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The λ-analogue degenerate Changhee polynomials and numbers

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Abstract

Kim introduced the Changhee polynomials and numbers, and some interesting identities and properties of these polynomials. Also, a family of Changhee polynomials such as q-Changhee, weighted Changhee, modified Changhee, degenerate Changhee and so on is studied by many researchers. Recently, Kwon-Park also introduced modified degenerate Changhee polynomials and derived some interesting identities and properties (see [12]). In this paper, we introduce the λ-analogue degenerated Changhee polynomials and derive some new and interesting identities and properties of those polynomials.

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1. Introduction

When we use p as a fixed odd prime number, $\mathbb{Z}_p$ refers to the ring of p-adic integers, $\mathbb{Q}_p$ to the field of p-adic rational numbers, and $\mathbb{C}_p$ to the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be in $\mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p < 1$.

For $f \in C(\mathbb{Z}_p)$, the *p-adic invariant integral on $\mathbb{Z}_p$* is defined by T.Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [1-16])}, \tag{1.1}$$

where $C(\mathbb{Z}_p)$ is the the space of continuous functions on $\mathbb{Z}_p$.

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity;

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.2}$$

In particular, if $n = 1$, then

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.3}$$

The Stirling numbers of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l \ (x \geq 0), \text{ (see [12])}. \tag{1.4}$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \quad \text{(see [12])}.$$

Note that

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!}, \ (n \geq 0), \quad \text{(see [12])}.$$

As is well-known, *Euler polynomials of order r* are defined by the generating function to be

$$\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^r e^{xt} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{1.5}$$

In the special case, $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the *Euler numbers of order* r.

By (1.5), we have

$$E_n^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1+\cdots+x_r+x)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\ (n \geq 0),\ (see[4,7,9]). \tag{1.6}$$

Now, we define *modified Euler polynomials* which are given by the generating function to be

$$\int_{\mathbb{Z}_p} q^{-y} e^{(x+y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \widehat{E}_n(x)\frac{t^n}{n!}. \tag{1.7}$$

When $x = 0$, $\widehat{E}_n = \widehat{E}_n(0)$ are called the *modified Euler numbers*.

Also, we define *modified Euler polynomials of order r* which are given by the generating function to be

$$\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)} e^{(x_1+\cdots+x_r+x)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \sum_{n=0}^{\infty} \widehat{E}_n(x)\frac{t^n}{n!}. \tag{1.8}$$

When $x = 0$, $\widehat{E}_n = \widehat{E}_n(0)$ are called the *modified Euler numbers of order r* .

Kim et al. (see [3]) defined the *Changhee polynomials* as follows:

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{2+t}(1+t)^x,$$

and authors (see [11]) defined the *modified degenerate Euler of order r* polynomials as follows:

$$\sum_{n=0}^{\infty} \xi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r (1+\lambda)^{\frac{t}{\lambda}x}. \tag{1.9}$$

Recently, Changhee numbers and polynomials are introduced by Kim et. al. in [3], and by many researchers, which are generalized and obtained many new and interesting properties (see [2,3,8,10,12,14,15,16]). Recently, Kwon-Park also introduced modified degenerate Changhee polynomials and derived some interesting identities and properties(see [12]). In this paper, we consider the λ-analogue weighted degenerate Changhee polynomials and numbers by using the p-adic invariant integral, and derive some new and interesting identities and properties of those polynomials.

2. The λ-analogue degenerate Changhee polynomials and numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$.

In the viewpoint of (1.3), we define *the λ-analogue degenerate Changhee polynomials* which are given by the generating function to be

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.10}$$

In the special case, $x = 0$, $\widehat{Ch}_{n,\lambda} = \widehat{Ch}_{n,\lambda}(0)$ are called *the λ-analogue degenerate Changhee numbers*.

Now, we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\
=&\int_{\mathbb{Z}_p} q^{-y}\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n (x+y)^n(\log(1+t))^n\frac{1}{n!} \qquad (2.11)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p} q^{-y}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)d\mu_{-1}(y)\right)\frac{t^n}{n!}.
\end{aligned}$$

Where $S_1(n, m)$ is the Stirling numbers of the first kind.

From (2.10) and (2.11), we are able to derive the following theorem.

Theorem 2.1. For each $n \in \mathbb{N} \cup \{0\}$, $\widehat{Ch}_{n,\lambda}(x)$ can be written as

$$\widehat{Ch}_{n,\lambda}(x) = \sum_{m=0}^{n}\int_{\mathbb{Z}_p} q^{-y}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)d\mu_{-1}(y). \qquad (2.12)$$

Then, by using (2.10), we are able to calculate

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)d\mu_{-1}(y)\\
=&\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p} q^{-y}(x+y)^m d\mu_{-1}(y) \qquad (2.13)\\
=&\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\widehat{E}_m(x).
\end{aligned}$$

Where $\widehat{E}_m(x)$ is the weighted Euler polynomials.

From (2.13), we are able to derive the following theorem.

Theorem 2.2. For each $n \in \mathbb{N} \cup \{0\}$, $\widehat{Ch}_{n,\lambda}(x)$ can be written as

$$\widehat{Ch}_{n,\lambda}(x) = \sum_{m=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\widehat{E}_m(x). \qquad (2.14)$$

By substituting t as $e^t - 1$ in (2.10), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\
=&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}t}d\mu_{-1}(y),
\end{aligned} \qquad (2.15)$$

and

$$\begin{aligned}&\sum_{m=0}^{\infty}\widehat{Ch}_{m,\lambda}(x)\frac{1}{m!}(e^t-1)^m\\ =&\sum_{m=0}^{\infty}\widehat{Ch}_{m,\lambda}(x)\frac{1}{m!}m!\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}(x)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned}\tag{2.16}$$

where $S_2(n,m)$ are the Stirling numbers of the second kind.

The left hand side of (2.15) is given by

$$\int_{\mathbb{Z}_p}q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}t}d\mu_{-1}(y)=\sum_{n=0}^{\infty}\xi_{n,\lambda}(x)\frac{t^n}{n!}.\tag{2.17}$$

By comparing on the both sides (2.15), (2.16) and (2.17), we obtain the following theorem.

Theorem 2.3. For each $n\in\mathbb{N}\cup\{0\}$, $\xi_{n,\lambda}(x)$ can be written as

$$\xi_{n,\lambda}(x)=\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}(x)S_2(n,m).\tag{2.18}$$

Note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}q^{-y}(1+\lambda)^{\frac{x+y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&\int_{\mathbb{Z}_p}q^{-y}(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&(1+\lambda)^{\frac{x}{\lambda}\log(1+t)}\int_{\mathbb{Z}_p}q^{-y}(1+\lambda)^{\frac{y}{\lambda}\log(1+t)}d\mu_{-1}(y)\\ =&\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}x^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\widehat{Ch}_{k,\lambda}\frac{t^k}{k!}\right)\\ =&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}x^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)\widehat{Ch}_{n-m,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.19}$$

The following theorem is obtained from (2.19).

Theorem 2.4. For each $n\in\mathbb{N}\cup\{0\}$, $\widehat{Ch}_{n,\lambda}(x)$ can be written as

$$\widehat{Ch}_{n,\lambda}(x)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}x^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l S_1(m,l)\widehat{Ch}_{n-m,\lambda}.\tag{2.20}$$

Now, we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m (x+y)^m S_1(n,m)d\mu_{-1}(y)\\
&=\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p} q^{-y}(x+y)^m d\mu_{-1}(y)\\
&=\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\sum_{l=0}^{m}\binom{m}{l}x^{m-l}\int_{\mathbb{Z}_p} q^{-y}y^l d\mu_{-1}(y)\\
&=\sum_{l=0}^{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\binom{m}{l}x^{m-l}\widehat{E}_l
\end{aligned} \tag{2.21}$$

Therefore, by using (2.21), the following theorem can be derived.

Theorem 2.5. For each $n \in \mathbb{N}\cup\{0\}$, $\widehat{Ch}_{n,\lambda}(x)$ can be written as

$$\widehat{Ch}_{n,\lambda}(x)=\sum_{l=0}^{m}\left(\frac{\log(1+\lambda)}{\lambda}\right)^m S_1(n,m)\binom{m}{l}x^{m-l}\widehat{E}_l. \tag{2.22}$$

From now on, we define *the λ-analogue degenerate Changhee polynomials of order r* as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.23}$$

When $x=0$, $\widehat{Ch}_{n,\lambda}^{(r)}=\widehat{Ch}_{n,\lambda}^{(r)}(0)$ are called *the λ-analogue degenerate Changhee numbers of order r*.

Now, we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}\log(1+t)}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}\left(\frac{\log(1+t)}{\lambda}\right)^m (x_1+\cdots+x_r+x)^m S_1(n,m)d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\left(\frac{\log(1+t)}{\lambda}\right)^m S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(x_1+\cdots+x_r+x)^m d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\
&=\sum_{n=0}^{\infty}\left(\frac{\log(1+t)}{\lambda}\right)^m S_1(n,m)\widehat{E}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.24}$$

Therefore, by (2.24), we are able to derive the following theorem.

Theorem 2.6. For each $n \in \mathbb{N}\cup\{0\}$ and $r\in\mathbb{N}$, $\widehat{Ch}_{n,\lambda}^{(r)}(x)$ can be written as

$$\widehat{Ch}_{n,\lambda}^{(r)}(x)=\left(\frac{\log(1+t)}{\lambda}\right)^m S_1(n,m)\widehat{E}_{n,\lambda}^{(r)}(x). \tag{2.25}$$

By substituting t as $e^t - 1$ in (2.23), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &=\sum_{m=0}^{\infty}\widehat{Ch}_{m,\lambda}(x)^{(r)}\frac{1}{m!}(e^t-1)^m\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}^{(r)}(x)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.26}$$

The left hand side of (2.26) is given by

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_r)}(1+\lambda)^{\frac{x_1+\cdots+x_r+x}{\lambda}t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)=\sum_{n=0}^{\infty}\xi_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.27}$$

By comparing on the both sides (2.26) and (2.27), we obtain the following theorem.

Theorem 2.7. For each $n \in \mathbb{N}\cup\{0\}$, $\xi_{n,\lambda}^{(r)}(x)$ can be written as

$$\xi_{n,\lambda}^{(r)}(x)=\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}^{(r)}(x)S_2(n,m). \tag{2.28}$$

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