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Solution of Nonlinear Singular Initial Value Problem by Differential Transform Method Powered by Adomian Polynomial

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Abstract

In this paper, Differential Transform Method is used as never before to solve nonlinear singular initial value problems represented by certain classes of Emden–Fowler type equations. Unlike the common method of using Differential Transform Method alone to solve a nonlinear differential equation, in this work Adomian Polynomial is used to decompose the nonlinear terms and hence this makes the computation of nonlinear terms very simple. It is observed that the result obtained with the proposed new approach is in good agreement with the exact solution. The advantages of this technique are proved as well.

AMS subject classification: 34G20, 44A99.
Keywords: Emden–Fowler type equation, Adomian Polynomial, Differential Transform Method.

1. Introduction

In recent years, the studies of singular initial-value problems (IVPs) of the type

$$y^{''} + 2x^{-1}y^{'} + x^n = 0, \qquad y(0) = 1, \quad y^{'}(0) = 0 \tag{1}$$

have seeked the attention of many mathematicians and physicists [1, 2, 3, 4, 5, 6]. In this paper, our aim is to study the IVPs of the form

$$y^{''} + p(x)y^{'} + q(x, y(x)) = 0, \qquad y(0) = a, \quad y^{'}(0) = b, \quad x > 0 \tag{2}$$

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The case $q = f(x)g(y)$ corresponds to the Emden-Fowler equations. The Emden-Fowler type of equations are second-order singular initial valued order ordinary differential equations (ODEs) which have been used to model several phenomena such as thermal explosions, stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and thermionic currents in mathematical physics and astrophysics [7, 8, 9]. For variety of forms of $g(y)$, many researchers have investigated the applications of Emden-Fowler equation in various scientific fields.

The function $p(x)$ in (2) may be singular at $x = 0$. The problem (2) extends some well-known IVPs in the literature [10, 11, 12, 13, 14]. In the case of $b = 0$, the existence of the solution for problem (2) has been studied in [15], where the author demonstrated the importance of the condition $b = 0$. Authors in [16], have found the conditions for $p(x)$ and $q(x, y(x))$ to guarantee the existence of the solution for any $b(\in \Re) \neq 0$. Keeping these conditions in view, in this paper we have solved Emden-Fowler type equations by Differential Transform Method, where nonlinear terms are decomposed by using Adomian Polynomial and we call it as Differential Transform Method powered by Adomian Polynomial (DTMAP).

Many methods including numerical and perturbation methods have been used to solve the Emden-Fowler type equations. The approximate solutions to the Emden-Fowler type equations were presented by Shawagfeh [17] and Wazwaz [18, 19] using the Adomian decomposition method (ADM). Also Wazwaz applied ADM to solve the time dependent Emden-Fowler type of equations [20]. Liao solved Lane-Emden type equations by applying homotopy analysis method (HAM)[21]. In [22, 23], the variational iteration method (VIM) [24, 25] is used to solve Emden-Fowler type of equations. Recently, Parand, Dehghan, Rezaei and Ghaderi applied Hermite function collocation (HFC) method [26].

2. Differential Transform Method Powered by Adomian Polynomial

Definition 2.1. Let $y(x)$ be the original analytic function and differentiated continuously in the domain of interest. Then Differential Transform of y(x) is defined as:

$$Y_k = \frac{1}{k!}\Big[\frac{d^k x}{xt^k} y(x)\Big]_{x=0} \tag{3}$$

where $y(x)$ is the original function and Y_k is the transformed function.

Definition 2.2. Differential inverse transform of Y_k is defined as:

$$y(x) = \sum_{k=0}^{\infty} Y_k x^k \tag{4}$$

Combining (3) and (4) we may write

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\left(\frac{d^k y}{dx^k}\right)\right]_{x=0} \tag{5}$$

This implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. Instead, relative derivatives are calculated by a recurrence relation which are described by the transformed equations of the original functions. Some fundamental transformations, which can be readily obtained are listed in the following table.

Table 1: Fundamental Operations in Reduced Differential Transform Method (RDTM)

Original form	Transformed form
$y(x) = w(x) \pm v(x)$	$Y_k = W_k \pm V_k$
$y(x) = \alpha w(x)$	$Y_k = \alpha W_k$
$u(x) = \frac{d^m}{dx^m} w(x)$	$Y_k = \frac{(k+m)!}{k!} W_{k+m}$
$y(x) = x^n$	$Y_k = \delta(k-n)$
	where $\delta(k-n) = \begin{cases} 1 & \text{if k=n} \\ 0 & \text{otherwise} \end{cases}$
$y(x) = w(x)v(x)$	$Y_k = \sum_{r=0}^{k} W_r V_{k-r}$

To illustrate the basic concepts of the DTMAP, we consider a general nonlinear ordinary differential equation with initial conditions of the form

$$Dy(x) + Ny(x) = g(x) \tag{6}$$

with initial conditions

$$\frac{d^i y(0)}{dx^i} = c_i, \quad i = 0, 1, 2, \ldots, m-1$$

where D is the m^{th} order linear differential operator $D = \frac{d^m}{dx^m}$, N represents the general nonlinear differential operator and g(x) is the source term.

According to DTM, we can construct the following iteration formula:

$$(k+1)(k+2)\cdots(k+m)Y_{k+m} = G_k - NY_k$$

with initial condition

$$Y_i = c_i, \quad i = 0, 1, 2, \ldots, m-1$$

But according to DTMAP, we construct the iteration formula as

$$(k+1)(k+2)\cdots(k+m)Y_{k+m} = G_k - A_k \tag{7}$$

with initial condition

$$Y_i = c_i, \quad i = 0, 1, 2, \ldots, m-1 \tag{8}$$

The Adomian Polynomial A_k defined as

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}[N(\sum_{i=0}^{k}\lambda^i Y_i(x,t))]|_{\lambda=0} \tag{9}$$

is the decomposition of the nonlinear operator Ny. The general formula (9) can be decomposed as follows:

$A_0 = N(Y_0)$

$A_1 = Y_1 N(Y_0)$

$A_2 = Y_2 N^{'}(Y_0) + \frac{1}{2!}Y_1^2 N^{''}(Y_0)$

$A_3 = Y_3 N^{'}(Y_0) + Y_1 Y_2 N^{''}(Y_0) + \frac{1}{3!}Y_1^3 N^{'''}(Y_0), \ldots$

Substituting (8) and (9) into (7) and then by iteration we obtain the succeeding values of Y_k. Then, the inverse transformation of the set of values $\{Y_k\}_{k=0}^{n}$ gives the n-term approximation to solution as follows:

$$y_n(x) = \sum_{k=0}^{n} Y_k x^k \tag{10}$$

Therefore the exact solution of the problem is given by

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{11}$$

3. Applications

Example 3.1. Consider the standard Emden-Fowler equation

$$y^{''} + \frac{2}{x}y^{'} + y^n = 0 \tag{12}$$

subject to the initial conditions

$$y(0) = 1, \quad y^{'}(0) = 0 \tag{13}$$

Multiplying bith sides of equation (12) by x,

$$xy^{''} + y^{'} + xy^n = 0 \tag{14}$$

By using above theorem of DTM and the DTMAP we obtained the following recurrence relation

$$Y_{k+1} = -\frac{A_{k-1}}{(k+1)(k+2)}, \quad k \geq 1 \tag{15}$$

where A_k represented the Adomian Polynomial applied for decomposing the nonlinear term such that

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\Big(\sum_{i=0}^{k}\lambda^i Y_i\Big)^n \tag{16}$$

From Eq. (3), the initial conditions given in Eq. (13) can be transformed as

$$Y_0 = 1, \quad Y_1 = 0 \tag{17}$$

Substituting Eq. (15) and Eq. (16) into Eq. (14) and then by direct iteration steps we obtain the following:

$$\begin{aligned}
&Y_0 = 1, \quad Y_1 = 0, \quad Y_2 = -\frac{1}{6}, \quad Y_3 = 0, \quad Y_4 = \frac{n}{120}, \quad Y_5 = 0, \quad Y_6 = -\frac{n(8n-5)}{3(7!)},\\
&Y_7 = 0, \quad Y_8 = \frac{n\left(122n^2 - 183n + 70\right)}{9(9!)}, \quad Y_9 = 0,\\
&Y_{10} = -\frac{n\left(5032n^3 - 12642n^2 + 10805n - 3150\right)}{45(11!)}, \ldots
\end{aligned} \tag{18}$$

For $n = 0$ and $n = 1$ Eq. (11) gives the exact solution

$$y(x) = 1 - \frac{1}{6}x^2$$

$$y(x) = \frac{\sin x}{x}$$

Example 3.2. Consider the following Emden-Fowler type equation

$$y'' + \frac{2}{x}y' + 4(2e^y + e^{y/2}) = 0, \quad x \geq 0 \tag{19}$$

subject to the initial conditions

$$y(0) = 0, \quad y'(0) = 0 \tag{20}$$

Multiplying both sides of Eq. (19) by x and then applying the above definitions and formulas of DTM and DTM powered by Adomian Polynomial, we obtain the following recurrence relation:

$$Y_{k+1} = -\frac{4A_{k-1}}{(k+1)(k+2)}, \quad k \geq 1 \tag{21}$$

and the transformed initial condition is

$$Y_0 = 0, \quad Y_1 = 0 \tag{22}$$

The nonlinear operator $N(y) = 2e^y + e^{y/2}$ is decomposed by using the general formula of Adomian Polynomial as defined in Eq. (9) and may be written as follows:

$$A_0 = 2e^{Y_0} + e^{Y_0/2}$$

$$A_1 = Y_1\left(2e^{Y_0} + e^{Y_0/2}\right)$$

$$A_2 = Y_2\left(2e^{Y_0} + \frac{1}{2}e^{Y_0/2}\right) + Y_1^2\frac{1}{2!}\left(2e^{Y_0} + \frac{1}{2^2}e^{Y_0/2}\right)$$

$$A_3 = Y_3\left(2e^{Y_0} + \frac{1}{2}e^{Y_0/2}\right) + Y_1Y_2\left(2e^{Y_0} + \frac{1}{2^2}e^{Y_0/2}\right) + \frac{1}{3!}Y_1^3\left(2e^{Y_0} + \frac{1}{2^3}e^{Y_0/2}\right)$$

$$A_4 = Y_4\left(2e^{Y_0} + \frac{1}{2}e^{\frac{Y_0}{2}}\right) + Y_1Y_3\left(2e^{Y_0} + \frac{1}{2^2}e^{\frac{Y_0}{2}}\right) + \frac{1}{2}Y_1^2Y_2\left(2e^{Y_0} + \frac{1}{2}e^{\frac{Y_0}{2}}\right)$$
$$+\frac{1}{2}Y_2^2\left(2e^{Y_0} + \frac{1}{2^3}e^{\frac{Y_0}{2}}\right) + \frac{1}{4!}Y_1^4\left(2e^{Y_0} + \frac{1}{2^4}e^{\frac{Y_0}{2}}\right), \ldots$$

Using the Eq. (22) and Adomian Polynomial defined above in Eq. (21) and then from Eq. (11) we obtain the solution in the form

$$y(x) = -2\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \frac{x^{10}}{5} - \cdots\right) \tag{23}$$

Hence the solution may be written in the form

$$y(x) = -2\ln\left(1 + x^2\right) \tag{24}$$

Example 3.3. We finally consider the following Emden-Fowler type equation

$$y^{''} + \frac{2}{x}y^{'} - 6y = 4y\ln y, \quad x \geq 0 \tag{25}$$

subject to the initial conditions

$$y(0) = 1, \quad y^{'}(0) = 0 \tag{26}$$

Multiplying both sides of Eq. (25) by x and then applying the above definitions and formulas of DTM and DTMAP, we obtain the following recurrence relation:

$$Y_{k+1} = \frac{4A_{k-1} + 6Y_{k-1}}{(k+1)(k+2)}, \quad k \geq 1 \tag{27}$$

and the transformed initial condition is

$$Y_0 = 1, \quad Y_1 = 0 \tag{28}$$

The nonlinear operator $N(y) = y\ln y$ is decomposed by using the general formula of Adomian Polynomial as defined in Eq. (9) and may be written as follows:

$$A_0 = Y_0\ln(Y_0)$$

$$A_1 = Y_1(\ln(Y_0) + 1)$$

$$A_2 = \frac{Y_1^2}{2Y_0} + Y_2 (\ln (Y_0) + 1)$$

$$A_3 = -\frac{Y_1^3}{6Y_0^2} + \frac{Y_2 Y_1}{Y_0} + Y_3 (\ln (Y_0) + 1), \ldots$$

Using the Eq. (28) and Adomian Polynomial defined above in Eq. (27) we obtain the terms: $Y_2 = 1$, $Y_3 = 0$, $Y_4 = \frac{1}{2}$, $Y_5 = 0$, $Y_6 = \frac{1}{6}$, $Y_7 = 0$, $Y_8 = \frac{1}{24}$, and so on. This results the series solution of Eq. (25) as follows:

$$y(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \tag{29}$$

Hence the exact solution of Eq. (25) is

$$y(x) = e^{x^2} \tag{30}$$

4. Conclusion

In this study,a class of singular initial value problems represented by Emden-Fowler types equations are solved by successfully applying Differential Transform Method powered by Adomian Polynomial. It is observed that, when nonlinear terms are decomposed by using Adomian Polynomial, instead of DTM, the recurrence relation are obtained in very simplified form. This makes the further computations very straightforward and simple.The chosen numerical examples emphasized our belief that DTMAP is a powerful technique to handle any types of linear and nonlinear initial value problems.

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