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Turbulence and Devaney's Chaos in Interval

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Abstract

In this paper, we show that Logistic Map and Tent Map are Devaney Chaotic on the interval. Also we show the relation between turbulent function and Devaney chaos using these maps.

Keywords: Turbulent Function, Devaney Chaos

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1. INTRODUCTION

After the introduction of first mathematical definition of chaos in 1975 by Li and Yorke [2], several other definitions in various contexts were introduced. The most popular among all such definitions is the one given by Robert. L. Devaney [1]. In Devaney's definition of chaos these are three important factors, namely, Transitivity, Dense periodicity and Sensitivity. This paper will be dealt with the relation between turbulent function and the Devaney Chaos.

2. PRELIMINARIES

2.1. Definition: Devaney Chaos [1]

Let V be a set. $f: V \to V$ is said to be chaotic on V if

- f has sensitive dependence on initial conditions
- f is topologically transitive
- periodic points are dense in V

2.2. Definition: Turbulent Function [3]

Let $f: X \to X$ be a continuous function. We say that f is a turbulent function if there exist compact subintervals J, K with at most one common point such that $J \cup K \subseteq f(J) \cap f(K)$.

2.3. Definition: Topologically Transitive [1]

$f: J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$

2.4. Definition: Sensitive Dependence on Initial Conditions [1]

$f: J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighbourhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

3. MAIN RESULTS

In order to prove the relationship between Turbulent Function and Devaney Chaos, we need some existing results. Also we need to check whether the given function is turbulent, Devany Chaotic with the help of logistic and Tent Maps

3.1. Lemma [3] (Chapter II, p.26)

If f is turbulent, then there exist points $a, b, c \in I$ such that $f(b) = f(a) = a$, $f(c) = b$ and either $a < c < b, f(x) > a \text{ for } a < x < b$

$$x < f(x) < b \text{ for } a < x < c$$

$$\text{or } b < c < a, f(x) < a \text{ for } b < x < a$$

$$b < f(x) < x \text{ for } c < x < a$$

3.2. Corollary [3] (Chapter IV, p.75)

If J is a subinterval of I which contains no periodic point of f then, for any $x \in I$, the points of the trajectory $\gamma(x)$ which lie J form a strictly monotonic.

3.3. Examples

Let $f:[0,1] \to [0,1]$ be a continuous function. The given function is a logistic function, $L(x) = 4x\,(1 - x)$. Now we take compact subintervals J, K with at most one common point, $J = [0,1/2]$, $K = [1/2, 1]$. Then$f\ (J) = [0,1]\ and\ f(K) = [0,1]$, so $J \cup K \subseteq f(J) \cap f(K)$.Then the condition for a function to be turbulent is satisfied and so the function f is turbulent.

3.4. Example: Let f be a continuous function from interval I to itself, here the given function is Tent function defined as $T(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$

Now we take a pair of open sets say $U = (0, 1/4)\ and\ V = (1/2, 1)$ in$I$. In order to prove that the function is topologically transitive, we have to show that $f^k(U) \cap V \neq \emptyset$ for some $k > 0$.

$f\ (U) \Rightarrow f(0, 1/4) = (0, 1/2)$; $f^2(0, 1/2) = (0, 1)$

$\Rightarrow f^2(0,1/2) \cap (1/2, 1) \neq \emptyset$ Then the function is topologically transitive with respect to the Tent Map.

3.5. Example: Let the given function be the Tent Map, we have to show that the map is sensitive dependence on initial conditions. So with $\delta = .5$, N be the neighbourhood of $4/9$. We know that $8/9$ comes inside this neighbourhood and $n \geq 0$, we have to check

$|f(x) - f(y)| = |8/9 - 2/9| = 2/3 > \delta$ and therefore the function is *s.d.i.c.*

3.6. Example: Let the given function be the Tent map, we have to prove that the function agrees the density of periodic points. In order to prove the function has got density of periodic points, for this we prove two conditions:

- $f(x) \notin (x, f^2(x))$
- $f^3(x) = x$

First condition can be proved using Corollary 3.2., since the trajectory is not strictly monotonic, we can say that the interval has got at least one periodic point and with the help of second condition we can prove the density.

Let $x = 4/9$

Then $T(x) = 8/9$; $T\big(T(x)\big) = T^2(8/9) = 2/9$; $T^3(2/9) = 4/9$

So $x = 4/9$ is a periodic point with period three and also we have seen that

$T^2(x) < x < T(x)$, which contradicts the Corollary. So the above conditions are satisfied, then the function has got the density of periodic points.

Now we shall check the relationship between turbulent function and Devaney Chaos.

3.7. Theorem

If $f: I \to I$ be a continuous map and f is turbulent, then f is topologically transitive.

Proof: If f is a turbulent function, so it must satisfy the necessary and sufficient condition for the turbulent function which is given in Lemma 3.1.

If f is turbulent, then there exist points $a, b, c \in I$ such that $f(b) = f(a) = a$, $f(c) = b$ and either $a < c < b, f(x) > a \text{ for } a < x < b$

$$x < f(x) < b \text{ for } a < x < c$$

$$\text{or } b < c < a, f(x) < a \text{ for } b < x < a$$

$$b < f(x) < x \text{ for } c < x < a$$

Suppose the pair of open sets be (a, c) and (c, b), So from the necessary condition for turbulence we have

$$a < c < b,$$

$$f(x) > a \text{ for } a < x < b$$

$$x < f(x) < b \text{ for } a < x < c$$

So after successive iterations the elements of (a, c) will enter to the second open set (c, b)

So we can conclude that f is Turbulent $\Rightarrow f$ is Topologically Transitive.

3.8. Theorem

If $f: I \to I$ be a continuous map and f is turbulent, then f has sensitive dependence on initial conditions.

Proof: We know that If f is turbulent, then there exist points $a, b, c \in I$ such that $f(b) = f(a) = a$, $f(c) = b$. So there exist compact subintervals J, K with at most one common point. Let $J = [\alpha, \beta]$ and $K = [\gamma, \delta]$, where $\beta = \gamma$. Then we assume that $f(\beta) \neq \beta$. Now $[\alpha, \delta]$ be the neighbourhood of β with $\delta > 0$. Since the function is turbulent, we have $|f(\beta) - f(\alpha)| = \delta_1 > \delta$. Then f has sensitive dependence on initial conditions.

3.9 Theorem

If $f: I \to I$ be a continuous map and f is turbulent, then f has dense set of periodic points.

Proof: In order to prove the dense set of periodic points, we have to show that each open interval contains a periodic point. Let U be an open set in I. Since f is turbulent, then it implies that $J \cup K \subseteq f(J) \cap f(K)$. Given any $n \in N$, $\exists\, x_n$ such that $f^n(x_n) = x_n$.

Suppose there are no periodic points in U.

$$\forall\, x \in U, f^n(x) \neq x, \quad \forall\, n$$

$$\Rightarrow \nexists\, y, z \in J, K \; such\; that\; f^n(y) = f^m(z) = x \quad \forall\, n, m$$

$$\Rightarrow f(x) = f^{n+1}(y) = f^{m+1}(z)$$

$$\Rightarrow f^k(x) = f^{n+k}(y) = f^{m+k}(z)$$

$$\Rightarrow O_f(x) = \{\, \mathrm{x}, f(x), f^2(x), \dots \dots, f^k(x)\} \cap \mathrm{U} = \emptyset$$

So we have arrived with a contradiction that f is not transitive. But if f is turbulent then f is transitive. So U has periodic points and therefore the set of periodic points of f are dense in I.

4. CONCLUSION

We have proved that the given maps, tent and logistic maps are Devaney chaotic and also proved the relation between turbulent function and Devaney chaos. Further studies can be conducted in higher dimensions.

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