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Approximation of fixed points in 2-Hilbert spaces

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Abstract

It is well known that from any point x in a real uniformly convex Banach space X to any of its closed convex subsets B, there exists a unique point $\bar{x}$ in B called the metric projection of x on B which has the minimal distance to x. We extend the notion of metric projection to uniformly convex 2-Banach spaces. We also extend the definition of weakly contractive in 2-normed space and proved the fixed point theorem of nonself maps in 2-Hilbert spaces. Moreover, we defined uniform smoothness of 2-normed spaces and proved some theorems relating uniform smoothness and uniform convexity in 2-normed spaces.

AMS subject classification:
Keywords:

1. Introduction

S. Gähler, in 1965, has initiated the notion of linear 2-normed spaces, [9]. Gähler, White, Diminnic, Newton, and other various workers have studied the geometric structure of linear 2-normed space. The first two workers extended the concept to 2-Banach spaces, where A. White [1] established Hahn-Banach theorem in a 2-Banach space. Also, the concept of uniform convexity of 2-normed spaces has been defined by M.E. Newton [4]. Now in this paper, we extend the notion of uniform smoothness of Banach spaces to the 2-normed spaces. Besides, we also extend some theorems to the case of uniformly convex uniformly smooth 2-normed spaces. Moreover, we prove the existence of nearest point in these spaces and the approximation of fixed points in 2-Hilbert spaces.

Definition 1.1. [1,7,9] Let X be a real linear space of $dimX \geq 2$ and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
2. $\|x, y\| = \|y, x\| \; \forall x, y \in X$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\| \; \forall x, y \in X$ and $\alpha \in R$,
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\| \; \forall x, y, z \in X$.

$\|., .\|$ is called a 2-norm on X and $(X, \|., .\|)$ is called a linear 2-normed space.

A sequence $\{x_n\}$ is called a Cauchy sequence in a linear 2-normed space $(X, \|., .\|)$ if $\lim_{n,m\to\infty} \|x_n - x_m, c\| = 0$ for all $c \in X$. A sequence $\{x_n\}$ is convergent to an element x in a linear 2-normed space $(X, \|., .\|)$ if $\lim_{n\to\infty} \|x_n - x, c\| = 0$ for all $c \in X$. If every Cauchy sequence in X converges to a point of X, then a linear 2-normed space $(X, \|., .\|)$ is said to be a 2-Banach space.

Definition 1.2. **[3,7]** Let X be a real linear space with $dimX > 1$ and let $\langle ., ./. \rangle$ be a real valued function on $X \times X \times X$ which satisfies the following conditions:

1. $\langle x, x/y \rangle \geq 0$ and $\langle x, x/y \rangle = 0$ iff x and y are linearly dependent,
2. $\langle x, x/y \rangle = \langle y, y/x \rangle$ for all $x, y \in X$,
3. $\langle x, y/z \rangle = \langle y, x/z \rangle$ for all $x, y, z \in X$,
4. $\langle \alpha x, y/z \rangle = \alpha \langle x, y/z \rangle$ for all $x, y, z \in X$ and scalars $\alpha \in R$,
5. $\langle x_1 + x_2, y/z \rangle = \langle x_1, y/z \rangle + \langle x_2, y/z \rangle$ for all $x_1, x_2, y, z \in X$.

$\langle ., ./. \rangle$ is called a 2-inner product and $(X, \langle ., ./. \rangle)$ is called a real 2-inner product space.

Every 2-inner product space $(X, \langle ., ./. \rangle)$ is a linear 2-normed space with the 2-norm $\|x, y\| = \sqrt{\langle x, x/y \rangle}$. This norm satisfies the Parallelogram Law ($\|x + y, z\|^2 + \|x - y, z\|^2 = 2(\|x, z\|^2 + \|y, z\|^2)$). On the other hand, if the 2-norm satisfies the Parallelogram Law, then it generates the 2-inner product space with the 2-inner product $\langle x, y/z \rangle = 1/4(\|x + y, z\|^2 - \|x - y, z\|^2)$. Any complete 2-inner product space H is said to be a 2-Hilbert space. The Cauchy-Schwarz's inequality takes the form: $\langle x, y/z \rangle \leq \sqrt{\langle x, x/z \rangle}\sqrt{\langle y, y/z \rangle} = \|x, z\| \|y, z\|$.

Definition 1.3. **[7]** Let $(X, \|., .\|)$ be a linear 2-normed space. Let L, M be linear manifolds of X. A functional $f : L \times M \to R$ is said to be bilinear functional if it satisfies the following conditions:

1. $f(a + c, b + d) = f(a, b) + f(a, d) + f(c, b) + f(c, d)$ for all $(a, b), (c, d) \in L \times M$,
2. $f(\alpha a, \beta b) = \alpha\beta f(a, b)$ for all $\alpha, \beta \in R$ and all $(a, b) \in L \times M$.

A functional $f : L \times M \to R$ is said to be bounded functional if there is a real number $K > 0$ such that $|f(a,b)| \leq K\|a,b\|$ for all $(a,b) \in L \times M$. Then, the number

$$\|f\| = \inf\{K > 0 : |f(a,b)| \leq K\|a,b\|; (a,b) \in L \times M\}$$

is called the norm of the bilinear bounded functional f.

Remark 1.4. For any $c \neq 0$, let $V(c)$ be the linear manifold generated by c. By $(X \times V(c))^*$ or X_c^* we denote the set of all bounded bilinear functionals $f : X \times V(c) \to R$ equipped with the norm

$$\|f\| = \sup\left\{\frac{|f(x,c)|}{\|x,c\|} : \|x,c\| \neq 0, (a,c) \in X \times V(c)\right\}.$$

Thus, the space $(X_c^*, \|.\|)$ is a Banach space.

Definition 1.5. [7] A functional $f : L \times M \to R$ is said to be continuous at $(a,b) \in L \times M$, where L, M are linear manifolds of a linear 2-normed space, if for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\min\left(\max_{(x,y)\in L\times M}\left\{\|x-a,b\|, \|x,y-b\|\right\}, \max_{(x,y)\in L\times M}\left\{\|x-a,y\|, \|a,y-b\|\right\}\right) < \delta.$$

This implies, $|f(x,y) - f(a,b)| < \varepsilon$. A functional f is said to be continuous if it is continuous at each point of its domain $L \times M$.

One can see that, the 2-norm $\|.,.\|$ is a continuous functional.

Theorem 1.6. [5,7] Let $(X, \|.,.\|)$ be a linear 2-normed space and x_o in X be a non-zero element. Let $c \in X$ be an element such that x_o and c are linearly independent. Then, there exists a bounded bilinear functional f with domain X_c^* such that:

1. $f(x_o, c) = \|x_o, c\|$,
2. $\|f\| = 1$.

Definition 1.7. [8] If X be a linear 2-normed space, then a mapping $A : X \to X$ is said to be nonexpansive if A satisfies $\|Ax - Ay, c\| \leq \|x - y, c\|$ for any $x, y, c \in X$.

Lemma 1.8. [2,6] Let $\{\lambda_n\}$ be a sequence of nonnegative numbers, and $\{\alpha_n\}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. If the recursive inequality $\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n)$ holds, where $\psi(\lambda)$ is a continuous strictly increasing function for all $\lambda \geq 0$ with $\psi(0) = 0$, then:

1. $\lambda_n \to 0$ as $n \to \infty$,

2. the estimate of the convergence rate $\lambda_n \leq \Phi^{-1}(\Phi(\lambda_1) - \sum_{j=1}^{n-1} \alpha_j)$ is satisfied, where Φ is defined by $\Phi(t) = \int \frac{dt}{\psi(t)}$, and Φ^{-1} is the inverse function to Φ.

Definition 1.9. [4] A linear 2-normed space $(X, \|.,.\|)$ is said to be uniformly convex if for every ε in $(0, 2]$ and $c \neq 0$ in X, there exists a $\delta(\varepsilon, c)$ such that $\|x, c\| \leq 1$, $\|y, c\| \leq 1$ and $\|x - y, c\| \geq \varepsilon$ imply that $\left\| \frac{1}{2}(x + y), c \right\| \leq 1 - \delta(\varepsilon, c)$.

Example 1.10. As R^3 is a 2-Banach space, defined the norm

$$\|x, y\| = \left[(bf - ce)^2 + (cd - af)^2 + (ae - bd)^2\right]^{\frac{1}{2}},$$

where $x = (a, b, c)$, $y = (d, e, f)$. Then, the space $(R^3, \|.,.\|)$ is uniformly convex linear 2-normed space.

2. Main Results

Definition 2.1. A linear 2-normed space X is said to be uniformly smooth if for a given $\varepsilon > 0$ and $c \neq 0$ there exists a $\delta(\varepsilon, c) > 0$ such that if $\|x, c\| = 1$ and $\|y, c\| \leq \delta$, then

$$\|x + y, c\| + \|x - y, c\| < 2 + \varepsilon \|y, c\|.$$

Definition 2.2. Let $(X, \|.,.\|)$ be a linear 2-normed space with $\dim X \geq 2$. The modulus of smoothness of X is the function $\rho_X : [0, \infty) \times X \to [0, \infty)$ defined by

$$\rho_X(\tau, c) := \sup \left\{ \frac{\|x + y, c\| + \|x - y, c\|}{2} - 1 : \|x, c\| = 1; \|y, c\| = \tau \right\}.$$

Theorem 2.3. A linear 2-normed space $(X, \|.,.\|)$ is uniformly smooth if and only if $\lim_{t \to 0} \frac{\rho_X(t, c)}{t} = 0$ for any $0 \neq c \in X$.

Proof. Assume that $(X, \|.,.\|)$ is uniformly smooth and $c \neq 0$ and if $\varepsilon > 0$ is given, then there exists $\delta > 0$ such that

$$\frac{\|x + y, c\| + \|x - y, c\|}{2} - 1 < \frac{\varepsilon}{2} \|y, c\|$$

for every x and y in X with $\|x, c\| = 1$ and $\|y, c\| = \delta$. This implies that $\rho_X(t, c) \leq \frac{\varepsilon}{2} t$ for every $t < \delta$.

Conversely, for all $c \neq 0$ and given $\varepsilon > 0$ suppose that there exists $\delta > 0$ such that $\rho_X(t, c) < \frac{\varepsilon}{2} t$ for every $t < \delta$. Let x and y be in X with $\|x, c\| = 1$ and $\|y, c\| = \delta$.

Then, with $t = \|y, c\|$ we have $\|x + y, c\| + \|x - y, c\| < 2 + \varepsilon\|y, c\|$ and thus the space is uniformly smooth. ■

Now, we prove one of the fundamental links between Lindenstrauss duality formulas.

Proposition 2.4. Let $(X, \|., .\|)$ be a 2-Banach space. For every $\tau > 0$, $c \neq 0$, x in X with $\|x, c\| = 1$ and f in X_c^* with $\|f\| = 1$, we have

(a) $\rho_X(\tau, c) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X_c^*}(\varepsilon) : 0 < \varepsilon \leq 2\right\}$,

(b) $\rho_{X_c^*}(\tau) = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_X(\varepsilon, c) : 0 < \varepsilon \leq 2\right\}$.

Proof. (a) Given $\tau > 0$, $c \neq 0$ in X and let f, g be in X_c^* with $\|f\| = \|g\| = 1$. For any $\eta > 0$, and using the definition of $\|.\|$ in X there exist x_o, y_o in X with $\|x_o, c\| = \|y_o, c\| = 1$ such that

$$\|f + g\| - \eta \leq (f + g)(x_o, c) \quad \textit{and} \quad \|f - g\| - \eta \leq (f - g)(y_o, c).$$

By adding both inequalities, we get

$$\|f + g\| + \tau\|f - g\| - 2 \;\leq\; f(x_o + \tau y_o, c) + g(x_o - \tau y_o, c) - 2 + \eta(1 + \tau).$$

Knowing that in a 2-Banach space, $\|x, c\| = \sup\{|f(x, c)| : \|f\| = 1\}$, we get

$$\begin{aligned} \|f + g\| + \tau\|f - g\| - 2 &\leq \|x_o + \tau y_o, c\| + \|x_o - \tau y_o, c\| - 2 + \eta(1 + \tau) \\ &\leq 2\rho_X(\eta, c) + \eta(1 + \tau). \end{aligned}$$

If $0 < \varepsilon \leq \|f - g\|$, we have

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau, c) - \frac{\eta}{2}(1 + \tau) \leq 1 - \left\|\frac{f + g}{2}\right\|,$$

that leads to

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau, c) + \frac{\eta}{2}(1 + \tau) \leq \delta_{X_c^*}.$$

Now, since $\eta > 0$ is arbitrary, we conclude that

$$\frac{\tau\varepsilon}{2} - \rho_X(\tau, c) \leq \delta_{X_c^*} \quad \text{for every } \varepsilon \text{ in } (0, 2].$$

Therefore,

$$\sup\left\{\frac{\varepsilon\tau}{2} - \delta_{X_c^*}(\varepsilon) : 0 < \varepsilon \leq 2\right\} \leq \rho_X(\tau).$$

On the other hand, if $\tau > 0$, $c \neq 0$ in X and x, y in X with $\|x, c\| = 1 = \|y, c\|$, then by Theorem 1.6 there exist f_o, g_o in X_c^* with $\|f_o\| = 1 = \|g_o\|$ such that

$$f_o(x + \tau y, c) = \|x + \tau y, c\| \quad \textit{and} \quad g_o(x - \tau y, c) = \|x - \tau y, c\|.$$

Hence,

$$\begin{aligned}\|x+\tau y,c\|+\|x-\tau y,c\|-2 &= (f_o+g_o)(x,c)+\tau(f_o-g_o)(y,c)-2\\ &\leq \|f_o+g_o\|+\tau|(f_o-g_o)(y,c)|-2.\end{aligned}$$

Thus, if we defined $\varepsilon_o := |(f_o-g_o)(y,c)|$, then $0<\varepsilon_o\leq\|x_o-y_o,c\|\leq 2$ and

$$\begin{aligned}\frac{\|x+\tau y,c\|+\|x-\tau y,c\|}{2}-1 &\leq \frac{\tau\varepsilon_o}{2}-\left(1-\left\|(f_o+g_o)/2\right\|\right)\\ &\leq (\tau\varepsilon_o)/2-\delta_{X_c^*}(\varepsilon).\end{aligned}$$

Therefore,

$$\rho_X(\tau,c)\leq\sup\{\tau\varepsilon_o/2-\delta_{X_c^*}(\varepsilon):0<\varepsilon\leq 2\}.$$

(b) The proof of second formula is similar to (a). ■

Proposition 2.5. Let $(X,\|.,.\|)$ be a 2-Banach space and $c\neq 0$ in X. Then, X is uniformly smooth if and only if X_c^* is a uniformly convex normed space.

Proof. We will prove both directions of this proposition by contradiction.
"$\to$" If X_c^* is not uniformly convex where $c\neq 0$ in X, then there exists ε_o in $(0,2]$ and $c_o\neq 0$ with $\delta_{X_{c_o}^*}(\varepsilon_o)=0$, and by using Proposition 2.4 (b), we obtain for every $\tau>0$,

$$0<\frac{\varepsilon_o}{2}\leq\frac{\rho_X(\tau,c_o)}{\tau},\quad \textit{hence}\quad \lim_{t\to 0}\frac{\rho_X(\tau,c_o)}{\tau}\neq 0,$$

which means that X is not uniformly smooth.
"$\leftarrow$" Assume that X is not uniformly smooth. Then, there exists $c_o\neq 0$ with $\lim_{t\to 0}\frac{\rho_X(t,c_o)}{t}\neq 0$. Thus, there exists $\varepsilon>0$ such that for every $\delta>0$ we can find t_δ with $0<t_\delta<\delta$ and $t_\delta\varepsilon\leq\rho_X(t_\delta,c_o)$ for some $c_o\neq 0$. Then, there exists a sequence $\{\tau_n\}$ such that $0<\tau_n\leq 1$, $\tau_n\to 0$ and $\rho_X(\tau_n,c_o)\geq\frac{\varepsilon}{2}\tau_n$. By Using Proposition 2.4 (b), for every $n\in N$ there exists ε_n in $(0,2]$ such that

$$\frac{\varepsilon}{2}\tau_n<\delta_{X^*c_o}(\varepsilon_n)\leq\frac{\tau_n}{2}(\varepsilon_n-\varepsilon).$$

In particular $\varepsilon<\varepsilon_n$ and $\delta_{X_{c_o}^*}(\varepsilon_n)\to 0$. Given the fact that $\delta_{X_{c_o}^*}(\varepsilon_n)$ is a non-decreasing function, we get $\delta_{X_{c_o}^*}(\varepsilon_n)<\delta_{X_c^*}(\varepsilon_n)\to 0$. Therefore, $X_{c_o}^*$ is not uniformly convex. ■

Theorem 2.6. Let $(X,\|.,.\|)$ be a 2-Banach space and $c\neq 0$ in X. Then, X is uniformly convex if and only if X_c^* is a uniformly smooth normed space.

Proof. Using Proposition 2.4 (a) and interchanging the role of X and X_c^*, we get the proof of this theorem. ■

Now, we show that, from any point x in a linear 2-normed space X to a closed convex subset B, we can get a unique nearest point $\bar{x}$ in B such that $\|x-\bar{x},c\|=\inf_{\xi\in B}\|x-\xi,c\|$ in which x does not depend on $c\neq 0$.

Theorem 2.7. Let X be a real uniformly convex 2-Banach space and B be a nonempty closed convex proper subset of X. Then, the metric projection is well defined on B, i.e.,

$$\forall x \in X, \exists!\, \bar{x} \in B : \forall\, (0 \neq)\, c \in X \Rightarrow \|x - \bar{x}, c\| = \inf_{\xi \in B} \|x - \xi, c\|.$$

Claim. Let $\{x_n\}$ be a sequence in the real uniformly convex 2-Banach space X satisfying:

$$\lim_{n,m\to\infty} \|x_n + x_m, c\| = 2 \lim_{n\to\infty} \|x_n, c\| \quad \text{for all} \quad 0 \neq c \in X.$$

Then, $\{x_n\}$ is a Cauchy sequence in X.

Proof of the claim. If $\lim_{n\to\infty} \|x_n, c\| = l_c$, then

$$\lim_{n\to\infty} \left\| \frac{x_n}{l_c}, c \right\| = 1 \quad and \quad \lim_{n,m\to\infty} \left\| \frac{(x_n + x_m)}{l_c}, c \right\| = 2 \quad \text{for all} \quad 0 \neq c \in X.$$

Suppose contrarily that the sequence $\{x_n\}$ is not a Cauchy sequence in X. Then,

$$\|x_n - x_m, c\| \nrightarrow 0 \quad as \quad n, m \to \infty \quad \text{for some} \quad c \neq 0,$$

i.e., there exist ε_c and two subsequences of natural numbers n_k, m_k such that

$$\|x_{n_k} - x_{m_k}, c\| \geq \varepsilon_c, \quad \text{and hence} \quad \left\| \frac{x_{n_k} - x_{m_k}}{l_c}, c \right\| \geq \frac{\varepsilon_c}{l_c} > 0.$$

Since $(X, \|.,.\|)$ is uniformly convex 2-Banach space, then $\exists\, \delta_c > 0$ such that

$$\left\| \frac{x_{n_k} + x_{m_k}}{2l_c}, c \right\| \leq 1 - \delta_c.$$

Taking limit as $n_k, m_k \to \infty$, we get $1 \leq 1 - \delta_c$ which is impossible because $\delta_c > 0$, this contradiction completes the proof of the claim. ■

Proof of the theorem. Let $d_c = \inf\{\|x - \xi, c\| : \xi \in B\}$ for all $0 \neq c \in X$. Then, there exists a sequence $\{\xi_n\}$ in B such that

$$d_c \leq \|x - \xi_n, c\| < d_c + \frac{1}{n}, \quad \text{and hence} \quad \lim_{n\to\infty} \|x - \xi_n, c\| = d_c. \tag{2.1}$$

Since B is convex, then $\frac{\xi_n + \xi_m}{2} \in B \quad \forall n, m \in N$. Then,

$$d_c \leq \left\| x - \frac{\xi_n + \xi_m}{2}, c \right\| < d_c + \frac{1}{n} + \frac{1}{m}.$$

Then,

$$\lim_{n,m\to\infty} \|x - \xi_n + x - \xi_m, c\| = 2d_c \quad as \quad n, m \to \infty. \tag{2.2}$$

From (2.1), (2.2) and using the claim, then $\{x-\xi_n\}$ is a Cauchy sequence, i.e., $\|x-\xi_n-(x-\xi_m), c\| \to 0$ for all $c \neq 0$ that gives $\|\xi_n-\xi_m, c\| \to 0$ for all $c \neq 0$. Therefore, $\{\xi_n\}$ is a Cauchy sequence in the 2-Banach space X, hence there exist $\bar{x} \in X$ such that $\xi_n \to \bar{x}$. Since B is closed in X and $\xi_n \in B$, then $\bar{x} \in B$, hence $x-\xi_n \to x-\bar{x}$. Using the continuity of the 2-norm, then

$$\|x-x_n, c\| \to \|x-\bar{x}, c\| = d_c.$$

Now, we prove the uniqueness of $\bar{x}$: Suppose contrarily that there exists

$$\bar{y} \in B : \|x-\bar{y}, c\| = \inf\{\|x-\xi, c\| : \xi \in B\} = d_c \quad and \quad \bar{y} \neq \bar{x}.$$

Then, $\|\bar{y}-\bar{x}, c\| = \varepsilon_c > 0$ for some $0 \neq c \in X$. Since B is convex, then $\frac{\bar{x}+\bar{y}}{2} \in B$ and so

$$d_c \leq \left\|x-\frac{\bar{x}+\bar{y}}{2}, c\right\| \leq \frac{d_c}{2}+\frac{d_c}{2} = d_c.$$

Hence, $\left\|x-\frac{\bar{x}+\bar{y}}{2}, c\right\| = d_c$ where $\bar{y} \neq \frac{\bar{x}+\bar{y}}{2} \neq \bar{x}$.
Since X is uniformly convex and

$$\left\|\frac{x-\bar{x}}{d_c}-\frac{x-\bar{y}}{d_c}, c\right\| = \frac{\varepsilon_c}{d_c}.$$

Then, $\exists\, \delta_c > 0$ such that

$$\left\|x-\frac{\bar{x}+\bar{y}}{2d_c}, c\right\| \leq 1-\delta_c.$$

Therefore,

$$d_c = \left\|x-\frac{\bar{x}+\bar{y}}{2}, c\right\| \leq d_c - d_c\delta_c.$$

This is impossible because $\delta_c > 0$. Therefore, $\bar{x}$ is unique. ■

Lemma 2.8. Let B be a nonempty closed convex subset of a 2-Hilbert space X and P_B the metric projection from X onto B. Then, P_B is nonexpansive.

Proof. Is easy. ■

Definition 2.9. Let B be a nonempty subset of a real 2-Hilbert space H. A mapping $A : B \to B$ is called weakly contractive of the class $C_{(\psi(t))}$; if there exists a continuous and nondecreasing function $\psi(t)$ defined on R^+ with $\phi(t) > 0\ \forall\, t > 0$, $\psi(0) = 0$ and $\lim_{n\to\infty} \psi(t) = +\infty$; such that

$$\|Ax-Ay, c\| \leq \|x-y, c\| - \psi(\|x-y, c\|) \quad \forall x, y, c \in B.$$

Theorem 2.10. Let B be a closed convex subset of 2-Hilbert space H and let $A : B \to H$ be a weakly contractive map of the class $\psi(t)$ with strictly increasing function $\psi(t)$. Suppose that the set $F(A)$ of all fixed points of the map A is nonempty set, and for $x_1 \in B$ consider the iteration $x_{n+1} = P_B A x_n$, $n \geq 0$. Then, $\{x_n\}$ and $\{Ax_n\}$ are bounded in H, and $\{x_n\}$ strongly converges to some point $x^* \in F(A)$, and the estimate

$$\|x_n - x^*, c\| \leq \Phi^{-1}\big(\Phi(\|x_1 - x^*, c\| - (n-1))\big) \quad \forall c \in H$$

is satisfied, where Φ is defined by $\Phi(t) = \displaystyle\int \frac{dt}{\psi(t)}$.

Proof. Since P_B is nonexpansive and A is weakly contractive, then $\forall c \in H$

$$\begin{aligned} \|x_{n+1} - x^*, c\| &\leq \|Ax_n - Ax^*, c\| \\ &\leq \|x_n - x^*, c\| - \psi(\|x_n - x^*, c\|). \end{aligned} \tag{2.3}$$

Thus, the sequence of positive numbers $\{\lambda_n\}$ defined by $\lambda_n := \|x_n - x^*, c\|$ satisfies

$$0 \leq \lambda_{n+1} \leq \lambda_n - \psi(\lambda_n) \leq \lambda_n.$$

This implies that the sequence is non-increasing and bounded below by 0, thus converges to some λ. As $n \to \infty$, we get $\lambda \leq \lambda - \psi(\lambda) \leq \lambda$, then $0 \leq -\psi(\lambda) \leq 0$. By the hypothesis of ψ, we get $\psi(0) = 0$, i.e., $\lambda_n \to 0$, i.e., $\|x_n - x^*, c\| \to 0\ \forall\, c \in H$, thus x_n is convergent to x^*. Further, by Lemma 1.8 we have the estimate:

$$\lambda_n \leq \Phi^{-1}(\Phi(\lambda_1) - (n-1)) \qquad \forall n \geq 1.$$

From (2.3),

$$\|x_{n+1} - x^*, c\| \leq \|x_n - x^*, c\|.$$

Hence by induction,

$$\|x_n - x^*, c\| \leq \|x_1 - x^*, c\|. \tag{2.4}$$

Since

$$\|x - y, c\|^2 = \|x, c\|^2 - 2\langle x, y/c\rangle + \|y, c\|^2 \quad \textit{and} \quad |\langle x, y/c\rangle| \leq \|x, c\|\|y, c\|,$$

then

$$(\|x, c\| - \|y, c\|)^2 \leq \|x - y, c\|^2 \leq (\|x, c\| + \|y, c\|)^2. \tag{2.5}$$

Therefore, by (2.4) and (2.5),

$$\|x_n, c\| - \|x^*, c\| \leq \|x_1, c\| + \|x^*, c\|.$$

Since x_1, x^* are fixed, this implies that

$$\|x_n, c\| \leq \|x_1, c\| + 2\|x^*, c\|,$$

then sequence $\{x_n\}$ is bounded. Using (2.3) − (2.4), we get

$$\|Ax_n - Ax^*, c\| \leq \|x_1 - x^*, c\|. \tag{2.6}$$

Therefore, by (2.5) and (2.6),

$$(\|Ax_n, c\| - \|Ax^*, c\|)^2 \leq \|Ax_n - Ax^*, c\|^2 \leq (\|x_1, c\| + \|x^*, c\|)^2.$$

Since $\{x_n\}$ is bounded, then

$$\|Ax_n, c\| - \|x^*, c\| \leq \|x_1, c\| + \|x^*, c\|,$$

thus

$$\|Ax_n, c\| \leq \|x_1, c\| + 2\|x^*, c\|.$$

Therefore, the sequence $\{Ax_n\}$ is bounded. ■

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