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Common fixed points for F-dominated mappings

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Abstract

In this article, the concept of F-contraction has been extended by introducing F-dominated mappings. A common fixed point theorem has been proved for two commuting self mappings one of which is F-dominated by the other. The main result of the article extends Jungck's common fixed point theorem as well as Wardowski's fixed point theorem. Example is provided to justify the generality of the work.

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1. Introduction

In [1], Pfeffer proved that an involution r of a circle S has a fixed point if and only if there exists a free involution $(-r)$ of S which commutes with r. This result depicts an interdependence between the commuting mapping and the fixed point concepts. Jungck [2] further highlighted this interdependence in a more general context by proving a common fixed point theorem for two commuting mappings on a complete metric space. Banach's fixed point theorem [4] reduced to a particular case of Jungck's result when one of the two mappings is taken as the identity mapping.

Further, in 2012, Wardowski [3] introduced what he called as an F-contraction and proved that an F-contraction mapping on a complete metric space has a unique fixed point. Here F is a real valued function defined on the set of positive real numbers and it satisfies certain conditions.

For more study on F-contractions and for some of the work on common fixed points, one may refer to [7] and [10, 9, 8, 5, 6] respectively.

In this article, an extension of above mentioned results has been worked out by presenting a common fixed point result for two commuting mappings on a complete metric space such that one of them is F-dominated by the other.

2. Preliminaries

Throughout the article, X will represent a metric space with metric d, $\mathbb{R}$ will denote the set of real numbers and $\mathbb{N}$ will denote the set of natural numbers.

Let $\mathcal{F}$ be the set of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F1) F is strictly increasing, that is, $F(a) < F(b)$ whenever $a, b \in \mathbb{R}^+$ and $a < b$.

(F2) For any sequence of positive real numbers a_n we have $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists a real number k with $0 < k < 1$ and $\lim_{a\to 0^+} a^k F(a) = 0$.

These mappings have been used by Wardowski [3] to define an F-contraction which generalizes the Banach's contraction [4].

Definition 2.1. [3] For $F \in \mathcal{F}$, a mapping $f : X \to X$ is said to be an F-contraction if there exists a number $\tau > 0$ such that for all x_1, x_2 in X,

$$fx_1 \neq fx_2 \Rightarrow \tau + F(d(fx_1, fx_2)) \leq F(d(x_1, x_2))$$

Remark 2.2. [3] For $F \in \mathcal{F}$, if $f : X \to X$ is an F-contraction then we have for all x_1, x_2 in X, $d(fx_1, fx_2) \leq d(x_1, x_2)$.

3. Main Results

Definition 3.1. For $F \in \mathcal{F}$, a mapping $f : X \to X$ is said to be dominated by another mapping $g : X \to X$ if there exists a number $k \in [0, 1)$ such that for all x_1, x_2 in X,

$$d(fx_1, fx_2) \leq k\, d(gx_1, gx_2) \tag{3.1}$$

Definition 3.2. For $F \in \mathcal{F}$, a mapping $f : X \to X$ is said to be F-dominated by another mapping $g : X \to X$ if there exists a number $\tau > 0$ such that for all x_1, x_2 in X,

$$fx_1 \neq fx_2 \Rightarrow \begin{cases} gx_1 \neq gx_2 \text{ and} \\ \tau + F\,(d(fx_1, fx_2)) \leq F\,(d(gx_1, gx_2)) \end{cases}$$

Remark 3.3. For $F \in \mathcal{F}$, if $f : X \to X$ is F-dominated by $g : X \to X$ then we have for all x_1, x_2 in X, $d(fx_1, fx_2) \leq d(gx_1, gx_2)$.

In the Definition 3.2, we obtain a variety of dominations for various types of mapping F. Consider the following examples:

Example 3.4. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln a$. Clearly F satisfies all the three conditions for being an F-contraction, specially $(F3)$ for any $k \in (0, 1)$. A mapping $f : X \to X$ is F-dominated by another mapping $g : X \to X$ if and only if there exists a number $\tau > 0$ such that for all x_1, x_2 in X,

$$d(fx_1, fx_2) \leq e^{-\tau} d(gx_1, gx_2). \tag{3.2}$$

In particular, if a mapping $f : X \to X$ is dominated by another mapping $g : X \to X$, then by (3.1), we see that (3.2) holds for $\tau = \ln(1/k)$. So the mapping $f : X \to X$ is F-dominated by the mapping $g : X \to X$. Conversely, if a mapping $f : X \to X$ is F-dominated by another mapping $g : X \to X$, then by (3.2), we observe that (3.1) holds for $k = e^{-\tau}$. So $f : X \to X$ becomes dominated by $g : X \to X$.

Example 3.5. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln a + a$. It is clear that F satisfies all the three conditions for being an F-contraction. A mapping $f : X \to X$ is F-dominated by another mapping $g : X \to X$ if and only if there exists a number $\tau > 0$ such that for all x_1, x_2 in X,

$$d(fx_1, fx_2) e^{d(fx_1, fx_2) - d(gx_1, gx_2)} \leq e^{-\tau} d(gx_1, gx_2) \tag{3.3}$$

In particular, if there exists a number $k \in [0, 1)$ such that for all x_1, x_2 in X,

$$d(fx_1, fx_2) e^{d(fx_1, fx_2) - d(gx_1, gx_2)} \leq k\, d(gx_1, gx_2) \tag{3.4}$$

then by (3.4), we see that (3.3) holds for $\tau = \ln(1/k)$. So the mapping $f : X \to X$ is F-dominated by the mapping $g : X \to X$. Conversely, if a mapping $f : X \to X$ is F-dominated by another mapping $g : X \to X$, then by (3.3), we observe that (3.4) holds for $k = e^{-\tau}$.

Example 3.6. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = -1/\sqrt{a}$. Then F satisfies all the three conditions for being an F-contraction. A mapping $f : X \to X$ is said to be F-dominated by another mapping $g : X \to X$ if and only if there exists a number $\tau > 0$ such that for all x_1, x_2 in X, we have

$$d\,(fx_1, fx_2) \leq \left(1 + \tau\sqrt{d(gx_1, gx_2)}\right)^{-2} d(gx_1, gx_2). \tag{3.5}$$

Example 3.7. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(a) = \ln(a^2 + a)$. Then F satisfies all the three conditions for being an F-contraction. A mapping $f : X \to X$ is said to be F-dominated by another mapping $g : X \to X$ if and only if there exists a number $\tau > 0$ such that for all x_1, x_2 in X, we have

$$d(fx_1, fx_2)\,(d(fx_1, fx_2) + 1) \leq e^{-\tau} d(gx_1, gx_2)\,(d(gx_1, gx_2) + 1)\,. \tag{3.6}$$

In particular, if there exists a number $k \in [0, 1)$ such that for all x_1, x_2 in X we have

$$d(fx_1, fx_2)\,(d(fx_1, fx_2) + 1) \leq k\, d(gx_1, gx_2)\,(d(gx_1, gx_2) + 1) \tag{3.7}$$

then by (3.7), we see that (3.6) holds for $\tau = \ln(1/k)$. So the mapping $f : X \to X$ is F-dominated by the mapping $g : X \to X$. Conversely, if a mapping $f : X \to X$

is F-dominated by another mapping $g : X \to X$, then by (3.6), we observe that (3.7) holds for $k = e^{-\tau}$.

Remark 3.8. Let F_1, $F_2 \in \mathcal{F}$ be arbitrary. Suppose $F_1(a) \leq F_2(a)$ for all $a > 0$ and a mapping $G = F_2 - F_1$ is nondecreasing, then a mapping $f : X \to X$ is F_2-dominated by another mapping $g : X \to X$ whenever $f : X \to X$ is F_1-dominated by $g : X \to X$.

Indeed, by Definition 3.2, there exists a number $\tau > 0$ such that for all x_1, x_2 in X,

$$fx_1 \neq fx_2 \Rightarrow \begin{cases} gx_1 \neq gx_2 \text{ and} \\ \tau + F_1(d(fx_1, fx_2)) \leq F_1(d(gx_1, gx_2)) \end{cases}$$

Since F_1 is nondecreasing, this gives $d(fx_1, fx_2) \leq d(gx_1, gx_2)$. Now

$$\begin{aligned} \tau + F_2(d(fx_1, fx_2)) &= \tau + F_1(d(fx_1, fx_2)) + G(d(fx_1, fx_2)) \\ &\leq F_1(d(gx_1, gx_2)) + G(d(gx_1, gx_2)) = F_2(d(gx_1, gx_2)). \end{aligned}$$

Let $F_1(a) = \ln a$ for all $a > 0$ and $F_2(a) = \ln a + a$ for all $a > 0$. Clearly F_1, $F_2 \in \mathcal{F}$, $F_2 - F_1$ is nondecreasing and $F_1(a) \leq F_2(a)$ for all $a > 0$. So, a mapping $f : X \to X$ is F_2-dominated by another mapping $g : X \to X$ whenever $f : X \to X$ is F_1-dominated by $g : X \to X$. But converse need not be true. Example 3.12 justifies this fact.

Theorem 3.9. Let X be a complete metric space, $F \in \mathcal{F}$ and self mappings $f : X \to X$ and $g : X \to X$ satisfy the following conditions:

(a) f is F-dominated by g,

(b) f and g commute,

(c) g is continuous and

(d) $f(X) \subseteq g(X)$.

Then, there exists a unique common fixed point of f and g.

Proof. Choose an element $x_0 \in X$ arbitrarily. Since $f(X) \subseteq g(X)$, therefore, there exists an element $x_1 \in X$ such that $fx_0 = gx_1$. Again, since $f(X) \subseteq g(X)$, there exists an element $x_2 \in X$ such that $fx_1 = gx_2$. This process can be continued. Proceeding inductively, it can easily be asserted that there exists a sequence $\{x_n\}$ in X satisfying

$$fx_{n-1} = gx_n \tag{3.8}$$

for all positive integers n. This is because of the fact that $f(X) \subseteq g(X)$. Let $d_n = d(fx_n, fx_{n+1})$ for all nonnegative integer values of n. If $fx_n = fx_{n+1}$ for some nonnegative integer n then by (3.8), we obtain $fu = gu$ for $u = x_{n+1} \in X$. Let us now assume that the sequence $\{fx_n\}$ is such that any two consecutive terms are distinct. That is, $fx_n \neq fx_{n+1}$ for all nonnegative integers n. So $d_n > 0$ for all nonnegative integers n. Since f is F-dominated by g, there exists a number $\tau > 0$ such that for every positive integer n,

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \ldots \leq F(d_0) - n\tau. \tag{3.9}$$

By (3.9), we get $\lim_{n\to\infty} F(d_n) = -\infty$ which together with $(F2)$ gives

$$\lim_{n\to\infty} d_n = 0. \tag{3.10}$$

By using $(F3)$, we can find a $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0. \tag{3.11}$$

By (3.9), the following holds for all positive integers n:

$$d_n^k F(d_n) - d_n^k F(d_0) \leq d_n^k (F(d_0) - n\tau) - d_n^k F(d_0) = -d_n^k n\tau \leq 0. \tag{3.12}$$

Letting $n \to \infty$ in (3.12), and using (3.10) and (3.11), we get

$$\lim_{n\to\infty} n d_n^k = 0 \tag{3.13}$$

Now, by (3.13), there exists a positive integer p such that $nd_n^k \leq 1$ for all $n \geq p$. Consequently we have for all $n \geq p$,

$$d_n \leq 1/n^{1/k} \tag{3.14}$$

Let us choose positive integers m and n such that $m > n \geq p$. By (3.14), we get

$$d(fx_m, fx_n) \leq d_{m-1} + d_{m-2} + \ldots + d_n < \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} (1/i^{1/k}).$$

By the convergence of the series $\sum_{i=1}^{\infty}(1/i^{1/k})$, we obtain that $\{fx_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, therefore, there exists an element $u \in X$ such that $fx_n \to u$ as $n \to \infty$. Because of commutativity of f and g and for all positive integers n, we have,

$$gfx_n = fgx_n = ffx_{n-1}.$$

Continuity of g and hence of f by Remark 3.3, now gives $fu = gu$. Thus irrespective of the nature of the sequence $\{fx_n\}$, we get an element $u \in X$ such that $fu = gu$. Since f and g commute, we get $ffu = fgu = gfu$. If $ffu \neq fu$ then by F-domination of f by g, we obtain,

$$\tau + F(d(ffu, fu)) \leq F(d(gfu, gu)) = F(d(ffu, fu))$$

But this is not true. So we must have $ffu = fu$. Thus, fu becomes a common fixed point of f and g.

To prove uniqueness, suppose that x and x^* are any two common fixed points of f and g. Then we have $fx^* = x^* = gx^*$ and $fx = x = gx$. If $x \neq x^*$, then by F-domination of f by g we get,

$$\tau + F(d(x, x^*)) = \tau + F(d(fx, fx^*)) \leq F(d(gx, gx^*)) = F(d(x, x^*))$$

But this is not true. So we must have $x = x^*$. ■

The following result due to Jungck [2] for common fixed point of two self mappings on a complete metric space can be proved by taking $F(a) = \ln a$ in Theorem 3.9.

Corollary 3.10. [2] Let (X, d) be a complete metric space and the mappings $f : X \to X$ and $g : X \to X$ satisfy the following conditions:

(a) There exists a number $\alpha \in [0, 1)$ satisfying $d(fx, fy) \leq \alpha\, d(gx, gy)$ for all $x, y \in X$.

(b) f and g are commuting.

(c) g is continuous.

(d) $f(X) \subseteq g(X)$.

Then, there exists a unique common fixed point of f and g.

The following fixed point result due to Wardowski [3] for the fixed point of a self mapping on a complete metric space can be proved by taking $g = I_X$, the identity mapping on X, in Theorem 3.9.

Corollary 3.11. [3] Let (X, d) be a complete metric space and $f : X \to X$ be an F-contraction. Then there exists a unique fixed point of f.

Example 3.12. Consider the sequence $\{s_n\}$ given by $s_n = 1 + 2 + \ldots + n = n(n + 1)/2$ for all $n \in \mathbb{N}$. Let $X = \{s_n : n \in \mathbb{N}\}$. Let us consider Euclidean metric d on X. Then (X, d) is a complete metric space.

Let $f : X \to X$ and $g : X \to X$ be defined by

$$f(s_n) = \begin{cases} s_1 \text{ if } n \text{ is odd,} \\ s_{n-1} \text{ if } n \text{ is even} \end{cases} \quad \text{and } g(s_n) = \begin{cases} s_1 \text{ if } n \text{ is odd,} \\ s_{n+1} \text{ if } n \text{ is even} \end{cases}$$

Then both f and g are continuous on X. Also, it is easy to verify that for all $m, n \in \mathbb{N}$, $fs_m \neq fs_n \Rightarrow gs_m \neq gs_n$. Now, for any $s_n \in X$, $fgs_n = fs_1 = s_1 = gs_1 = gfs_n$ if n is odd and $fgs_n = fs_{n+1} = s_1 = gs_{n-1} = gfs_n$ if n is even. So f and g are commuting. Further, we have $f(X) = g(X) = \{s_{2n-1} : n \in \mathbb{N}\}$. Take $F_1 \in \mathcal{F}$ as in Example 3.4. The mapping f is not F_1-dominated by g (which means f is not dominated by g). Indeed if we take s_{2n}, s_{2n+1} in X then

$$\frac{d(fs_{2n}, fs_{2n+1})}{d(gs_{2n}, gs_{2n+1})} = \frac{|s_{2n-1} - s_1|}{|s_{2n+1} - s_1|} = \frac{2n^2 - n - 1}{2n^2 + 3n} \to 1 \text{ as } n \to \infty.$$

Hence the common fixed point theorem due to Jungck [2] can not be applied. Further,

$$d(fs_3, fs_4) = d(s_1, s_3) = 5 > 4 = d(s_3, s_4),$$
$$d(gs_1, gs_2) = d(s_1, s_3) = 5 > 2 = d(s_1, s_2)$$

and Remark 2.2 imply that f and g are not F-contractions for any $F \in \mathcal{F}$. Thus the fixed point theorem due to Wardowski [3] is also not applicable. Let $F_2 \in \mathcal{F}$ be taken as in Example 3.5. We obtain that f is F_2-dominated by g with $\tau = 2$. To see this, consider the following calculations: First, let us choose s_k, s_l in X. If both k and l are odd then we get $f(s_k) = s_1 = f(s_l)$. If k is even but l is odd then we get $fs_k = s_{k-1}$ and $fs_l = s_1$. So for $k > 2$, $fs_k \neq fs_l$. Thus for $k > 2$, we have

$$\begin{aligned}&\frac{d(fs_k, fs_l)}{d(gs_k, gs_l)} e^{d(fs_k, fs_l) - d(gs_k, gs_l)}\\ =&\frac{s_{k-1} - s_1}{s_{k+1} - s_1} e^{s_{k-1} - s_{k+1}} = \frac{k^2 - k - 2}{k^2 + 3k} e^{-2k-1} < e^{-2}.\end{aligned}$$

Now assume that both k and l are even. Then we obtain $fs_k = s_{k-1}$, $fs_l = s_{l-1}$. So $fs_k \neq fs_l$ whenever $k \neq l$. Consequently, for $k > l$, we have

$$\begin{aligned}&\frac{d(fs_k, fs_l)}{d(gs_k, gs_l)} e^{d(fs_k, fs_l) - d(gs_k, gs_l)}\\ =&\frac{s_{k-1} - s_{l-1}}{s_{k+1} - s_{l+1}} e^{s_{k-1} - s_{l-1} - s_{k+1} + s_{l+1}}\\ =&\frac{k + l - 1}{k + l + 3} e^{-2(k-l)} < e^{-2}.\end{aligned}$$

Thus f is F_2-dominated by g. Hence all the conditions of Theorem 3.9 are satisfied. We observe that s_1 is a unique common fixed point of f and g.

4. Competing interests

The author declares that there are no competing interests involved in publication of the article.

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