}(\theta-\tau_l(\theta))|\big]
\end{aligned}$$

$$
\begin{aligned}
&> \big[(1-\nu\lambda)a_i^- - \lambda\big]K\gamma_i - (1-\nu\lambda)a_i^+ \int_{\theta-\alpha_i(\theta)}^{\theta} \hat{e}_\lambda(\theta,\omega)K\gamma_i \nabla\omega \\
&-\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}^+ L_j^g \hat{e}_\lambda(0,\theta-\tau_j(\theta))K\chi_j \\
&+\sum_{j=1}^{l} e_{ijl}^+ (Q_L L_j^p + P_j L_l^q)\hat{e}_\lambda(0,\theta-\tau_j(\theta))K\chi_j\big] \\
&= \big\{\big[(1-\nu\lambda)a_i^- - (\lambda + (1-\nu\lambda)a_i^+ \int_{\theta-\alpha_i(\theta)}^{\theta} \hat{e}_\lambda(\theta,\omega)\nabla\omega)\big]\gamma_i \\
&-\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\chi_j[b_{ij}^+ L_j^g \hat{e}_\lambda(0,\theta-\tau_j(\theta)) \\
&+\sum_{l=1}^{m} e_{ijl}^+ (Q_L L_j^p + P_j L_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))\big]\big\}K > 0,
\end{aligned}
$$

which is also a contraction. Thus, (i) is not valid.

Let (ii) hold, from H.6 and equation (4.5), we obtain

$$
\begin{aligned}
K\gamma_i = \; & |X_i^\nabla(\theta)| \le (1-\nu\lambda)a_i(\theta)|X_i(\theta)| \\
&+\lambda|X_i(\theta)| + a_i(\theta)(1-\nu\lambda)\int_{\theta-\alpha_i(\theta)}^{\theta} \hat{e}_\lambda(\theta,\omega)|X_i^\nabla(\omega)|\nabla\omega \\
&+\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}(\theta)L_j^g \hat{e}_\lambda(0,\theta-\tau_j(\theta))|Y_j(\theta-\tau_j(\theta))| \\
&+\sum_{l=1}^{m} e_{ijl}(\theta)(Q_L L_j^p + P_j L_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))|Y_j^\nabla(\theta-\tau_l(\theta))|\big] \\
&< [(1-\nu\lambda)a_i^+ + \lambda]K\gamma_i + (1-\nu\lambda)a_i^+ \int_{\theta-\alpha_i(\theta)}^{\theta} \hat{e}_\lambda(\theta,\omega)K\gamma_i\nabla\omega \\
&+\hat{e}_\lambda(\rho(\theta),0)\sum_{j=1}^{m}\big[b_{ij}^+ L_j^g \hat{e}_\lambda(0,\theta-\tau_j(\theta))K\chi_j \\
&+\sum_{l=1}^{m} e_{ijl}^+ (Q_L L_j^p + P_j L_l^q)\hat{e}_\lambda(0,\theta\tau_l(\theta))K\chi_j\big] \\
&= \big\{\big[(1-\nu\lambda)a_i^+ + (\lambda + (1-\nu\lambda)a_i^+ \int_{\theta-a_i(\theta)}^{\theta} \hat{e}_\lambda(\theta,\omega)\nabla\omega)] \\
&+\hat{e}_\lambda(\rho(\theta),0)/\gamma_i \sum_{j=1}^{m}\chi_j[b_{ij}^+ L_j^g \hat{e}_\lambda(0,\theta-\tau_j(\theta)) \\
&+\sum_{l=1}^{m} e_{ijl}^+ (Q_L L_j^p + P_j L_l^q)\hat{e}_\lambda(0,\theta-\tau_l(\theta))\big]\big\}K\gamma_i \\
&< K\gamma_i,
\end{aligned}
$$

which is a contraction. So (ii) is not valid. Applying similar arguments, one can also show that (iii) and (iv) do not hold true. From the above discussions, it follows that

$$\begin{cases} |x_i(t) - x_i^*(t)| \le e_{\ominus\nu\lambda}(t,0)K\gamma_i, \ t \in \mathbb{T}, \ i = 1,2,\ldots,n, \\ |y_i(t) - y_i^*(t)| \le e_{\ominus\nu\lambda}(t,0)K\chi_j, \ t \in \mathbb{T}, \ j = 1,2,\ldots,m. \end{cases}$$

Thus, the almost periodic solution on time scale of system (1.1) is exponentially stable. The proof is finished. ∎

In this section, an example is given to illustrate the effectiveness of the obtained results in previous sections.

Example 3.3. In view of system (1.1), consider the following neutral–type BAM neural networks with delays in the neutral derivative and leakage terms of the form:

$$\begin{cases} x_1^{\nabla}(t) = -a_1(t)x_i(t-\alpha_1(t)) + \sum\limits_{j=1}^{2} b_{1j}(t)g_j(y_j(t-\tau_j(t))) \\ \qquad + \sum\limits_{j=1}^{2}\sum\limits_{l=1}^{2} e_{1jl}(t)p_j(y_j^{\nabla}(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_1(t), \\ x_2^{\nabla}(t) = -a_2(t)x_2(t-\alpha_2(t)) + \sum\limits_{j=1}^{2} b_{2j}(t)g_j(y_j(t-\tau_j(t))) \\ \qquad + \sum\limits_{j=1}^{2}\sum\limits_{l=1}^{2} e_{2jl}(t)p_j(y_j^{\nabla}(t-\tau_j(t)))q_l(y_l(t-\tau_l(t))) + I_2(t), \\ y_1^{\nabla}(t) = -d_1(t)y_1(t-\beta_1(t)) + \sum\limits_{i=1}^{2} c_{1i}(t)f_i(x_i(t-\delta_i(t))) \\ \qquad + \sum\limits_{i=1}^{2}\sum\limits_{l=1}^{2} s_{1il}(t)u_i(x_i^{\nabla}(t-\delta_i(t)))v_l(x_l(t-\delta_l(t))) + J_1(t), \\ y_2^{\nabla}(t) = -d_2(t)y_2(t-\beta_2(t)) + \sum\limits_{i=1}^{2} c_{2i}(t)f_i(x_i(t-\delta_i(t))) \\ \qquad + \sum\limits_{i=1}^{2}\sum\limits_{l=1}^{n} s_{2il}(t)u_i(x_i^{\nabla}(t-\delta_i(t)))v_l(x_l(t-\delta_l(t))) + J_2(t), \end{cases} \tag{5.1}$$

where

$$\begin{bmatrix} a_1(t) & a_2(t) \\ d_1(t) & d_2(t) \end{bmatrix} = \begin{bmatrix} 0.6 + 0.3\sin t & 0.7 + 0.3\cos t \\ 0.8 + 0.4\sin t & 0.7 + 0.5\cos t \end{bmatrix}$$

$$\begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.09 + 0.03\sin t & 0.04 + 0.02\cos t \\ 0.07 + 0.02\sin t & 0.06 + 0.03\cos t \end{bmatrix}$$

$$\begin{bmatrix} e_{111}(t) & e_{112}(t) \\ e_{121}(t) & e_{122}(t) \end{bmatrix} = \begin{bmatrix} 0.08 + 0.03\sin t & 0.06 + 0.01\cos t \\ 0.05 + 0.01\sin t & 0.07 + 0.02\cos t \end{bmatrix}$$

$$\begin{bmatrix} e_{211}(t) & e_{212}(t) \\ e_{221}(t) & e_{222}(t) \end{bmatrix} = \begin{bmatrix} 0.07+0.02\sin t & 0.05+0.01\cos t \\ 0.08+0.03\sin t & 0.05+0.02\cos t \end{bmatrix}$$

$$\begin{bmatrix} s_{111}(t) & s_{112}(t) \\ s_{121}(t) & s_{122}(t) \end{bmatrix} = \begin{bmatrix} 0.06+0.02\sin t & 0.07+0.03\cos t \\ 0.08+0.04\sin t & 0.07+0.02\cos t \end{bmatrix}$$

$$\begin{bmatrix} s_{211}(t) & s_{212}(t) \\ s_{221}(t) & s_{222}(t) \end{bmatrix} = \begin{bmatrix} 0.09+0.04\sin t & 0.07+0.02\cos t \\ 0.08+0.02\sin t & 0.06+0.01\cos t \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1(t) & \alpha_2(t) \\ \beta_1(t) & \beta_2(t) \end{bmatrix} = \begin{bmatrix} 0.04+0.02\sin t & 0.05+0.03\cos t \\ 0.03+0.01\sin t & 0.04+0.01\cos t \end{bmatrix}$$

$$\begin{bmatrix} \tau_1(t) & \tau_2(t) \\ \delta_1(t) & \delta_2(t) \end{bmatrix} = \begin{bmatrix} 0.05+0.02\sin t & 0.04+0.03\cos t \\ 0.06+0.03\sin t & 0.04+0.02\cos t \end{bmatrix}$$

$$\begin{bmatrix} I_1(t) & I_2(t) \\ J_1(t) & J_2(t) \end{bmatrix} = \begin{bmatrix} 0.0027\sin t & 0.0035\cos t \\ 0.0054\sin t & 0.0069\cos t \end{bmatrix}, \begin{bmatrix} g_1(y_1) & g_2(y_2) \\ f_1(x_1) & f_2(x_2) \end{bmatrix} = \begin{bmatrix} 3\sin\frac{\sqrt{3}}{3}y_1 & 3\cos\frac{\sqrt{5}}{3}y_2 \\ 2\sin\frac{\sqrt{2}}{3}x_1 & 2\cos\frac{\sqrt{7}}{3}x_2 \end{bmatrix}$$

$$\begin{bmatrix} p_1(y_1) & p_2(y_2) \\ q_1(y_1) & q_2(y_2) \end{bmatrix} = \begin{bmatrix} 3\sin\frac{\sqrt{3}}{3}y_1 & 3\cos\frac{\sqrt{5}}{3}y_2 \\ 3\sin\frac{\sqrt{2}}{3}y_1 & 3\cos\frac{\sqrt{7}}{3}y_2 \end{bmatrix}, \begin{bmatrix} u_1(x_1) & u_2(x_2) \\ v_1(x_1) & v_2(x_2) \end{bmatrix} = \begin{bmatrix} 0.45\sin\frac{\sqrt{3}}{3}x_1 & 0.75\cos\frac{\sqrt{5}}{3}x_2 \\ 0.45\sin\frac{\sqrt{2}}{3}x_1 & 0.75\cos\frac{\sqrt{7}}{3}x_2 \end{bmatrix}.$$

Consider the case $\mathbb{T} = \mathbb{R}$ with $\rho(t) = t$ and $\nu(t) = 0$. Let $\gamma_1 = \gamma_2 = \chi_1 = \chi_2 = 1$ and $\lambda = 0.003$. Then, it is not difficult to check that all assumptions H.1–H.6 are satisfied. Hence, from Theorem 3.1 and Theorem 3.2, we can conclude that system (5.1) has exactly one solution on almost periodic time scale which is exponentially stable.

Remark 3.4. In view of the above example, one can easily figure out that the proposed results in [39–41] can not be applied to comment on the dynamic behavior of system (5.1). Therefore, our theorems extend the existing results in the literature.

4. A concluding remark

This paper considers neutral type BAM neural networks with delays imposed in the neutral derivative and leakage terms on time scales. It has been evidenced that the presence of time delay in BAM neural networks has unshakable consequences in the hardware implementation. Indeed, the time delay has capability to distabilize the system and to cause oscillation in the networks. A primary objective of this paper is to see how influential these days on system (1.1) will be! The exponential dichotomy of linear differential equation on time scales, Banach fixed point theorem and the method of lyapunov functional are the main techniques used to prove the main results in this paper.

It is worth mentioning here that the paper in [39] dealt with cellular networks of the form

$$\begin{aligned} x_i^{\nabla}(t) &= -c_i(t)x_i(t-\eta_i(t)) + \sum_{j=1}^{n} a_{ij}(t)\widetilde{g}_j(x_j(t-\tau_{ij}(t))) \\ &+ \sum_{j=1}^{n} b_{ij}(t)\int_0^{\infty} k_{ij}(u)g_j(x_j(t-u))\nabla u + I_i,\ i=1,2,\ldots,n. \quad (6.19) \end{aligned}$$

The authors studied the existence and global stability of () with delays in the leakage terms. in [39], on the other hand, the current author considered neutral–type BAM neural networks involving a leakage term with distributed delay. However, it does not include a regulator transmission functions which noticeably play an important role in measuring the rate of activation functions. In recent paper [41], Du et al, investigated a neutral–type neural networks with distributed leakage delay of the form

$$\begin{aligned} (A_i x_i)^{\nabla}(t) &= -a_i(t)\int_0^{\infty} k_i(s)x_i(t-s)\nabla s + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} d_{ij}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i,\ i=1,2,\ldots,n, \end{aligned}$$

where $(A_i x_i)(t) = x_i(t) - \sum_{j=1}^{n} c_{ij}(t)x_i(t-\delta_{ij}(t))$. A main feature of this paper is the incorporation of operator A in the description of the networks and the implementation of new technique based on discrete–continuous analysis method. The authors claim that teh conditions which are proposed in this paper to ensure the existence and stability of solutions of system (1.1) are unlike those established in [39–41] and never been considered before. Indeed, they are entirely new, have their own merit and extend those existing in the literature.

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