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Toeplitz determinant for Some Subclasses of Analytic Functions

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Abstract

In this present investigation, we obtain the coefficient bounds using symmetric Toeplitz determinants $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ for the functions belonging to the subclass $\mathcal{M}_\alpha$.

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1. Introduction

Let $\mathcal{A}$ denote the family of normalized analytic functions in the open unit disk $\mathcal{D} = \{z \in \mathcal{C} : \ |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \, z \in \mathcal{D}. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathcal{D}$.

In the univalent function theory, an extensive focus has been given to estimate the bounds of Hankel matrices. Hankel determinants play a vital role in different branches and have many applications [9].

The closer relation from the Hankel determinants are the Toeplitz determinants. A Toeplitz determinant can be thought of as an 'upside-down' Hankel determinant, in that Hankel determinant have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. A good summary of the applications of Toeplitz determinant to a wide range of areas of pure and applied mathematics can also be found in [9].

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [2, 3] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

and define the symmetric Toeplitz determinant $T_q(n)$ as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

in particular,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

In this paper, we consider the symmetric Toeplitz determinants and obtain the estimates of that determinants and whose elements are the coefficients of a_n of starlike function.

We can see the brief history for the problem of finding best possible bounds $big||a_{n+1}| - |a_n||$ for the function $f \in S$, [8]. It is well-known that $||a_{n+1}| - |a_n|| \leq k$; however, finding exact values of the constant k for S and its subclasses has proved difficult. It is clear from the definition that finding estimates for $T_n(q)$ is related to finding bounds for $A(n) := |a_{n+1} - a_n|$. However, the function $h(z) = \dfrac{z}{(1+z)^2}$ shows that the best possible upper bound obtainable for $A(n)$ is $2n + 1$, and so obtaining bounds for $A(n)$ is different to finding bounds for $||a_{n+1}| - |a_n||$.

In this present paper, we obtaining the coefficient bounds for the symmetric Toeplitz determinant $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ when $f(z)$ is starlike.

2. Definitions and Preliminary Results

Definition 2.1. Let f be analytic in $\mathcal{D}$ and be given by (1.1). Then a function f is starlike and Convex if, and only if,

$$Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > 0.$$

$$Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\} > 0.$$

We denote the class of starlike functions by S^{*} and convex functions by C respectively.

Definition 2.2. Let α be real and suppose that $f(z)$ is defined by (1.1) with $f(z).f^{'}(z) \neq 0$ in $0 < |z| < 1$ if

$$Re\left\{\frac{zf^{'}(z)+\alpha z^{2}f^{''}(z)}{(1-\alpha)f(z)+\alpha zf^{'}(z)}\right\} > 0, \quad 0 \leq \alpha < 1, z \in \mathcal{D}$$

We let the class of these functions be defined by $\mathcal{M}_{\alpha}$.

The class $\mathcal{M}_{\alpha}$ is motivated and studied by [7]. Also $\mathcal{M}_{0} \equiv S^{*}$, the class of starlike functions and $\mathcal{M}_{1} \equiv C$, the class of convex functions.

3. Preliminary Results

Let $\mathcal{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

which are regular in the open unit disc $\mathcal{D}$ and satisfy $Rep(z) > 0$ for any $z \in \mathcal{D}$. Here $p(z)$ is called the Caratheodory function [1].

Lemma 3.1. [1],[8] Let the function $p \in \mathcal{P}$ be given by the series then the sharp estimate $|c_n| \leq 2, n = 1, 2, \ldots$ holds. the inequality is sharp for each n.

Lemma 3.2. [5],[6] The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$Let the function $p \in \mathcal{P}$ be given by (1.1), then

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x, $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)(1 - |x|^2)\eta$$

for some complex value η, $|\eta| \leq 1$.

4. Main Results

Theorem 4.1. Let $0 \le \alpha < 1$, and if the function $f(z)$ be of the form (1.1) belongs to the class $\mathcal{M}_\alpha$, then

$$T_2(2) = \left|a_3^2 - a_2^2\right| \le \frac{9}{(1+2\alpha)^2} - \frac{4}{(1+\alpha)^2}.$$

Proof. Let the function $f \in \mathcal{M}_\alpha$, there exists a $p \in \mathcal{P}$ such that

$$\left[zf^{'}(z) + \alpha z^2 f^{''}(z)\right] = \left[(1-\alpha)f(z) + \alpha f^{'}(z)\right] p(z).$$

then equating the coefficients, we obtain

$$a_2 = \frac{c_1}{1+\alpha};\ \ a_3 = \frac{c_2}{2(1+2\alpha)} + \frac{c_1^2}{2(1+2\alpha)};\ \ a_4 = \frac{c_3}{3(1+3\alpha)} + \frac{c_1c_2}{2(1+3\alpha)} + \frac{c_1^3}{6(1+3\alpha)} \tag{4.2}$$

Thus we have

$$\left|a_3^2 - a_2^2\right| = \left|\frac{c_1^4}{4(1+2\alpha)^2} - \frac{c_1^2}{(1+\alpha)^2} + \frac{c_2c_1^2}{2(1+2\alpha)^2} + \frac{c_2^2}{4(1+2\alpha)^2}\right|$$

substituting for c_2, and c_3 using Lemma 3.2, we obtain

$$\left|a_3^2 - a_2^2\right| = \left|\frac{9c_1^4}{16(1+2\alpha)^2} - \frac{c_1^2}{(1+\alpha)^2} + \frac{3c_1^2xX}{8(1+2\alpha)^2} + \frac{x^2X^2}{16(1+2\alpha)^2}\right|.$$

By Lemma 3.1, we have $|c_1| \le 2$. For convenience of notation,we take $c_1 = c$ and we may assume without loss of generality that $c \in [0, 2]$. Applying the triangle inequality with $X = 4 - c^2$.

$$\left|a_3^2 - a_2^2\right| \le \left|\frac{9c^4}{16(1+2\alpha)^2} - \frac{c^2}{1+\alpha)^2}\right| + \frac{3c^2|x|X}{8(1+2\alpha)^2} + \frac{|x|^2X^2}{16(1+2\alpha)^2} =: \phi(|x|).$$

Differentiating $\phi(|x|)$ and clearly $\phi^{'}(|x|) > 0$ on $[0, 1]$ and so $\phi(|x|) \le \phi(1)$. Hence

$$\begin{aligned}\left|a_3^2 - a_2^2\right| &\le \left|\frac{9c^4}{16(1+2\alpha)^2} - \frac{c^2}{(1+\alpha)^2}\right| + \frac{3c^2X}{8(1+2\alpha)^2} + \frac{X^2}{16(1+2\alpha)^2}\\ &= \left|\frac{9c^4}{16(1+2\alpha)^2} - \frac{c^2}{(1+\alpha)^2}\right| + \frac{1}{(1+2\alpha)^2}\left(1 + c^2 - \frac{5}{16}c^4\right).\end{aligned}$$

Trivially we can show that this expression $\phi(|x|)$ has a maximum value $\frac{9}{(1+2\alpha)^2} - \frac{4}{(1+\alpha)^2}$ on $[0, 2]$, when $c = 2$. ■

Remark 4.2. [4] For $\alpha = 0$, we get the sharp bound as $T_2(2) = \left|a_3^2 - a_2^2\right| \leq 5$.

Theorem 4.3. Let $0 \leq \alpha < 1$, and if the function $f(z)$ be of the form (1.1) belongs to the class $\mathcal{M}_\alpha$, then

$$T_2(3) = \left|a_4^2 - a_3^2\right| \leq \frac{16}{(1+3\alpha)^2} - \frac{9}{(1+2\alpha)^2}.$$

Proof. Using (4.2) and Lemma 3.1 to express c_2 and c_3 in terms of c_1, we obtain, with $X = 4 - c_1^2$ and $Z = (1 - |x|^2)\eta$,

$$\begin{aligned}|a_4^2 - a_3^2| = \Big| &-\frac{9}{16(1+2\alpha)^2}c_1^4 + \frac{1}{4(1+3\alpha)^2}c_1^6 - \frac{3}{8(1+2\alpha)^2}c_1^2 xX \\ &+\frac{5}{12(1+3\alpha)^2}c_1^4 xX - \frac{1}{12(1+3\alpha)^2}c_1^4 x^2 X - \frac{1}{16(1+2\alpha)^2}x^2X^2 \\ &+\frac{25}{144(1+3\alpha)^2}c_1^2 x^2 X^2 - \frac{5}{72(1+3\alpha)^2}c_1^2 x^3 X^2 + \frac{1}{144(1+3\alpha)^2}c_1^2 x^4 X^2 \\ &+\frac{1}{6(1+3\alpha)^2}c_1^3 XZ + \frac{5}{36(1+3\alpha)^2}c_1 x X^2 Z - \frac{1}{36(1+3\alpha)^2}c_1 x^2 X^2 Z \\ &+\frac{1}{36(1+3\alpha)^2}X^2Z^2\Big|\end{aligned}$$

As in the proof of Theorem 4.1, letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Thus applying the triangle inequality, we obtain

$$\begin{aligned}|a_4^2 - a_3^2| \leq &\left|\frac{1}{4(1+3\alpha)^2}c^6 - \frac{9}{16(1+2\alpha)^2}c^4\right| + \frac{3}{8(1+2\alpha)^2}c^2|x|X \\ &+\frac{5}{12(1+3\alpha)^2}c^4|x|X + \frac{1}{12(1+3\alpha)^2}c^4|x|^2X + \frac{1}{16(1+2\alpha)^2}|x|^2X^2 \\ &+\frac{25}{144(1+3\alpha)^2}c^2|x|^2X^2 + \frac{5}{72(1+3\alpha)^2}c^2|x|^3X^2 + \frac{1}{144(1+3\alpha)^2}c^2|x|^4X^2 \\ &+\frac{1}{6(1+3\alpha)^2}c^3XZ + \frac{5}{36(1+3\alpha)^2}c|x|X^2Z + \frac{1}{36(1+3\alpha)^2}c|x|^2X^2Z \\ &+\frac{1}{36(1+3\alpha)^2}X^2Z^2 \\ &=: \phi(c, |x|).\end{aligned}$$

Note that $X = 4 - c^2$ and $Z = 1 - |x|^2$.

Now to find the maximum value of ϕ over the region $\mathcal{D}$, differentiate ϕ with respect to $|x|$, we get,

$$\frac{\partial\phi}{\partial|x|} = \frac{3}{8(1+2\alpha)^2}c^2(4-c^2) + \frac{5}{12(1+3\alpha)^2}c^4(4-c^2) - \frac{1}{3(1+3\alpha)^2}c^3(4-c^2)|x|$$
$$+\frac{1}{8(1+2\alpha)^2}(4-c^2)^2|x| + \frac{25}{72(1+3\alpha)^2}c^2(4-c^2)^2|x|$$
$$-\frac{5}{18(1+3\alpha)^2}c(4-c^2)^2|x|^2 + \frac{5}{24(1+3\alpha)^2}c^2(4-c^2)^2|x|^2$$
$$-\frac{1}{18(1+3\alpha)^2}c(4-c^2)^2|x|^3 + \frac{1}{36(1+3\alpha)^2}c^2(4-c^2)^2|x|^3$$
$$+\frac{1}{6(1+3\alpha)^2}c^4(4-c^2)|x| + \frac{5}{36(1+3\alpha)^2}c(4-c^2)^2(1-|x|^2)$$
$$-\frac{1}{9(1+3\alpha)^2}|x|(1-|x|^2) + \frac{1}{18(1+3\alpha)^2}c(4-c^2)^2|x|(1-|x|^2).$$

Considering the discriminant of the resulting quadratic expression in $|x|$, which shows that, $\phi'(c,|x|) \geq 0$ for $|x| \in [0,1]$ and fixed $c \in [0,2]$. It thus follows that $\phi(c,|x|)$ increases with $|x|$, and so $\phi(c,|x|) \leq \phi(c,1)$. Hence

$$|a_4^2 - a_3^2| \leq \left|\frac{1}{4(1+3\alpha)^2}c^6 - \frac{9}{16(1+2\alpha)^2}c^4\right| + \frac{3}{8(1+2\alpha)^2}c^2(4-c^2)$$
$$+\frac{1}{3(1+3\alpha)^2}c^4(4-c^2) + \frac{1}{16(1+3\alpha)^2}(4-c^2)^2 + \frac{1}{4(1+3\alpha)^2}c^2(4-c^2)^2.$$

Now by usual calculations, we obtain this expression has maximum value $\frac{16}{(1+3\alpha)^2} - \frac{9}{(1+2\alpha)^2}$, which completes the proof. ■

Remark 4.4. [4] When $\alpha = 0$, Theorem 4.3 reduced to $T_2(3) = |a_4^2 - a_3^2| \leq 7$ and this bound is sharp.

Theorem 4.5. Let $0 \leq \alpha < 1$, and if the function $f(z)$ be of the form (1.1) belongs to the class $\mathcal{M}_\alpha$, then

$$|T_3(2)| = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right|$$
$$\leq \left[\frac{2}{(1+\alpha)} - \frac{4}{1+3\alpha}\right]\left[\frac{4}{(1+\alpha)^2} - \frac{18}{(1+2\alpha)^2} + \frac{8}{(1+\alpha)(1+3\alpha)}\right].$$

Proof. Using the same techniques as in Theorem 4.1 and Theorem 4.3, we obtain

$$|a_2 - a_4| = \left|\frac{c_1}{(1+\alpha)} - \frac{c_3}{3(1+3\alpha)} - \frac{c_1c_2}{2(1+3\alpha)} - \frac{c_1^3}{6(1+3\alpha)}\right|$$

by using triangle inequality with $c_1 = c$,

$$|a_2 - a_4| \leq \left| \frac{c}{(1+\alpha)} - \frac{c^3}{2(1+3\alpha)} \right| + \frac{c|x|X}{6(1+3\alpha)} + \frac{c|x|^2X}{12(1+3\alpha)} + \frac{XZ}{6(1+3\alpha)} + \frac{c|x|X}{4(1+3\alpha)}$$

It is easy exercise to show that $|a_2 - a_4| \leq \frac{2}{(1+\alpha)} - \frac{4}{1+3\alpha}$ and from (4.2), we get

$$\begin{aligned} \left|(a_2^2 - 2a_3^2 + a_2a_4)\right| = & \left| \frac{1}{(1+\alpha)^2} c_1^2 - \left[\frac{9}{8(1+2\alpha)^2} + \frac{1}{2(1+\alpha)(1+3\alpha)} \right] c_1^4 \right. \\ & - \left[\frac{3}{4(1+2\alpha)^2} - \frac{5}{12(1+\alpha)(1+3\alpha)} \right] c_1^2 xX \\ & - \frac{1}{12(1+\alpha)(1+3\alpha)} c_1^2 x^2 X \\ & \left. - \frac{1}{8(1+2\alpha)^2} x^2 X^2 + \frac{1}{6(1+\alpha)(1+3\alpha)} c_1 XZ \right| \end{aligned}$$

Choosing $c_1 = c \in [0, 2]$, applying triangle inequality and simplifying, we obtain

$$\begin{aligned} \left|(a_2^2 - 2a_3^2 + a_2a_4)\right| = & \left| \frac{1}{(1+\alpha)^2} c^2 - \left[\frac{9}{8(1+2\alpha)^2} + \frac{1}{2(1+\alpha)(1+3\alpha)} \right] c^4 \right| \\ & + \left[\frac{3}{4(1+2\alpha)^2} - \frac{5}{12(1+\alpha)(1+3\alpha)} \right] c^2(4-c^2)|x| \\ & + \frac{1}{12(1+\alpha)(1+3\alpha)} c^2(4-c^2)|x|^2 \\ & + \frac{1}{8(1+2\alpha)^2} (4-c^2)^2 |x|^2 \\ & + \frac{1}{6(1+\alpha)(1+3\alpha)} c(4-c^2)(1-|x|^2) := \mu(c, |x|). \end{aligned} \tag{4.3}$$

Now to find the maximum of the function $\mu(c, |x|)$ given in (4.3) on the closed region $[0, 2] \times [0, 1]$. Differentiating $\mu(c, |x|)$ partially with respect to $|x|$, and equating to 0 we get,

$$\begin{aligned} \frac{\partial u}{\partial |x|} = & \left[\frac{3}{4(1+2\alpha)^2} - \frac{5}{12(1+\alpha)(1+3\alpha)} \right] c^2(4-c^2) \\ & + \frac{1}{6(1+\alpha)(1+3\alpha)} c^2(4-c^2)|x| \\ & + \frac{1}{4(1+2\alpha)^2} (4-c^2)^2 |x| - \frac{1}{3(1+\alpha)(1+3\alpha)} c(4-c^2)|x| \end{aligned}$$

which gives that $c = 2$, is a contradiction. Further to find the maximum of $\mu(c, |x|)$, we need to consider only the end points of $[0, 2] \times [0, 1]$.

when $c = 0$,

$$\mu(0, |x|) \le \frac{2}{(1+2\alpha)^2}|x|^2 \le \frac{2}{(1+2\alpha)^2}.$$

when $c = 2$,

$$\mu(2, |x|) = \frac{4}{(1+\alpha)^2} - \frac{18}{(1+2\alpha)^2} - \frac{8}{(1+\alpha)(1+3\alpha)}.$$

when $|x| = 0$,

$$\mu(c, 0) = \left|\frac{1}{(1+\alpha)^2}c^2 - \left[\frac{9}{8(1+2\alpha)^2} + \frac{1}{2(1+\alpha)(1+3\alpha)}\right]c^4\right| + \frac{1}{6(1+\alpha)(1+3\alpha)}c(4-c^2),$$

which has the maximum value $\frac{4}{(1+\alpha)^2} - \frac{18}{(1+2\alpha)^2} - \frac{8}{(1+\alpha)(1+3\alpha)}$ on $[0, 2]$.

Also when $|x| = 1$,

$$\mu(c, 1) = \left|\frac{1}{(1+\alpha)^2}c^2 - \left[\frac{9}{8(1+2\alpha)^2} + \frac{1}{2(1+\alpha)(1+3\alpha)}\right]c^4\right| + \left[\frac{3}{4(1+2\alpha)^2} - \frac{1}{3(1+\alpha)(1+3\alpha)}\right]c^2(4-c^2) + \frac{1}{8(1+2\alpha)^2}(4-c^2)^2$$

which has the maximum value $\frac{4}{(1+\alpha)^2} - \frac{18}{(1+2\alpha)^2} - \frac{8}{(1+\alpha)(1+3\alpha)}$ on $[0, 2]$ and completes the proof of the Theorem 4.5. ■

Remark 4.6. For $\alpha = 0$, we agree with the results of Thomas et al. [4].

$$|T_3(2)| = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)\right| \le 12.$$

Theorem 4.7. Let $0 \le \alpha < 1$, and if the function $f(z)$ be of the form (1.1) belongs to the class $\mathcal{M}(\alpha)$, then

$$|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \le 1 + \frac{24}{(1+\alpha)^2(1+2\alpha)} - \frac{9}{(1+2\alpha)^2} - \frac{8}{(1+\alpha)^2}.$$

Proof. Expanding the determinant $T_3(1)$ by using (4.2) and Lemma 3.1, we obtain

$$\begin{aligned} |T_3(1)| &= \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \\ &= \left|1 + \left[\frac{3}{2(1+\alpha)^2(1+2\alpha)} - \frac{9}{16(1+2\alpha)^2}\right]c_1^4 - \frac{2}{(1+\alpha)^2}c_1^2 \right. \\ &\quad \left. - \frac{1}{8(1+2\alpha)^2}xc_1^2X - \frac{1}{16(1+2\alpha)^2}|x|^2X^2\right| \end{aligned}$$

As earlier, without loss of generality we can assume that $c_1 = c$, where $c \in [0, 2]$. and $X = (4 - c^2)$. Then, by using triangle inequality and the fact that $|x| = 1$ we obtain

$$|T_3(1)| \leq \left|1 + \left[\frac{3}{2(1+\alpha)^2(1+2\alpha)} - \frac{9}{16(1+2\alpha)^2}\right]c^4 - \frac{2}{(1+\alpha)^2}c^2\right| + \frac{1}{8(1+2\alpha)^2}c^2(4-c^2) + \frac{1}{16(1+2\alpha)^2}(4-c^2)^2.$$

Now it is easy to show that this expression has a maximum value at $c = 2$ is

$$1 + \frac{24}{(1+\alpha)^2(1+2\alpha)} - \frac{9}{(1+2\alpha)^2} - \frac{8}{(1+\alpha)^2}$$

which completes the proof. ■

Remark 4.8. For $\alpha = 0$, Theorem 4.7 reduced to the sharp bound as $|T_3(1)| = \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| \leq 8$.

References

[1] C. Caratheodory, Uber den variabilitatsbereich der Fourier'schen Konstanten Von Positiven harmonischen, Rend. Circ. Mat. Palermo., 32(1911), 193–217.

[2] Ch. Pommerenke, On the Hankel determinants of univalent functions, Mathematika., 14 (1967) 108–112.

[3] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc., 41 (1966) 111–122.

[4] D. K. Thomas, S. Abdul Halim, Toeplitz Matrices Whose Elements are the Coefficients of starlike and Close-to-Convex Functions, Bull. Malays. Math. sci. Soc., (2016), 193–217.

[5] J. Libera, E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer. Math. Soc., 87(2)(1983), 251–257.

[6] J. Libera, E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc., 85(2)(1982), 225–230.

[7] O. Altindas, On a subclass 0f certain stralike functions with negative coefficients, Math. Japan., 36(3)(1991), 489–495.

[8] P. L. Duren, Univalent functions, (Springer-Verlag, 1983), 114–115. Mat. Sb., 37(79)(195) 471–476.

[9] Ye. Ke, and Lek-Heng Lim, "Every matrix is a product of Toeplitz matrices." Foundations of Computational Mathematics., 16.3 (2016): 577–598.