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Solving Fuzzy Sequential Linear Programming Problem by Fuzzy Frank Wolfe Algorithm

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Abstract

In this paper an algorithm is proposed to solve Fuzzy sequential non-linear programming problem. This algorithm is applicable to the problem when the objective function is of non-linear and the constraints are all of linear. Initially, the Fuzzy sequential linear programming problem is converted in to fuzzy linear programming problem using fuzzy Frank Wolfe algorithm and then it is solved by Fourier Motzkin Elimination Method.

Key words: Triangular fuzzy numbers, Fuzzy Linear system of equations, Fourier Motzkin Elimination Method, Sequential Linear Programming Problem.

AMS subject classification: 03E72, 90C70, 15A39, 90C27

1. INTRODUCTION

Sequential Linear Programming problem is originally proposed by Griffith and Stewart [8]. Bhavikatti [2] suggested several progresses to the method to make sure that the method can be used nearly as a black box for real time problems. A upright discussion of the earlier growths can be found in Himmelblaue [9] who calls the method as approximate programming. Sequential Linear Programming (SLP) [1, 6, 12] is one of the dominant methods for solving nonlinear optimization problems. The

SLP [3] consists in linearising the constraints and the objective function in the neighbourhood of a design vector and solving the resulting linear programming problem to get a new design vector. The linearization of non-linear equations and the solution of linear programming problem is sustained in a sequence till optimal solution is reached.

The concept of fuzzy numbers was first introduced and investigated by Zadeh [14], etc. One of the major applications of fuzzy arithmetic is treating linear systems and their parameters that are all partially represented by fuzzy number. Friedman etal [7] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system.

In this paper, all the variables are considered as fuzzy variables. In solving the fuzzy linear programming problem, to linearize the objective function we applied the Fuzzy Frank Wolfe algorithm [6] and then it is solved by Fourier Motzkin elimination method [11]. Some basic definitions about the fuzzy concept, the fuzzy variables and fuzzy linear programming problem are given for the clear understanding of the nature of the problem considered here. We also explained the Fuzzy Frank – Wolfe algorithm and numerical examples are illustrated by the proposed method.

2. PRELIMINARIES

2.1. Fuzzy set [12]: A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(\mathrm{x}, \mu_{\mathrm{A}}(\mathrm{x})) : \mathrm{x} \in \mathrm{A}, \mu_{\mathrm{A}}(\mathrm{x}) \in [0,1]\}$. In the pair $(\mathrm{x}, \mu_{\mathrm{A}}(\mathrm{x}))$,the first element x belongs to the classical set A and the second element $\mu_{\mathrm{A}}(\mathrm{x})$ belongs to the interval [0, 1] called *membership function*.

2.2. Fuzzy number [12]: A fuzzy set $\tilde{A}$ on R must possess at least the following three properties to qualify as a fuzzy number:

(i) $\tilde{A}$ must be a normal fuzzy set;
(ii) $\alpha_{\tilde{A}}$ must be a closed interval for every $\alpha \in [0, 1]$; and
(iii) The support of $\tilde{A}$ must be bounded.

2.3. Triangular fuzzy number[12]: It is a fuzzy number represented with three points as follows: $\tilde{A} = (a_1, a_2, a_3)$. This representation is interpreted as membership functions and holds the following conditions:

(i) a_1 to a_2 is an increasing function;
(ii) a_2 to a_3 is a decreasing function; and
(iii) $a_1 \le a_2 \le a_3$.

$$\mu_A(x)=\begin{cases}0 & for\ x\prec a_1\\ \dfrac{x-a_1}{a_2-a_1} & for\ a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & for\ a_2\le x\le a_3\\ 0 & for\ x\succ a_3\end{cases}$$

2.4. Operation of triangular fuzzy number using function principle [12]:

Let $\tilde{A}=(a_1,a_2,a_3)$ and $\tilde{B}=(b_1,b_2,b_3)$ Then

- **(i) Addition**: $\tilde{A}+\tilde{B}=(a_1+b_1,a_2+b_2,a_3+b_3)$.
- **(ii) Subtraction**: $\tilde{A}-\tilde{B}=(a_1-b_3,a_2-b_2,a_3-b_1)$.
- **(iii) Multiplication**:
  $\tilde{A}\times\tilde{B}=(\min(a_1b_1,a_1b_3,a_3b_1,a_3b_3),a_2b_2,\max(a_1b_1,a_1b_3,a_3b_1,a_3b_3))$
- **(iv) Division**: $\tilde{A}/\tilde{B}=(\min(a_1b_1,a_1b_3,a_3b_1,a_3b_3),a_2b_2,\max(a_1b_1,a_1b_3,a_3b_1,a_3b_3))$.
- **(v) Scalar Multiplication:**

$$K(\tilde{A})=(Ka_1,Ka_2,Ka_3), if\ K\ is\ positive\ and$$
$$K(\tilde{A})=(Ka_3,Ka_2,Ka_1), if\ K\ is\ negative$$

2.5. Fuzzy linear system of equations

Consider the *m* × *n* fuzzy linear system of equations [11]:

$$a_{11}\times x_{11}+a_{12}\times x_{12}+.......+a_{1n}\times x_{1n}=\tilde{b}_1$$
$$a_{21}\times x_{21}+a_{22}\times x_{22}+.......+a_{2n}\times x_{2n}=\tilde{b}_2$$
.
.
$$a_{m1}\times x_{m1}+a_{m2}\times x_{m2}+.......+a_{mn}\times x_{mn}=\tilde{b}_m$$

The matrix form of the above equations is $A\times x=\tilde{b}$, where the coefficient matrix *A* is (a_{ij}) where i = 1 to m and j = 1 to *n*, x is a fuzzy variable and $\tilde{b}$ is also a fuzzy variable.

2.6. Graded mean integration method

The graded mean integration method [4] is used to defuzzify the triangular fuzzy number. The representation of triangular fuzzy number is $\tilde{A}=(a_1,a_2,a_3)$ and its defuzzified value is obtained by $A=\dfrac{a_1+4a_2+a_3}{6}$.

3. AN ALGORITHM TO SOLVE THE FUZZY SEQUENTIAL LINEAR PROGRAMMING PROBLEM

3.1 Proposed Algorithm

Step 1: Assume an initial point $x^{(0)}$, two convergence parameters $\in$ *and* δ. Set an iteration Counter t =0.

Step 2: Calculate $\nabla f(x^{(t)})$. *If* $\left\|\nabla f(x^{(t)})\right\| \leq \in$. Terminate; Else go to step 3.

Step 3: Frame the Fuzzy Linear Programming Problem as

Maximize *or* Minimize $f(x^{(t)}) + \nabla f(x^{(t)})(x - x^{(t)})$

Subject to constraints

$$g_j(x^{(t)}) + \nabla g_j(x^{(t)})(x - x^{(t)}) \geq 0 \;\; ; j = 1, 2....J$$

$$g_j(x^{(t)}) + \nabla g_j(x^{(t)})(x - x^{(t)}) \leq 0 \;\; ; j = 1, 2....J$$

$$h_k(x^{(t)}) + \nabla h_k(x^{(t)})(x - x^{(t)}) = 0 \;\; ; k = 1, 2....K$$

$$x^{(L)} \leq x_i \leq x^{(U)}$$

Step 4: In the above fuzzy linear programming problem include the objective function in the constraints, for maximization problem, change the equal '=' in the objective as '≤' and (for minimization problem, change '≥').

Step 5: Eliminate the variable one by one in the order as $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n.)$

i. Divide each equation by its modulus value of $\tilde{x}_1$ coefficient or all the equations.

ii. Now we have three classes of $\tilde{x}_1$ coefficient, i.e ,' -1' or ' $+1$' or '0' linear equations.

iii. Adding or subtracting any two classes of equations to eliminate $\tilde{x}_1$.

Step 6: Repeat the above process until all the 'n'fuzzy variables are eliminated.

Step 7:.After eliminating all the 'n'fuzzy variables, we get the $\tilde{Z}$ values and substitute the $\tilde{Z}$ in above ,we get the values of fuzzy variables in back to back substitution. Then we get $y^{(t)}$ be the optimal solution to the above fuzzy LPP.

Step 8: Find $\tilde{\alpha}^{(t)}$ that minimizes $f((x^{(t)}) + \tilde{\alpha}(\tilde{y}^{(t)} - x^{(t)})$ in the range α ∈ (0, 1).

Step 9: Calculate $x^{(t+1)} = x^{(t)} + \tilde{\alpha}^{(t)}(\tilde{y}^{(t)} - x^{(t)})$

Step 10: If $\left\|x^{(t+1)} - x^{(t)}\right\| \leq \delta \left\|x^{(t)}\right\|$ and if

$$\left\|f(x^{(t+1)}) - f(x^{(t)})\right\| \leq \in \left\|f(x^{(t)})\right\|$$

Terminate. Else t = t+1 and go to step 2.

4. NUMERICAL EXAMPLE

Consider the Fuzzy Sequential Linear Programming Problem with triangular fuzzy number is,

Maximize $f(x) = 5x_1 - x_1^2 + 8x_2 - 2x_2^2$

Subject to

$$3x_1 + 2x_2 \le (5,6,7)$$

Non negative restriction

$$x_1 \ge (0,0,0)$$

$$x_2 \ge (0,0,0)$$

Applying the above algorithm step by step we get the following solutions:

Initial Point t=0

$$Max\ f(x) = f(x^{(0)}) + \nabla f(x^{(0)})(x - x^{(0)})$$

$$\nabla f(x^{(t)}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}_{(x_1,x_2)}$$

$$\nabla f(x^{(0)}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 5-2x_1 \\ 8-4x_1 \end{pmatrix}_{(0,0)} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$f(x^{(0)}) = 0$$

$$Max\ f(x) = 0 + \begin{pmatrix} 5 \\ 8 \end{pmatrix}\begin{pmatrix} x_1 - 0 \\ x_2 - 0 \end{pmatrix}$$

$$Max\ f(x) = 0 + \begin{pmatrix} 5x_1 \\ 8x_2 \end{pmatrix}$$

$$Max\ f(x) = 5x_1 + 8x_2$$

Linearized Fuzzy problem is given by

$$Max\ f(x) = 5x_1 + 8x_2$$

Subject to

$$3x_1 + 2x_2 \le (5,6,7)$$

Non negative restrictions are

$$x_1 \ge (0,0,0)$$

$$x_2 \ge (0,0,0)$$

Include the objective function in the constraints, for maximization problem, change the equal '=' in the objective as '≤' and (for minimization problem, change '≥').

$$\left.\begin{aligned} &\tilde{z} \le 5x_1 + 8x_2 \\ &3x_1 + 2x_2 \le (5,6,7) \\ &x_1 \ge (0,0,0) \\ &x_2 \ge (0,0,0) \end{aligned}\right\} \quad (4.1.1)$$

Change all the inequalities in the system as '≤' for maximization (and ≥ for minimization)

$$\left.\begin{aligned} &-5x_1 - 8x_2 + \tilde{z} \le (0,0,0) \\ &3x_1 + 2x_2 \le (5,6,7) \\ &-x_1 \le (0,0,0) \\ &-x_2 \le (0,0,0) \end{aligned}\right\} \quad (4.1.2)$$

To eliminate x_1, divide each coefficient of the system (4.1.2) by the coefficient of x_1 we have

$$\left.\begin{aligned} &-x_1 - 1.6x_2 + 0.2\tilde{z} \le (0,0,0) \\ &x_1 + 0.66x_2 \le (1.66, 2, 2.33) \\ &-x_1 \le (0,0,0) \\ &-x_2 \le (0,0,0) \end{aligned}\right\} \quad (4.1.3)$$

Rearranging the equations in (4.1.3), to eliminate x_2

$$\left.\begin{aligned} &-0.94x_2 + 0.2\tilde{z} \le (1.66, 2, 2.33) \\ &-1.6x_2 + 0.2\tilde{z} \le (0,0,0) \\ &0.66x_2 \le (1.66, 2, 2.33) \\ &-x_2 \le (0,0,0) \end{aligned}\right\} \quad (4.1.4)$$

Using the same procedure as the elimination of x_1, we get

$$\left.\begin{aligned} &-x_2 + 0.213\tilde{z} \le (1.76, 2.13, 2.47) \\ &-x_2 + 0.125\tilde{z} \le (0,0,0) \\ &x_2 \le (2.52, 3.03, 3.53) \\ &-x_2 \le (0,0,0) \end{aligned}\right\} \quad (4.1.5)$$

Now, the set equations were obtained by eliminating x_2

$$\left.\begin{array}{l} 0.088\tilde{z} \le (1.76, 2.13, 2.47) \\ 0.125\tilde{z} \le (2.52, 3.03, 3.53) \\ 0.125\tilde{z} \le (0, 0, 0) \\ 0.213\tilde{z} \le (4.27, 5.16, 6.00) \\ 0.213\tilde{z} \le (1.76, 2.13, 2.47) \\ 0 \le (2.52, 3.03, 3.53) \end{array}\right\} \quad (4.1.6)$$

From the above equation (4.1.6), we have

$$\left.\begin{array}{l} \tilde{z} \le (20, 24.20, 28.06) \\ \tilde{z} \le (20.16, 24.24, 28.24) \\ \tilde{z} \le (0, 0, 0) \\ \tilde{z} \le (20.04, 24.22, 28.16) \\ \tilde{z} \le (8.26, 10, 11.59) \end{array}\right\}$$

Now choosing the value for $\tilde{z}$ which satisfies all the constraints. So, the optimal solution is given by $\tilde{z} = (20, 24.20, 28.06)$.

Using the obtained $\tilde{z}$ in (4.1.5),

$$\begin{array}{l} -x_2 + 0.213(20, 24.20, 28.06) \le (1.76, 2.13, 2.47) \\ -x_2 + 0.125(20, 24.20, 28.06) \le (0, 0, 0) \\ x_2 \le (2.52, 3.03, 3.53) \\ -x_2 \le (0, 0, 0) \end{array}$$

We get,

$$\begin{array}{l} -x_2 + (4.26, 5.15, 5.97) \le (1.76, 2.13, 2.47) \\ -x_2 + (2.5, 3.02, 3.50) \le (0, 0, 0) \\ x_2 \le (2.52, 3.03, 3.53) \\ -x_2 \le (0, 0, 0) \end{array}$$

Then

$$\begin{array}{l} -x_2 \le (1.76, 2.13, 2.47) - (4.26, 5.15, 5.97) \\ -x_2 \le (0, 0, 0) - (2.5, 3.02, 3.50) \\ x_2 \le (2.52, 3.03, 3.53) \\ -x_2 \le (0, 0, 0) \end{array}$$

$\Rightarrow$

$$-x_2 \le (-4.21, -3.02, -1.79)$$
$$-x_2 \le -(2.5, 3.02, 3.50)$$
$$x_2 \le (2.52, 3.03, 3.53)$$
$$-x_2 \le (0, 0, 0)$$

$\Rightarrow$

$$-x_2 \le -(1.79, 3.02, 4.21)$$
$$-x_2 \le -(2.5, 3.02, 3.50)$$
$$x_2 \le (2.52, 3.03, 3.53)$$
$$-x_2 \le (0, 0, 0)$$

$\Rightarrow$

$$x_2 \ge (1.79, 3.02, 4.21)$$
$$x_2 \ge (2.5, 3.02, 3.50)$$
$$x_2 \le (2.52, 3.03, 3.53)$$
$$x_2 \ge (0, 0, 0)$$
$$\therefore (1.79, 3.02, 4.21) \le x_2 \le (2.52, 3.03, 3.53)$$

The defuzzified value x_2 on both sides is nearly 3.01, therefore, select any one

$$\therefore \quad x_2 = (2.52, 3.03, 3.53)$$

Substitute x_2 and $\tilde{z}$ in (4.1.3).

$$-x_1 - 1.6(2.52, 3.03, 3.53) + 0.2(20, 24.20, 28.06) \le (0, 0, 0)$$
$$x_1 + 0.66(2.52, 3.03, 3.53) \le (1.66, 2, 2.33)$$
$$-x_1 \le (0, 0, 0)$$
$$-(2.52, 3.03, 3.53) \le (0, 0, 0)$$

We get,

$$-x_1 - (4.03, 4.84, 5.64) + (4, 4.84, 5.61) \le (0, 0, 0)$$
$$x_1 + (1.66, 3.03, 3.53) \le (1.66, 2, 2.33)$$
$$-x_1 \le (0, 0, 0)$$
$$-(2.52, 3.03, 3.53) \le (0, 0, 0)$$

Then,

$$-x_1 \le (-1.61, 0, 1.61)$$
$$x_1 \le (-0.67, 0, 0.67)$$
$$x_1 \ge (0, 0, 0)$$

From the above equations, we get $(-1.61, 0, 1.61) \le x_1 \le (0,0,0)$

The defuzzified value of x_1 on both the inequalities are same.

$$x_1 = (0,0,0)$$

Then $y^{(t)}$ => $\therefore\ x_1 = (0,0,0), x_2 = (2.52, 3.03, 3.53) \Rightarrow \tilde{z} = (20, 24.20, 28.06)$.

Find $\tilde{\alpha}$:

$$f((x^{(t)}) + \tilde{\alpha}\,(\tilde{y}^{(t)} - x^{(t)})$$

$$\Rightarrow f\left\{\begin{pmatrix}(0,0,0)\\(0,0,0)\end{pmatrix} + \tilde{\alpha}\left(\begin{pmatrix}(0,0,0)\\(2.52,3.03,3.53)\end{pmatrix} - \begin{pmatrix}(0,0,0)\\(0,0,0)\end{pmatrix}\right)\right\}$$

$$\Rightarrow f\left\{\begin{pmatrix}(0,0,0)\\(0,0,0)\end{pmatrix} + \tilde{\alpha}\begin{pmatrix}(0,0,0)\\(2.52,3.03,3.53)\end{pmatrix}\right\}$$

$$\Rightarrow f\left\{\begin{pmatrix}(0,0,0)\\(0,0,0)\end{pmatrix} + \begin{matrix}(0,0,0)\tilde{\alpha}\\(2.52,3.03,3.53)\tilde{\alpha}\end{matrix}\right\}$$

$$\Rightarrow f\{(0,0,0)\tilde{\alpha}, (2.52,3.03,3.53)\tilde{\alpha}\}$$

$$f(x) = 5[(0,0,0)\tilde{\alpha}] - [(0,0,0)\tilde{\alpha}]^2 + 8[(2.52,3.03,3.53)\tilde{\alpha}] - 2[(2.52,3.03,3.53)\tilde{\alpha}]^2$$

$$= (20.08, 24.24, 28.24)\tilde{\alpha} - (12.5, 18.24, 24.5)\tilde{\alpha}^2$$

$$\frac{\partial f(x)}{\partial \alpha} = (20.08, 24.24, 28.24) - 2(12.5, 18.24, 24.5)\tilde{\alpha}$$

$$\Rightarrow (20.08, 24.24, 28.24) = 2(12.5, 18.24, 24.5)\tilde{\alpha}$$

$\Rightarrow \tilde{\alpha} = (0.41, 0.66, 1.21)$ in the range $\alpha \in (0, 1)$.

$$\Rightarrow f\{(0,0,0)\tilde{\alpha}, (2.52,3.03,3.53)\tilde{\alpha}\}$$

$$\Rightarrow f\{(0,0,0)(0.41,0.66,1.21), (2.52,3.03,3.53)(0.41,0.66,1.21)\}$$

$$\Rightarrow f\{(0,0,0), (2.81, 2.00, 1.45)\}$$

The Optimal solution is

$$\therefore\quad x_1 = (0,0,0)$$

$$x_2 = (2.81, 2.00, 1.45)$$

CONCLUSION

The proposed Fuzzy Sequential Linear Programming algorithm offers a suitable and well-organized method to solve the nonlinear programming problems. It provides not only the optimal solution but also near optimal solutions. So, it paves the way for the performance analysis of the solutions obtained. The optimal solutions obtained from the proposed algorithm are more approximate than many other solutions obtained from the existing methods.

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