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Fast transition layers of weakly compressible twophase flow equations near initial time[1]

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Abstract

We discuss the motion of weakly compressible twophase flow near the initial time. There exist fast transition-layers in higher order of the inner limit process. In the transitional regions, we derive formal asymptotic expansions for the solutions of compressible equations. Specifically, under the singular limit process, we find the fast transitional variables in closed form.

AMS subject classification: 76M45, 35B40, 35C20, 76Txx.
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1. Introduction

We discuss the motion of weakly compressible twophase flow models [2, 6, 7, 14, 15] near the initial time. The singular limit process of compressible twophase fluid flow is studied as the Mach number goes to zero. The incompressible limit of the compressible twophase flow equations is a time-singular and layer-type problem which requires advanced techniques in asymptotics [16]. The incompressible limit of the single phase compressible Euler or Navier-Stokes equations has been studied in higher space dimensions [3, 8, 17, 18]. A uniformly valid asymptotic expansion describing a singular limit process exists uniquely. Each order of asymptotic expansions for the solutions of the compressible flow representing the incompressible limit process has an independent existence, defined as proportional to a derivative of the compressible solution with respect

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to λ, the reciprocal of the Mach number, evaluated at the value $\lambda = 0$ of the expansion parameter. The uniformly valid outer limit asymptotic expansions have been derived [4, 13] for the compressible twophase flow solutions describing the fluids away from the initial time. The slow variables with a slow scale of motion exist in the outer expansions and have been determined through second order in closed form. This is the order the incompressible pressure first appears. This paper is concentrated on the inner limit process of weakly compressible twophase flow describing the fluids near the initial time. The fast variables in the inner limit expansions contain fast scale acoustical oscillations on the fast time scale. Moreover, in higher order in λ^{-1}, we are concerned here, more of fast transition-layers exist in the inner limit process. We derive the fast transition-layer expansions in the regions and determine the fast transitional variables which are used to derive uniformly valid inner limit asymptotic expansions by matching.

The compressible isentropic ideal twophase flow are nondimensionalized in the form of a nonlinear hyperbolic system

$$\frac{\partial \beta_k^\lambda}{\partial t} + v^{*\lambda}\frac{\partial \beta_k^\lambda}{\partial z} = 0, \tag{1.1}$$

$$\beta_k^\lambda\left(\frac{\partial \rho_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial \rho_k^\lambda}{\partial z}\right) + \beta_k^\lambda\rho_k^\lambda\frac{\partial v_k^\lambda}{\partial z} + \rho_k^\lambda(v_k^\lambda - v^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = 0, \tag{1.2}$$

$$\beta_k^\lambda\rho_k^\lambda\left(\frac{\partial v_k^\lambda}{\partial t} + v_k^\lambda\frac{\partial v_k^\lambda}{\partial z}\right) + \lambda^2\beta_k^\lambda\frac{\partial p_k^\lambda}{\partial z} + \lambda^2(p_k^\lambda - p^{*\lambda})\frac{\partial \beta_k^\lambda}{\partial z} = \beta_k^\lambda\rho_k^\lambda g(t) \tag{1.3}$$

for the volume fraction β_k^λ, velocity v_k^λ, density ρ_k^λ, and pressure p_k^λ of fluid k, depending on a large dimensionless parameter λ. Here the fluids are distinguished by a subscript k, *i.e.*, $k = 1$ for the light fluid and $k = 2$ for the heavy fluids, $g = g(t) > 0$ is the gravity, $p_k^\lambda = p_k(\rho_k^\lambda)$, an equation of state $p_k(\rho_k) = A_k\rho_k^{\gamma_k}$, $\gamma_k > 1$ is given with $\partial p_k/\partial\rho_k(\rho_k) > 0$ for $\rho_k > 0$ and the entropy A_k assumed to be constant within each fluid but $A_1 \neq A_2$. The parameter $\lambda = M^{-1}(\gamma A)^{-1/2}$ is the reciprocal of the Mach number, $M = |v_m|(\gamma p(\rho_m)/\rho)^{-1/2}$, the ratio of fluid speed to sound speed, where ρ_m is the mean density and $|v_m|$ is a typical mean fluid velocity which is the ratio of time units to space units.

The interfacial quantities v^* and p^* of twophase flow have been proposed [6, 7, 13, 14] by closure relations

$$q^* = \mu_1^q q_2 + \mu_2^q q_1, \quad q = v, p, \tag{1.4}$$

$$\mu_k^q(\beta_k, d_k^q) = \frac{\beta_k}{\beta_k + d_k^q\beta_{k'}}. \tag{1.5}$$

Here the primed index k' denotes the fluid complementary to fluid k, i.e., $k' = 3 - k$. We note that $\mu_1^q + \mu_2^q = 1$ and $\mu_k^q \geq 0$ and that $\mu_k^q/\beta_k$is continuous on $0 \leq \beta_k \leq 1$ and for all t. The μ_k^q thus depends on a single parameter d_k^q. A closure for the constitutive law $d_k^q(t)$ was proposed in [14, 15] and it was compared in a validation study based on simulation data [1, 19].

The incompressible flow in the volume fraction β_k^∞, velocity v_k^∞, and scalar pressure p_k^∞, defined by the equations

$$\begin{aligned}\frac{\partial \beta_k^\infty}{\partial t} + v^{*\infty}\frac{\partial \beta_k^\infty}{\partial z} &= 0,\\ \beta_k^\infty \frac{\partial v_k^\infty}{\partial z} + (v_k^\infty - v^{*\infty})\frac{\partial \beta_k^\infty}{\partial z} &= 0,\\ \beta_k^\infty \rho_k^\infty \left(\frac{\partial v_k^\infty}{\partial t} + v_k^\infty \frac{\partial v_k^\infty}{\partial z}\right) + \beta_k^\infty \frac{\partial p_k^\infty}{\partial z} + (p_k^\infty - p^{*\infty})\frac{\partial \beta_k^\infty}{\partial z} &= \beta_k^\infty \rho_k^\infty g(t)\end{aligned} \tag{1.6}$$

are considered as a background flow of the incompressible limit, $\lambda \to \infty$. Here ρ_k^∞ is the constant density of phase k. We assume that all state variables are piecewise C^1 functions with discontinuous derivatives at the mixing zone edges $z = Z_k^\infty(t)$ of incompressible flow. Analytic solutions of the incompressible problem (1.6) have been obtained in closed form [5, 6, 7].

Here we are concerned with the limiting motion of the solution $U_k^\lambda \equiv (\beta_k^\lambda, v_k^\lambda, \rho_k^\lambda, p_k^\lambda)^{tr}$ of the compressible equations (1.1)-(1.3) near the initial time as $\lambda \to \infty$. Specifically under the singular limit process, we derive inner limit asymptotic expansions for the solutions of the compressible equations. Jin [10], repeated in Sec. 3, discussed the inner terms, uniformly valid in space, in closed form to $O(\lambda^{-1})$ for β_k, ρ_k, p_k and to $O(1)$ for v_k. It showed that there is no fast scale acoustical oscillation in the asymptotic expansions of β_k, ρ_k, p_k through first order and in the order zero expansion of v_k. In [11], the result showed the evaluation of the inner terms through second order in the expansion of volume fraction, density and pressure, and first order in the expansion of velocity in the exterior domain and within the mixing zone. In higher order in λ^{-1}, we are concerned here, there are four transition-layers having new fast transitional variables. The main result of this paper is to derive the fast transition-layer expansions in the regions and to determine the fast transitional variables in closed form. In Sec. 4.1, we evaluate the fast transitional variables in the second fast transition-layer expansions. In Sec. 4.2, the first transition-layer expansions are determined up to higher order, we are concerned here. These fast transition-layer expansions will be used to determine the uniformly valid inner limit terms by matching the inner limit in the exterior and the incompressible mixing zone and the fast transition-layers expansions [12]. In Sec. 2 we present the asymptotic assumptions and conditions for the constitutive laws and the boundaries, which are required to derive formal asymptotic expansions for compressible solutions. For simplicity, we suppress superscript λ's of compressible variables from now on.

2. Preliminary Conditions

We specify boundary conditions for compressible and incompressible flow,

$$v_1(z^{+\infty}) = 0\ , \quad p_2(z^{-\infty}) = \text{const}, \tag{2.7}$$

where $z = z^{+\infty}(z = z^{-\infty})$ denotes the position of the upper (lower) wall of a finite but large domain $\mathcal{D}$.

We introduce the asymptotic expansions of compressible solutions in the form

$$\begin{aligned}
\beta_k &= \beta_k^{(0,s)} + \lambda^{-1}\beta_k^{(1,s)} + \lambda^{-2}\left(\beta_k^{(2,s)} + \beta_k^{(2,f)}\right) + O(\lambda^{-3}),\\
v_k &= v_k^{(0,s)} + \lambda^{-1}\left(v_k^{(1,s)} + v_k^{(1,f)}\right) + O(\lambda^{-2}), \qquad (2.8)\\
\rho_k &= \rho_k^{(0,s)} + \lambda^{-1}\rho_k^{(1,s)} + \lambda^{-2}\left(\rho_k^{(2,s)} + \rho_k^{(2,f)}\right) + O(\lambda^{-3}).
\end{aligned}$$

The equation of state gives the expansion

$$\begin{aligned}
p_k &= p_k^{(0,s)} + \lambda^{-1}p_k^{(1,s)} + \lambda^{-2}\left(p_k^{(2,s)} + p_k^{(2,f)}\right) + O(\lambda^{-3})\\
&= p_k(\rho_k^{(0,s)}) + \lambda^{-1}c_k^2(\rho_k^{(0,s)})\rho_k^{(1,s)}\\
&\quad + \lambda^{-2}\left(\frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\rho_k^{(0,s)})\rho_k^{(1,s)2} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,s)} + c_k^2(\rho_k^{(0,s)})\rho_k^{(2,f)}\right) + O(\lambda^{-3}),
\end{aligned} \qquad (2.9)$$

where $c_k^2(\rho) = \dfrac{\partial p_k}{\partial \rho_k}(\rho)$. The compressible solutions are assumed to have the initial conditions

$$\begin{aligned}
\beta_k(z,0) &= \beta_k^{\infty}(z,0) + \lambda^{-1}\beta_k^{(1,s)}(z,0) + \lambda^{-2}\beta_k^{(2,s)}(z,0),\\
v_k(z,0) &= v_k^{\infty}(z,0) + \lambda^{-1}\left[v_k^{(1,s)}(z,0) + v_k^{(1)}(z)\right],\\
\rho_k(z,0) &= \rho_k^{\infty}(z,0) + \lambda^{-2}\left[c_k^{-2}(\rho_k^{\infty})\rho_k^{\infty}(z,0) + \rho_k^{(2)}(z)\right], \qquad (2.10)\\
p_k(z,0) &= p_k(\rho_k^{\infty}) + \lambda^{-2}\left[p_k^{\infty}(z,0) + p_k^{(2)}(z)\right],
\end{aligned}$$

where $v_k^{(1)}(z)$, $\rho_k^{(2)}(z)$, and $p_k^{(2)}(z)$ belong to C^1 on $(-1)^k z \le (-1)^k Z_k(0)$ and $|||v_k^{(1)}(z)||| = O(1)$, $|||\rho_k^{(2)}(z)||| = O(1)$ and $|||p_k^{(2)}(z)||| = O(1)$. The variables $U_k^{(m,s)} \equiv \left(\beta_k^{(m,s)}, v_k^{(m,s)}, \rho_k^{(m,s)}, p_k^{(m,s)}\right)^{tr}$, $m = 0, 1, 2$, have a slow scale of motion and they have been determined in closed form through second order [13].

We denote $Z_k = Z_k(t)$ as the position of the mixing zone edge k, defined as the location of vanishing β_k and $V_k = dZ_k/dt$ as the velocity of the edge k. At edge k, the following boundary data holds $v_k = V_k(t)$ at $z = Z_k(t)$. The two phase flow model depends on the motions of the mixing zone edges Z_k and closure for the interfacial averages with the constitutive law d_k^q, $q = v, p$. The mixing zone edges Z_k are not well characterized for compressible flows. Thus the velocities or trajectories of the edges of the mixing zone must be provided as data. Here we asymptotically assume the velocities or trajectories of the edges of the mixing zone with a specific limit term. A uniformly valid asymptotic expansion for the compressible mixing zone edge is assumed as the following

$$Z_k(t) = Z_k^{(0,s)}(t) + \lambda^{-1}Z_k^{(1,s)}(t) + \lambda^{-2}\left(Z_k^{(2,s)}(t) + Z_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \qquad (2.11)$$

Here $\sum_{j=0}^{m} \lambda^{-j} Z_k^{(j,s)}$, $m = 0, 1, 2$, denotes the location of vanishing $\beta_k^{(m,s)}$. Thus Z_k and each of the expansion coefficients $Z_k^{(m,s)}$ and $Z_k^{(2,f)}$ are input to the model equations. We assume that the compressible edge moves faster than the incompressible edge with no initial perturbation. A similar assumption is applied to any finite number of terms in the expansion (2.10). We assume that the zero-th order term in the expansion (2.10) equals to the incompressible edge trajectory $Z_k^{\infty}(t)$. Thus we require

$$\begin{aligned} Z_k^{(0,s)}(t) = Z_k^{\infty}(t),\ Z_k(0) = Z_k^{\infty}(0), \\ (-1)^k Z_k^{(m,t)}(t) \geq 0,\ m = 1, 2,\ t = s, f. \end{aligned} \tag{2.12}$$

The variable $Z_k^{(2,f)}$ is oscillatory on the fast time scale $\tau \equiv \lambda t$ while the terms $Z_k^{(m,s)}$, $m = 0, 1, 2$, are the slow variables with a slow scale of motion. We assume that the fast variable $Z_k^{(2,f)}$ decays exponentially in τ away from the initial curve $t = 0$. The formally uniformly valid expansion (2.11) leads to the inner and outer expansion under the corresponding limit process. The reduced expansion is assumed in the derivation of inner and outer expansions for compressible solutions. Following the inner limit process $\lambda \to \infty$ with τ fixed $\neq \infty$, (2.11) leads to the inner limit expansion, valid near $t = 0$:

$$\begin{aligned} Z_k(\tau) &= \widehat{Z}_k^{(0)}(\tau) + \lambda^{-1}\widehat{Z}_k^{(1)}(\tau) + \lambda^{-2}\widehat{Z}_k^{(2)}(\tau) + O(\lambda^{-3}) \\ &= Z_k^{\infty}(0) + \lambda^{-1}\tau \dot{Z}_k^{\infty}(0) \\ &+ \lambda^{-2}\left(\frac{\tau^2}{2}\ddot{Z}_k^{\infty}(0) + \tau\dot{Z}_k^{(1,s)}(0) + Z_k^{(2,s)}(0) + Z_k^{(2,f)}(\tau)\right) + O(\lambda^{-3}), \end{aligned} \tag{2.13}$$

where $dZ_k/dt = \dot{Z}_k$ and $d^2Z_k/dt^2 = \ddot{Z}_k$. For details, refer to [9, 10]. The initial conditions associated with (2.12) are

$$\widehat{Z}_k^{(0)}(0) = Z_k^{\infty}(0),\ \widehat{Z}_k^{(m)}(0) = 0,\ m \geq 1. \tag{2.14}$$

Notice that the fast variable $Z_k^{(2,f)}$ consist of the inner term $\widehat{Z}_k^{(2)}$ minus common terms to order λ^{-2}. Under the outer limit process, $\lambda \to \infty$ with t fixed $\neq 0$, (2.11) leads to the outer expansion

$$Z_k(t) = Z_k^{\infty}(t) + \lambda^{-1}Z_k^{(1,s)}(t) + \lambda^{-2}Z_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{2.15}$$

valid away from the initial curve $t = 0$.

Since the edge velocity of the compressible flow satisfies $V_k = \dot{Z}_k = v_k(Z_k, t)$, it must have an asymptotic expansion associated with the expansion (2.10) in the form

$$\begin{aligned} V_k(t) =& V_k^{\infty}(t) + \lambda^{-1}\left(V_k^{(1,s)}(t) + V_k^{(1,f)}(t,\lambda)\right) \\ &+ \lambda^{-2}\left(V_k^{(2,s)}(t) + V_k^{(2,f)}(t,\lambda)\right) + O(\lambda^{-3}). \end{aligned} \tag{2.16}$$

The formal asymptotic expansion (2.16) leads to the inner limit expansion

$$\begin{aligned} V_k(\tau) &= \widehat{V}_k^{(0)}(\tau) + \lambda^{-1}\widehat{V}_k^{(1)}(\tau) + O(\lambda^{-2}) \\ &= V_k^{\infty}(0) + \lambda^{-1}\left(\tau\dot{V}_k^{\infty}(0) + V_k^{(1,s)}(0) + V_k^{(1,f)}(\tau)\right) + O(\lambda^{-2}). \end{aligned} \tag{2.17}$$

which is valid near the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with τ fixed $\neq \infty$, and to the outer limit expansion

$$V_k(t) = V_k^{(0,s)}(t) + \lambda^{-1}V_k^{(1,s)}(t) + \lambda^{-2}V_k^{(2,s)}(t) + O(\lambda^{-3}) \tag{2.18}$$

which is valid away from the initial curve $t = 0$ under the limit process $\lambda \to \infty$ with t fixed $\neq 0$.

The compressible constitutive laws $d_k^v(t)$ and $d_k^p(t)$ are asymptotically assumed with a specific limit term as follows

$$d_k^q(t,\lambda) = d_k^{q(0,s)}(t) + \lambda^{-1}\left(d_k^{q(1,s)}(t) + d_k^{q(1,f)}(t,\lambda)\right) + O(\lambda^{-2}),\ q = v, p, \tag{2.19}$$

where $d_k^{q(m,s)}(t)$, $d_k^{q(m,f)}(\tau) \in C([0,\infty))$, and we assume $d_k^{q(0,s)}(t) = d_k^{q\infty}(t)$. Similarly, we obtain [9] that the expansion (2.19) leads to the inner limit asymptotic expansion

$$\begin{aligned} d_k^q(\tau) &= \widehat{d}_k^{q(0)}(\tau) + \lambda^{-1}\widehat{d}_k^{q(1)}(\tau) + (\lambda^{-2}) \\ &= d_k^{q(0,s)}(0) + \lambda^{-1}\left(\tau\frac{dd_k^q(0,s)}{dt}(0) + d_k^{q(1,s)}(0) + d_k^{q(1,f)}(\tau)\right) + O(\lambda^{-2}), \end{aligned} \tag{2.20}$$

and to the outer limit expansion

$$d_k^q(t) = d_k^{q(0,s)}(t) + \lambda^{-1}d_k^{q(1,s)}(t) + \lambda^{-2}d_k^{q(2,s)}(t) + O(\lambda^{-3}),\ q = v, p. \tag{2.21}$$

3. Expansions and Transition Layers

We want to understand the motion of the fast variables by making change of variables to the fast time scale $\tau \equiv \lambda t$. Then the compressible equations (1.1)-(1.3) reduce to

$$\lambda\frac{\partial\beta_k}{\partial\tau} + v^*\frac{\partial\beta_k}{\partial z} = 0, \tag{3.22}$$

$$\beta_k\left(\lambda\frac{\partial\rho_k}{\partial\tau} + v_k\frac{\partial\rho_k}{\partial z}\right) + \beta_k\rho_k\frac{\partial v_k}{\partial z} + \rho_k(v_k - v^*)\frac{\partial\beta_k}{\partial z} = 0, \tag{3.23}$$

$$\beta_k\rho_k\left(\lambda\frac{\partial v_k}{\partial\tau} + v_k\frac{\partial v_k}{\partial z}\right) + \lambda^2\beta_k\frac{\partial p_k}{\partial z} + \lambda^2(p_k - p^*)\frac{\partial\beta_k}{\partial z} = \beta_k\rho_k g(t), \tag{3.24}$$

for $U_k(z,\tau)$. We introduce inner limit asymptotic expansions associated with the inner limit $\lambda \to \infty$ with τ fixed $\neq \infty$:

$$U_k(z,\tau) = \widehat{U}_k^{(0)}(z,\tau) + \lambda^{-1}\widehat{U}_k^{(1)}(z,\tau) + \lambda^{-2}\widehat{U}_k^{(2)}(z,\tau) + O(\lambda^-3), \tag{3.25}$$

where $\widehat{U}_k^{(m)}(z,\tau) \equiv \left(\widehat{\beta}_k^{(m)}, \widehat{v}_k^{(m)}, \widehat{\rho}_k^{(m)}, \widehat{p}_k^{(m)}\right)^{tr}$, $m = 0, 1, 2, \cdots$. The equation of state gives the expansion relations

$$\widehat{p}_k^{(0)} = p_k(\widehat{\rho}_k^{(0)}), \quad \widehat{p}_k^{(1)} = c_k^2(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(1)}, \quad \widehat{p}_k^{(2)} = \frac{1}{2}\frac{\partial^2 p_k}{\partial \rho_k^2}(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(1)2} + c_k^2(\widehat{\rho}_k^{(0)})\widehat{\rho}_k^{(2)} \tag{3.26}$$

between the terms in the expansions of ρ_k and p_k. From (3.25) and (3.26), the inner limit expansion for the mixing coefficient $\mu_k^q(\beta_k, d_k^q)$, $q = v, p$ is introduced. Refer to [10].

In higher order in λ^{-1}, there exist fast transitional layers in the inner expansion similarily to the transition regions in the outer asymptotic expansion [13]. The first order inner expansion is defined by five regions, $\widehat{\mathcal{E}}_k^{(1)} \cup \widehat{\mathcal{T}}_k^{(1)} \cup \widehat{\mathcal{M}} \cup \widehat{\mathcal{T}}_{k'}^{(1)} \cup \widehat{\mathcal{E}}_{k'}^{(1)}$, including two transition-layers through $z = \overline{\widehat{Z}_i^{(1)}}$, $i = k, k'$. In second order, we have two additional transition-layers extending out to $z = \overline{\widehat{Z}_i^{(2)}}$, so the second order expansion is uniquely defined by seven regions $\widehat{\mathcal{E}}_k^{(2)} \cup \widehat{\mathcal{T}}_k^{(2)} \cup \widehat{\mathcal{T}}_k^{(1)} \cup \widehat{\mathcal{M}} \cup \widehat{\mathcal{T}}_{k'}^{(1)} \cup \widehat{\mathcal{T}}_{k'}^{(2)} \cup \widehat{\mathcal{E}}_{k'}^{(2)}$. The regions are defined by

$$\widehat{\mathcal{E}}_{i'}^{(n)} = \left\{(z,t) : (-1)^i \overline{\widehat{Z}_i^{(n)}} \le (-1)^i z\right\}, \tag{3.27}$$

$$\widehat{\mathcal{T}}_{i'}^{(n)} = \left\{(z,t) : (-1)^i \overline{\widehat{Z}_i^{(n-1)}} \le (-1)^i z < (-1)^i \overline{\widehat{Z}_i^{(n)}}\right\}, \ n = 1, 2, \tag{3.28}$$

$$\widehat{\mathcal{M}} = \left\{(z,t) : \widehat{Z}_1^{(0)} < z < \widehat{Z}_2^{(0)}\right\}, \tag{3.29}$$

where

$$\overline{\widehat{Z}_k^{(m)}}(t) \equiv \sum_{j=0}^{m} \lambda^{-j} \widehat{Z}_k^{(j)}, \quad m = 0, 1, 2 \tag{3.30}$$

denotes the position of boundaries of the fast transition-layers.

With the new inner spatial variables

$$\widehat{\zeta}_i^{(n)} = \lambda^n \left(z - \overline{\widehat{Z}_i^{(n)}}\right), \ n = 1, 2, \quad i = k, k', \tag{3.31}$$

we make the change of variables from (z,τ) to $(\widehat{\zeta}_i^{(n)}, \tau)$, reducing (3.22)-(3.24) to the equations

$$\lambda \frac{\partial \beta_k}{\partial \tau} - \lambda^n \left(\overline{\widehat{V}_i^{(n)}} - v^*\right) \frac{\partial \beta_k}{\partial \widehat{\zeta}_i^{(n)}} = 0, \tag{3.32}$$

$$\beta_k \left\{\lambda \frac{\partial \rho_k}{\partial \tau} - \lambda^n \left(\overline{\widehat{V}_i^{(n)}} - v_k\right) \frac{\partial \rho_k}{\partial \widehat{\zeta}_i^{(n)}}\right\} + \lambda^n \beta_k \rho_k \frac{\partial v_k}{\partial \widehat{\zeta}_i^{(n)}} + \lambda^n \rho_k (v_k - v^*) \frac{\partial \beta_k}{\partial \widehat{\zeta}_i^{(n)}} = 0, \tag{3.33}$$

$$\beta_k \rho_k \left\{\lambda \frac{\partial v_k}{\partial \tau} - \lambda^n \left(\overline{\widehat{V}_i^{(n)}} - v_k\right) \frac{\partial v_k}{\partial \widehat{\zeta}_i^{(n)}}\right\} + \lambda^{n+2} \beta_k \frac{\partial p_k}{\partial \widehat{\zeta}_i^{(n)}} + \lambda^{n+2}(p_k - p^*) \frac{\partial \beta_k}{\partial \widehat{\zeta}_i^{(n)}} = \beta_k \rho_k g. \tag{3.34}$$

Here the edge velocity of the transition-layers is defined by

$$\overline{\widehat{V}_k^{(n)}}(t) \equiv \frac{d\overline{\widehat{Z}}_k^{(n)}}{dt} = \sum_{j=0}^{n} \lambda^{-j}\widehat{V}_k^{(j)}. \tag{3.35}$$

We assume the second transitional layer expansions of the form

$$U_k(\widehat{\zeta}_i^{(2)}, \tau) = U_k^{(0,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) + \lambda^{-1}U_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) + \lambda^{-2}U_k^{(2,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) + O(\lambda^{-3}) \tag{3.36}$$

in the region $(-1)^{i+1}\widehat{Z}_i^{(2)} \leq (-1)^i\widehat{\zeta}_i^{(2)} \leq 0$ and the first fast transitional layer expansions

$$U_k(\widehat{\zeta}_i^{(1)}, \tau) = U_k^{(0,ft)}(\widehat{\zeta}_i^{(1)}, \tau) + \lambda^{-1}U_k^{(1,ft)}(\widehat{\zeta}_i^{(1)}, \tau) + \lambda^{-2}U_k^{(2,ft)}(\widehat{\zeta}_i^{(1)}, \tau) + O(\lambda^{-3}) \tag{3.37}$$

in the region $(-1)^{i+1}\widehat{Z}_i^{(1)} \leq (-1)^i\widehat{\zeta}_i^{(1)} \leq 0$. Our concern is to derive the fast transition-layer asymptotic expansions (3.36) and (3.37) uniformly in the transition-layers $\widehat{\mathfrak{T}}_i^{(n)}$, $n = 1, 2$, $i = k, k'$ for the solutions of the compressible equations. The fast transitional variables of each order of λ^{-1} in the expansions (3.36) and (3.37) satisfy simple differential equations. Specifically, the second transitional variables $U_k^{(m,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) = \left(\beta_k^{(m,ftt)}, v_k^{(m,ftt)}, \rho_k^{(m,ftt)}, p_k^{(m,ftt)}\right)^{tr}$, $m = 0, 1, 2$, are solved through second order in Sec. 4.1. Using the first transitional terms $v_k^{(0,ft)}$, $\beta_k^{(m,ft)}$, $\rho_k^{(m,ft)}$, $p_k^{(m,ft)}$, $m = 0, 1$, previously determined in [10], we find the second order term $v_k^{(1,ft)}$, $\beta_k^{(2,ft)}$, $\rho_k^{(2,ft)}$, $p_k^{(2,ft)}$ in the first transition-layer expansions (3.37) in Sec. 4.2. These fast transitional variables are matched continuously order by order at the boundaries of these layers [12]. The matching process of inner limit and fast transitional expansions determines uniformly valid inner expansions in z.

3.1. Transitional Effective Variable

We solve a simple system of ODEs to be used in the analysis of the fast transitional terms $U_k^{(m,ftt)}$, $m = 0, 1, 2$, in Sec. 4.1 and $U_k^{(2,ft)}$ in Sec. 4.2. Let the effective transitional variable $q_k^{(t)}$ satisfy the system

$$\beta_k^{(t)}\frac{\partial q_k^{(t)}}{\partial \zeta_i} + \mu_k^{q(t)}\left(q_k^{(t)} - q_{k'}^{(t)}\right)\frac{\partial \beta_k^{(t)}}{\partial \zeta_i} = 0 \tag{3.38}$$

n the region $(-1)^{i+1}Z_i^{(1)} \leq (-1)^i\zeta_i \leq 0$. Here $\beta_k^{(t)}(\zeta_i, t)$ is a given C^1 function in the region and

$$\mu_k^{q(t)}(\beta_k^{(t)}, d_k^{q(0,s)}) = \frac{\beta_k^{(t)}}{\beta_k^{(t)} + d_k^{q(0,s)}\beta_{k'}^{(t)}} \tag{3.39}$$

satisfying $\mu_k^{q(t)} + \mu_{k'}^{q(t)} = 1$. This system can be decoupled and solved in closed form [13, 10] by the introduction of two linear combinations of the effective transitional variables as follows

Proposition 3.1. Let $q_k^{(t)}(\zeta_i, t)$ satisfy the system (3.38) in $(-1)^{i+1} Z_i^{(1)} \le (-1)^i \zeta_i \le 0$. Then the solution is

$$q_k^{(t)}(\zeta_i, t) = \frac{\mu_k^{q(t)}}{\beta_k^{(t)}} \left\{ \left[\beta_{k'}^{(t)}(-Z_i^{(1)}, t) - \beta_{k'}^{(t)} \right] q_{k'}^{(t)}(-Z_i^{(1)}, t) \right. \\ \left. + \left[\beta_k^{(t)}(-Z_i^{(1)}, t) + d_k^{q(t)} \beta_{k'}^{(t)} \right] q_k^{(t)}(-Z_i^{(1)}, t) \right\}. \tag{3.40}$$

3.2. Inner Terms up to First Order

The inner terms $\widehat{v}_k^{(0)}, \widehat{\beta}_k^{(m)}, \widehat{\rho}_k^{(m)}, \widehat{p}_k^{(m)}, m = 0, 1$, in the inner limit asymptotic expansions (3.25) have been found uniformly valid in z in closed form [10]. Specifically, the inner limit terms $\widehat{v}_k^{(0)}, \widehat{\beta}_k^{(0)}, \widehat{\rho}_k^{(0)}$ are completely determined by the initial data of incompressible flow in (2.10). Also it has been shown that there are no fast scale acoustical oscillation in the asymptotic expansions of β_k, ρ_k, p_k through first order and in the order zero expansion of v_k. The results are used in the evaluation of the higher order transitional terms in Secs. 4.1 and 4.2 and are summarized in the following theorem.

Theorem 3.2. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Then the compressible solutions have the uniformly valid inner expansions in z to $O(\lambda^{-1})$ for β_k, ρ_k, p_k and to $O(1)$ for v_k of the form

$$\begin{aligned} \beta_k &= \beta_k^{\infty}(z, 0) + \lambda^{-1} \widehat{\beta}_k^{(1)} + O(\lambda^{-2}), \\ v_k &= v_k^{\infty}(z, 0) + O(\lambda^{-1}), \\ \rho_k &= \rho_k^{\infty} + O(\lambda^{-2}), \\ p_k &= p^{(0)} + O(\lambda^{-2}), \end{aligned} \tag{3.41}$$

where $p^{(0)} = p_k(\rho_k^{\infty})$ and the inner term $\widehat{\beta}_k^{(1)}(z, \tau)$ satisfies

$$\widehat{\beta}_k^{(1)}(z, \tau) = \begin{cases} 0 & \text{in } \widehat{\mathcal{E}}_1^{(1)},\ \widehat{\mathcal{E}}_2^{(1)} \\ \lambda \left(z - Z_{k'}^{\infty}(0) \right) \dfrac{\partial \beta_k^{\infty}}{\partial z} (Z_{k'}^{\infty}(0) + (-1)^k 0, 0) & \\ + \tau \dfrac{\partial \beta_k^{\infty}}{\partial t} (Z_{k'}^{\infty}(0) + (-1)^k 0, 0) & \text{in } \widehat{\mathcal{T}}_k^{(1)} \\ \beta_k^{(1,s)}(z, 0) + \tau \dfrac{\partial \beta_k^{\infty}}{\partial t}(z, 0) & \text{in } \widehat{\mathcal{M}} \\ -\beta_{k'}^{(1,s)}(z, t) & \text{in } \widehat{\mathcal{T}}_{k'}^{(1)} \end{cases}. \tag{3.42}$$

4. Fast Transition-Layers Expansions

In [10] the zero-th order and the first order variables in the first transition-layer expansions (3.37) have been determined in closed form. We summarize the result in the following Lemma 4.1. Using these terms, we find the higher order terms $v_k^{(1,ft)}$, $\beta_k^{(2,ft)}$, $\rho_k^{(2,ft)}$, $p_k^{(2,ft)}$ in Sec. 4.2.

Lemma 4.1. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Then the transitional limit terms in the expansions (3.37) satisfies

$$\begin{aligned} U_k^{(0,ft)}\left(\widehat{\zeta}_i^{(1)},\tau\right) &= \left(\beta_k^{(0,ft)}, v_k^{(0,ft)}, \rho_k^{(0,ft)}, p_k^{(0,ft)}\right)^{tr} \\ &= \left(\delta_{ik'}, \delta_{ik} V_k^{\infty}(0), \rho_k^{\infty}, p^{(0)}\right) \end{aligned} \tag{4.43}$$

and the first order terms satisfy

$$p_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)},\tau\right) = \rho_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)},\tau\right) = 0, \tag{4.44}$$

$$\beta_k^{(1,ft)}(\widehat{\zeta}_i^{(1)},\tau) = \widehat{\zeta}_i^{(1)}\frac{\partial\beta_k^{\infty}}{\partial z}(Z_i^{\infty}(0)+(-1)^{i+1}0,0), \tag{4.45}$$

where the universal constant $p^{(0)} = p_k(\rho_k^{\infty})$ and δ_{ij} is the Kronecker symbol.

4.1. Second Fast Transition-Layer Expansions

We discuss the fast transition layers through $z = \overline{\widehat{Z}_i^{(2)}}$, $i = k, k'$, which define $\widehat{v}_k^{(1)}$, $\widehat{\beta}_k^{(2)}$, $\widehat{\rho}_k^{(2)}$, $\widehat{p}_k^{(2)}$ continuously in $\widehat{\mathfrak{I}}_{i'}^{(2)}$ by matching the inner limit expansions in the exterior and the second fast transitional expansions and matching the inner edge of the second fast transition-layer and outer edge of the first fast transition-layer. We substitute the fast transition-layer asymptotic expansions (3.36) into Eqs. (3.32)–(3.34), and equate powers of λ. Within a single power of λ, the transitional variables of each order in λ^{-1} are defined as a solution of simple differential equations in $(-1)^{i+1}\widehat{Z}_i^{(2)} \leq (-1)^i\widehat{\zeta}_i^{(2)} \leq 0$. Solution of the fast transitonal terms is needed to provide boundary conditions for the inner terms by matching.

We first determine the fast transitional limit solutions $U_k^{(0,ftt)}$ in the expansions (3.36). We isolate the order λ^2 terms in the interface and mass equation and the order λ^4 terms in the momentum equation. Since the coefficients of λ^2 and λ^4 must vanish, the leading order terms satisfy the coupled ODEs

$$O(\lambda^2) \quad : \quad \left(-\widehat{V}_i^{(0)} + v^{*(0,ftt)}\right) \frac{\partial \beta_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \tag{4.46}$$

$$\begin{aligned} O(\lambda^2) \quad : \quad & \beta_k^{(0,ftt)} \left(-\widehat{V}_i^{(0)} + v_k^{(0,ftt)}\right) \frac{\partial \rho_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \beta_k^{(0,ftt)} \rho_k^{(0,ftt)} \frac{\partial v_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} \\ & + \rho_k^{(0,ftt)} \mu_k^{v(0,ftt)} \left(v_k^{(0,ftt)} - v_{k'}^{(0,ftt)}\right) \frac{\partial \beta_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \end{aligned} \tag{4.47}$$

$$O(\lambda^4) \quad : \quad \beta_k^{(0,ftt)} \frac{\partial p_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \mu_k^{p(0,ftt)} \left(p_k^{(0,ftt)} - p_{k'}^{(0,ftt)}\right) \frac{\partial \beta_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0 \tag{4.48}$$

for $U_k^{(0,ftt)}(\widehat{\zeta}_i^{(2)}, t)$ in the region $(-1)^{i+1}\widehat{Z}_i^{(2)} \leq (-1)^i \widehat{\zeta}_i^{(2)} \leq 0$. We note that for $q = v, p$,

$$\mu_k^{q(0,ftt)} = \frac{\beta_k^{(0,ftt)}}{\beta_k^{(0,ftt)} + \widehat{d}_k^{q(0)} \beta_{k'}^{(0,ftt)}}. \tag{4.49}$$

The matching conditions to $O(1)$ are

$$\begin{aligned} \beta_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= \beta_k^{(0,ft)} \left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = \delta_{ik'}, \\ v_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= v_k^{(0,ft)} \left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = \widehat{V}_k^{(0)} \delta_{ik}, \\ \rho_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= \rho_k^{(0,ft)} \left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = \rho_k^{\infty}, \\ p_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= p_k^{(0,ft)} \left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = p^{(0)}, \end{aligned} \tag{4.50}$$

where δ_{ij} is the Kronecker symbol. Here we used the order zero terms $U_k^{(0,ft)}$ and $\widehat{U}_k^{(0)}$ given in Theorem 3.2 and Lemma 4.1. These matching conditions give the boundary data at $\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}$ for the solutions of Eqs. (4.46)-(4.48).

Lemma 4.2. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Assume the boundary data (4.50). Then the fast transitional limit in the expansions (3.36) satisfies

$$\beta_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau\right) = \delta_{ik'}, \tag{4.51}$$

$$v_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau\right) = V_k^{\infty}(0) \delta_{ik}, \tag{4.52}$$

$$\rho_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau\right) = \rho_k^{\infty}, \tag{4.53}$$

$$p_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau\right) = p^{(0)}, \tag{4.54}$$

where the universal constant $p^{(0)} = p_k(\rho_k^{\infty})$ and δ_{ij} is the Kronecker symbol.

Proof. We first solve Eq. (4.48) for $p_k^{(0,ftt)}$ by using Proposition 3.1 with $q_k^{(t)} = p_k^{(0,ftt)}$ and the boundary data (4.50). Then we easily obtain the solution (4.54). The equation of state implies (4.53). Substituting (4.53) into Eq. (4.47) and multiplying the equation by $1/\rho_k^{(0,ftt)}$, it reduces to

$$\beta_k^{(0,ftt)} \frac{\partial v_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \mu_k^{v(0,ftt)} \left(v_k^{(0,ftt)} - v_{k'}^{(0,ftt)} \right) \frac{\partial \beta_k^{(0,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0. \tag{4.55}$$

To decouple $v_k^{(0,ftt)}$ and $\beta_k^{(0,ftt)}$, we use Proposition 3.1 with $q_k^{(t)} = v_k^{(0,ftt)}$. Thus,

$$\begin{aligned} v_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau \right) =& \frac{\mu_k^{v(0,ftt)}}{\beta_k^{(0,ftt)}} \left\{ \left(\beta_{k'}^{(0,ftt)}(-\widehat{Z}_i^{(2)}, \tau) - \beta_{k'}^{(0,ftt)} \right) v_{k'}^{(0,ftt)}(-\widehat{Z}_i^{(2)}, \tau) \right. \\ & \left. + \left(\beta_k^{(0,ftt)}(-\widehat{Z}_i^{(2)}, \tau) + \widehat{d}_k^{v(0)} \beta_{k'}^{(0,ftt)} \right) v_k^{(0,ftt)}(-\widehat{Z}_i^{(2)}, \tau) \right\}. \end{aligned} \tag{4.56}$$

From (4.50) and (2.17), (4.56) implies the solution

$$v_k^{(0,ftt)} \left(\widehat{\zeta}_i^{(2)}, \tau \right) = \widehat{V}_k^{(0)} \delta_{ik} = V_k^{\infty}(0) \delta_{ik} \tag{4.57}$$

in (4.52) which is independent of space. Substituting (4.52) into Eq. (4.46), we solve the quasilinear PDE for $\beta_k^{(0,ftt)}$. The characteristic speed satisfies $-\widehat{V}_i^{(0)} + v^{*(0,ftt)}$. Using (4.50), we obtain the solution $\beta_k^{(0,ftt)}$ is constant in space and time. ∎

Using (4.51)–(4.54), we define the fast transitional variables $U_k^{(1,ftt)}$ in the expansions (3.36). We isolate the order zero terms in the interface equation, the order λ terms and the order zero terms in the continuity equation for $i = k'$ and $i = k$, and the order λ^3 terms and the order λ^2 terms in the momentum equation for $i = k'$ and $i = k$, respectively. Since λ is arbitrary, the coefficients of λ^n, $n = 0, 1, 2, 3$, must vanish, leading to the equations

$$O(1) \;:\; \frac{\partial \beta_k^{(1,ftt)}}{\partial \tau} + \left(-\widehat{V}_i^{(1)} + v^{*(1,ftt)}\right) \frac{\partial \beta_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \tag{4.58}$$

$$O(\lambda) \;:\; -\widehat{V}_i^{(0)} \frac{\partial \rho_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \frac{\partial v_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \left(-\widehat{V}_{k'}^{(0)}\right) \frac{\partial \beta_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k', \tag{4.59}$$

$$O(1) \;:\; \beta_k^{(1,ftt)} \rho_k^{\infty} \frac{\partial v_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \mu_k^{v(1,ftt)} \widehat{V}_k^{(0)} \frac{\partial \beta_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k\,, \tag{4.60}$$

$$O(\lambda^3) \;:\; \frac{\partial p_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k', \tag{4.61}$$

$$O(\lambda^2) \;:\; \beta_k^{(1,ftt)} \frac{\partial p_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k \tag{4.62}$$

for $U_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, t)$ in $(-1)^{i+1}\widehat{Z}_i^{(2)} \le (-1)^i \widehat{\zeta}_i^{(2)} \le 0$. We notice

$$v^{*(0,ftt)} = \widehat{V}_i^{(0)} = V_k^{\infty}(0), \tag{4.63}$$

$$\begin{aligned} \mu_k^{q(1,ftt)} &= \frac{\partial \mu_k^q}{\partial \beta_k}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\beta_k^{(1,ftt)} + \frac{\partial \mu_k^q}{\partial d_k^q}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\widehat{d}_k^{q(1)} \\ &= \widehat{d}_{i'}^{q(0)} \beta_k^{(1,ftt)} = d_{i'}^{q\infty}(0)\beta_k^{(1,ftt)}, \quad q = v,\ p. \end{aligned} \tag{4.64}$$

The matching conditions to $O(\lambda^{-1})$ are

$$\begin{aligned} \beta_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= \beta_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = 0, \\ v_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= v_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)} = 0, \tau\right), \\ \rho_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= \rho_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = 0, \\ p_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)}, \tau\right) &= p_k^{(1,ft)}\left(\widehat{\zeta}_i^{(1)} = 0, \tau\right) = 0, \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} \beta_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = 0, \tau\right) &= \widehat{\beta}_k^{(1)}\left(\overline{\widehat{Z}_i^{(2)}}, \tau\right) = 0, \\ v_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = 0, \tau\right) &= \widehat{v}_k^{(1)}\left(\overline{\widehat{Z}_i^{(2)}}, \tau\right), \\ \rho_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = 0, \tau\right) &= \widehat{\rho}_k^{(1)}\left(\overline{\widehat{Z}_i^{(2)}}, \tau\right), \\ p_k^{(1,ftt)}\left(\widehat{\zeta}_i^{(2)} = 0, \tau\right) &= \widehat{p}_k^{(1)}\left(\overline{\widehat{Z}_i^{(2)}}, \tau\right). \end{aligned} \tag{4.66}$$

These matching conditions give the boundary conditions at $\zeta_i^{(2)} = -\widehat{Z}_i^{(2)}$ and 0 for the solutions of Eqs. (4.58)–(4.62). The conditions (4.65) match the inner edge of the second transition layer with the outer edge of the first transition layer while (4.66) match the inner limit in the exterior region and the second transitional expansions. We remark that the boundary data (4.65) used the first order terms $U_k^{(1,ft)}$ in Lemma 4.1. Notice that

$$\widehat{v}_k^{(1)}\left(\overline{\widehat{Z}_k^{(2)}}, \tau\right) = \widehat{V}_k^{(1)} \tag{4.67}$$

is given in (4.66).

Using (4.65) and (4.66), the first order transitonal terms are determined as the following

Lemma 4.3. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Assume the boundary data (4.65) and (4.66). Then we obtain the fast transitional variables

$$\beta_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) = 0, \tag{4.68}$$
$$\rho_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) = 0, \tag{4.69}$$
$$p_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) = 0, \tag{4.70}$$
$$v_k^{(1,ftt)}(\widehat{\zeta}_k^{(2)}, \tau) = \widehat{V}_k^{(1)}, \tag{4.71}$$
$$v_k^{(1,ftt)}(\widehat{\zeta}_{k'}^{(1)}, \tau) = v_k^{(1,ftt)}(\widehat{\zeta}_{k'}^{(2)} = 0, \tau) = v_k^{(1,ftt)}(\widehat{\zeta}_{k'}^{(2)} = -\widehat{Z}_{k'}^{(2)}, \tau). \tag{4.72}$$

Proof. We first solve Eqs. (4.61), (4.62) for $p_k^{(1,ftt)}$. Using (4.65), it is trivial to prove (4.70). Also the proof of (4.69) is complete by the equation of state.

Substituting the $\rho_k^{(1,ftt)}$ into (4.59) and multiplying the equation by $1/\rho_k^{\infty}$, it reduces to

$$\frac{\partial v_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} - \widehat{V}_{k'}^{(0)} \frac{\partial \beta_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k', \tag{4.73}$$

Multiplying Eq. (4.59) by $1/(\beta_k^{(1,ftt)}\rho_k^{\infty})$, we obtain the equation

$$\frac{\partial v_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \widehat{V}_k^{(0)} \widehat{d}_{k'}^{v(0)} \frac{\partial \beta_k^{(1,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k. \tag{4.74}$$

Eqs. (4.73) and (4.74) are solved to decouple $v_k^{(1,ftt)}$ and $\beta_k^{(1,ftt)}$ by using (4.66). The solution $v_k^{(1,ftt)}$ is expressed in terms of $\beta_k^{(1,ftt)}$ and the boundary data as the following

$$v_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)}, \tau) = \begin{cases} \widehat{V}_{k'}^{(0)} \beta_k^{(1,ftt)} + v_k^{(1,ftt)}(\widehat{\zeta}_i^{(2)} = 0, \tau) & \text{if } i = k' \\ -\widehat{V}_k^{(0)} \widehat{d}_{k'}^{v(0)} \beta_k^{(1,ftt)} + \widehat{V}_k^{(1)} & \text{if } i = k \end{cases}. \tag{4.75}$$

Using (4.75), we now solve BVP (4.58) for $\beta_k^{(1,ftt)}(\widehat{\zeta}_k^{(2)}, \tau)$, *i.e.*, $i = k$, by the method of characteristics. One calculation shows the characteristic speed

$$-\widehat{V}_k^{(1)} + v^{*(1,ftt)} = -2\widehat{V}_k^{(0)} \widehat{d}_{k'}^{v(0)} \beta_k^{(1,ftt)}. \tag{4.76}$$

The boundary data (4.65) and (4.66) imply the trivial solution $\beta_k^{(1,ftt)}$. Therefore, $v_k^{(1,ftt)}$ is determined in (4.75). ■

Next we define the second order variables $U_k^{(2,ftt)}$ in the fast transition-layer expansions (3.36). Using the zero-th and first order terms determined in Lemmas 4.2 and 4.3, we isolate the order λ^{-1} in the interface equation, the order zero terms and the order λ^{-2} terms in the continuity equation for $i = k'$ and $i = k$, and the order λ^2 terms and the order zero terms in the momentum equation for $i = k'$ and $i = k$, respectively. Since λ is arbitrary, the coefficients of λ^n, $n = -2, -1, 0, 2$, must vanish, leading to the equations

$$O(\lambda^{-1}) \quad : \quad \frac{\partial \beta_k^{(2,ftt)}}{\partial \tau} = 0, \tag{4.77}$$

$$O(1) \quad : \quad -\widehat{V}_i^{(0)} \frac{\partial \rho_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \frac{\partial v_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \left(-\widehat{V}_{k'}^{(0)}\right) \frac{\partial \beta_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k', \tag{4.78}$$

$$O(\lambda^{-2}) \quad : \quad \beta_k^{(2,ftt)} \rho_k^{\infty} \frac{\partial v_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} + \rho_k^{\infty} \mu_k^{v(2,ftt)} \widehat{V}_k^{(0)} \frac{\partial \beta_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k, \tag{4.79}$$

$$O(\lambda^{2}) \quad : \quad \frac{\partial p_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k', \tag{4.80}$$

$$O(1) \quad : \quad \beta_k^{(2,ftt)} \frac{\partial p_k^{(2,ftt)}}{\partial \widehat{\zeta}_i^{(2)}} = 0, \quad i = k \tag{4.81}$$

for $U_k^{(2,ftt)}(\widehat{\zeta}_i^{(2)}, t)$ in $(-1)^{i+1}\widehat{Z}_i^{(2)} \le (-1)^i \widehat{\zeta}_i^{(2)} \le 0$. From (4.64) and Lemma 4.3, we observe that

$$v^{*(1,ftt)} = \widehat{V}_k^{(1)}, \tag{4.82}$$

$$\mu_k^{q(1,ftt)} = 0. \tag{4.83}$$

Also, note that for $q = v, p$,

$$\begin{aligned}
\mu_k^{q(2,ftt)} =& \frac{\partial \mu_k^q}{\partial \beta_k}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\beta_k^{(2,ftt)} + \frac{1}{2}\frac{\partial^2 \mu_k^q}{\partial \beta_k^2}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\beta_k^{(1,ftt)2} \\
&+ \frac{\partial^2 \mu_k^q}{\partial d_k^q \partial \beta_k}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\widehat{d}_k^{q(1)}\beta_k^{(1,ftt)} + \frac{1}{2}\frac{\partial^2 \mu_k^q}{\partial d_k^{q2}}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\widehat{d}_k^{q(1)2} \\
&+ \frac{\partial \mu_k^q}{\partial d_k^q}(\beta_k^{(0,ftt)}, \widehat{d}_k^{q(0)})\widehat{d}_k^{q(2)} \\
=& \widehat{d}_{i'}^{q(0)} \beta_k^{(2,ftt)} = d_{i'}^{q\infty}(0)\beta_k^{(2,ftt)}.
\end{aligned} \tag{4.84}$$

We first determine $p_k^{(2,ftt)}$ by solving Eqs. (4.80) and (4.81).

Lemma 4.4. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Then

$$p_k^{(2,ftt)}(\hat{\zeta}_i^{(2)},\tau) = p_k^{(2,ftt)}(\widehat{\zeta}_i^{(2)} = -\widehat{Z}_i^{(2)},\tau) = p_k^{(2,ftt)}(\widehat{\zeta}_i^{(2)} = 0,\tau). \tag{4.85}$$

From Lemmas 4.3 and 4.4, the transitional variables $v_k^{(1,ftt)}$ and $p_k^{(2,ftt)}$ are constant in space and thus depend on boundary data. Here we assume the boundary data which will be proved by matching in [12].

4.2. Higher Order Terms in First Fast Transition-Layers Expansions

Using Lemma 4.1, we define the fast transitional variables $v_k^{(1,ft)}$, $\beta_k^{(2,ft)}$, $\rho_k^{(2,ft)}$, $p_k^{(2,ft)}$ in the first transition-layer expansions (3.37). We isolate the order λ^{-1} terms in the interface equation, the order zero terms and the order λ^{-1} terms in the continuity equation for $i = k'$ and $i = k$, and the order λ terms and the order zero terms in the momentum equation for $i = k'$ and $i = k$, respectively. Since λ is arbitrary, the coefficients of λ^m, $m = -1, 0, 1$ must vanish, leading to the equations

$$O(\lambda^{-1}) \ : \ \frac{\partial \beta_k^{(2,ft)}}{\partial \tau} + v^{*(1,ft)}\frac{\partial \beta_k^{(1,ft)}}{\partial \widehat{\zeta}_i^{(1)}} = 0, \tag{4.86}$$

$$O(1) \ : \ \rho_k^{\infty}\frac{\partial v_k^{(1,ft)}}{\partial \widehat{\zeta}_i^{(1)}} + \rho_k^{\infty}\left(v_k^{(0,ft)} - v_{k'}^{(0,ft)}\right)\frac{\partial \beta_k^{(1,ft)}}{\partial \widehat{\zeta}_i^{(1)}} = 0, \quad i = k', \tag{4.87}$$

$$O(\lambda^{-1}) \ : \ \beta_k^{(1,ft)}\rho_k^{\infty}\frac{\partial v_k^{(1,ft)}}{\partial \widehat{\zeta}_i^{(1)}} + \rho_k^{\infty}\mu_k^{v(1,ft)}\left(v_k^{(0,ft)} - v_{k'}^{(0,ft)}\right)\frac{\partial \beta_k^{(1,ft)}}{\partial \widehat{\zeta}_i^{(1)}} = 0, \quad i = k, \tag{4.88}$$

$$O(\lambda) \ : \ \frac{\partial p_k^{(2,ft)}}{\partial \widehat{\zeta}_i^{(1)}} = 0, \quad i = k', \tag{4.89}$$

$$O(1) \ : \ \beta_k^{(1,ft)}\frac{\partial p_k^{(2,ft)}}{\partial \widehat{\zeta}_i^{(1)}} = 0, \quad i = k \tag{4.90}$$

for $v_k^{(1,ft)}$, $\beta_k^{(2,ft)}$, $\rho_k^{(2,ft)}$, $p_k^{(2,ft)}$ in $(-1)^{i+1}\widehat{Z}_i^{(1)} \le (-1)^i\widehat{\zeta}_i^{(1)} \le 0$. We note that

$$v^{*(0,ft)}(\hat{\zeta}_i^{(1)},\tau) = \widehat{V}_i^{(0)}. \tag{4.91}$$

Eqs. (4.87) and (4.88) are easily solved for $v_k^{(1,ft)}$.

Lemma 4.5. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Assume the boundary data (4.65) and (4.66). Then the first order velocity satisfies

$$v_k^{(1,ft)}(\widehat{\zeta}_i^{(1)},\tau) = \begin{cases} \widehat{V}_{k'}^{(0)}\beta_k^{(1,ft)} + \widehat{v}_k^{(1)}(\overline{\widehat{Z}_{k'}^{(2)}},\tau) & \text{if } i = k' \\ -\widehat{V}_k^{(0)}\widehat{d}_{k'}^{v(0)}\beta_k^{(1,ft)} + \widehat{V}_k^{(1)} & \text{if } i = k \end{cases}, \tag{4.92}$$

where $\beta_k^{(1,ft)}$ was given in (4.45).

Proof. Solving ODEs (4.87) and (4.88) for $v_k^{(1,ft)}$, we obtain

$$v_k^{(1,ft)}(\widehat{\zeta}_i^{(1)},\tau) = \begin{cases} \widehat{V}_{k'}^{(0)}\beta_k^{(1,ft)} + v_k^{(1,ft)}(\widehat{\zeta}_i^{(1)}=0,\tau) & \text{if } i=k' \\ -\widehat{V}_k^{(0)}\widehat{d}_{k'}^{v(0)}\beta_k^{(1,ft)} + v_k^{(1,ft)}(\widehat{\zeta}_i^{(1)}=0,\tau) & \text{if } i=k \end{cases} \tag{4.93}$$

by use of Lemma 4.1. From (4.71), (4.65) and (4.66), we evaluate the boundary data

$$\begin{aligned} v_k^{(1,ft)}(\widehat{\zeta}_{k'}^{(1)}=0,\tau) &= v_k^{(1,ftt)}(\widehat{\zeta}_{k'}^{(2)}=-\widehat{Z}_{k'}^{(2)},\tau) \\ &= v_k^{(1,ftt)}(\widehat{\zeta}_{k'}^{(2)}=0,\tau) = \widehat{v}_k^{(1)}(\overline{\widehat{Z}_{k'}^{(2)}},\tau), \end{aligned} \tag{4.94}$$

$$v_k^{(1,ft)}(\widehat{\zeta}_k^{(1)}=0,\tau) = v_k^{(1,ftt)}(\widehat{\zeta}_k^{(2)}=-\widehat{Z}_k^{(2)},\tau) = \widehat{V}_k^{(1)}. \tag{4.95}$$

■

Solving Eqs. (4.88) and (4.90), we determine $p_k^{(2,ft)}$ in $(-1)^{i+1}\widehat{Z}_i^{(1)} \le (-1)^i\widehat{\zeta}_i^{(1)} \le 0$, which depends on boundary data.

Lemma 4.6. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Then

$$p_k^{(2,ft)}(\hat{\zeta}_i^{(1)},\tau) = p_k^{(2,ft)}(\widehat{\zeta}_i^{(1)}=-\widehat{Z}_i^{(1)},\tau) = p_k^{(2,ft)}(\widehat{\zeta}_i^{(1)}=0,\tau). \tag{4.96}$$

From Lemmas 4.1 and 4.5, we note that

$$\begin{aligned} v^{*(1,ft)}(\hat{\zeta}_i^{(1)},\tau) &= \mu_k^{v(0,ft)}v_{k'}^{(1,ft)} + \mu_{k'}^{v(0,ft)}v_k^{(1,ft)} + \mu_k^{v(1,ft)}v_{k'}^{(0,ft)} + \mu_{k'}^{v(1,ft)}v_k^{(0,ft)} \\ &= \widehat{V}_i^{(1)} + 2\frac{\widehat{V}_i^{(0)2}}{\widehat{V}_{i'}^{(0)}}\beta_i^{(1,ft)}. \end{aligned} \tag{4.97}$$

Using (4.97) and Lemma 4.1, we solve Eq. (4.86) for $\beta_k^{(2,ft)}$.

Lemma 4.7. Assume (2.7), (2.13), (2.17), (2.20), and the initial data (2.10). Then we obtain

$$\begin{aligned} \beta_k^{(2,ft)}(\widehat{\zeta}_k^{(1)},\tau) &= \beta_k^{(2,ft)}(\widehat{\zeta}_k^{(1)}, -\frac{\widehat{\zeta}_k^{(1)}}{\widehat{V}_k^{(0)}}) \\ &\quad - \left(\widehat{Z}_k^{(2)}(\tau) - \widehat{Z}_k^{(2)}(-\frac{\widehat{\zeta}_k^{(1)}}{\widehat{V}_k^{(0)}})\right)\frac{\partial\beta_k^\infty}{\partial z}(Z_k^\infty(0)+(-1)^{k'}0,0) \\ &\quad - 2\left(\tau + \frac{\widehat{\zeta}_k^{(1)}}{\widehat{V}_k^{(0)}}\right)\widehat{\zeta}_k^{(1)}\frac{\widehat{V}_k^{(0)2}}{\widehat{V}_{k'}^{(0)}}\left[\frac{\partial\beta_k^\infty}{\partial z}(Z_k^\infty(0)+(-1)^{k'}0,0)\right]^2 \\ &= -\beta_{k'}^{(2,ft)}(\widehat{\zeta}_k^{(1)},\tau). \end{aligned} \tag{4.98}$$

Proof. We solve the BVP (4.86) for $\beta_k^{(2,ft)}(\widehat{\zeta}_k^{(1)}, \tau)$ in $(-1)^{k'}\widehat{Z}_k^{(1)} \leq (-1)^k\widehat{\zeta}_k^{(1)} \leq 0$, *i.e.*, $i = k$. The solution is

$$\beta_k^{(2,ft)}(\widehat{\zeta}_k^{(1)}, \tau) = \beta_k^{(2,ft)}(\widehat{\zeta}_k^{(1)}, -\frac{\widehat{\zeta}_k^{(1)}}{\widehat{V}_k^{(0)}}) - \int_{-\widehat{\zeta}_k^{(1)}/\widehat{V}_k^{(0)}}^{\widehat{\zeta}_k^{(1)}} \left(v^{*(1,ft)} \frac{\partial \beta_k^{(1,ft)}}{\partial \widehat{\zeta}_k^{(1)}} \right) (\widehat{\zeta}_k^{(1)}, \tau) d\tau. \tag{4.99}$$

From (4.97) and (4.45), a calculation yields

$$\begin{aligned} v^{*(1,ft)} \frac{\partial \beta_k^{(1,ft)}}{\partial \widehat{\zeta}_k^{(1)}} &= \widehat{V}_k^{(1)} \frac{\partial \beta_k^{\infty}}{\partial z} (Z_k^{\infty}(0) + (-1)^{k'}0, 0) \\ &\quad + 2\widehat{\zeta}_k^{(1)} \frac{\widehat{V}_k^{(0)2}}{\widehat{V}_{k'}^{(0)}} \left[\frac{\partial \beta_k^{\infty}}{\partial z} (Z_k^{\infty}(0) + (-1)^{k'}0, 0) \right]^2. \end{aligned} \tag{4.100}$$

We note from (2.17) that the last term in (4.100) is constant in τ. Using the identity

$$\int v_k^{(1,ft)}(\widehat{\zeta}_k^{(1)} = 0, \tau) d\tau = \int \widehat{V}_k^{(1)}(\tau) d\tau = \widehat{Z}_k^{(2)}(\tau) \tag{4.101}$$

from (2.13), we obtain the solution (4.98). ■

In Lemmas 4.5-4.7, we have unknown boundary data of $v_k^{(1,ft)}$, $p_k^{(2,ft)}$ and $\beta_k^{(2,ft)}$. These will be provided by matching [12].

In conclusion, the fast transitional variables in (3.36) and (3.37) have been found up to second order, we are concerned, in the second transition-layers $\widehat{\mathfrak{T}}_i^{(2)}$, and the first transition-layers $\widehat{\mathfrak{T}}_i^{(1)}$, $i = k, k'$. These variables will be used to determine the higher order inner terms $\widehat{v}_k^{(1)}$, $\widehat{\beta}_k^{(2)}$, $\widehat{\rho}_k^{(2)}$ and $\widehat{p}_k^{(2)}$, uniformly valid in space.

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