Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 3 (2017), pp. 911–920

http://www.ripublication.com/gjpam.htm

Some identities of symmetry for the modified degenerate Carlitz q-Bernoulli polynomials under symmetry group of degree n

Joohee Jeong
Department of Mathematics Education,
Kyungpook National University,
Daegu 41566, S. Korea.

Dong-Jin Kang
Department of Computer Engineering,
Information Technology Services,
Kyungpook National University, Daegu 41566, S. Korea.

Hong Kyung Pak
Department of Medical Information Technology,
Daegu Haany University,
Kyungsan, 38610, S. Korea.

Seog-Hoon Rim[1]
Department of Mathematics Education,
Kyungpook National University,
Daegu 41566, S. Korea.

Abstract

Recently, Kim [9] studied some identities and properties of the degenerate Carlitz q-Bernoulli numbers and polynomials. Lee and Jang [20] defined the modified degenerate q-Bernoulli numbers and polynomials and they investigated some of the

[1]Corresponding Author.

identities and properties which are associated with such numbers and polynomials. Those polynomials are calculated from the generating functions and p-adic q-integral on $\mathbb{Z}_p$.

In [4], Kim-Kim introduced some interesting identities of symmetry for q-Bernoulli polynomials under symmetry group of degree n.

In this paper, we study the symmetry for the modified degenerate q-Bernoulli polynomials and derive some identities of symmetry for these polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$.

AMS subject classification: 11B68, 11S80, 05A19, 05A30.
Keywords: identities of symmetry, modified degenerate Carlitz q-Bernoulli polynomial, symmetry group of degree n, p-adic q-integral.

1. Introduction

It is well-known that the Bernoulli numbers B_n are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}=e^{Bt}$$
$$\text{(see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]).} \tag{1.1}$$

From (1.1), we can derive the well-known recurrence relation on Bernoulli numbers:

$$B_0=1, \qquad (B+1)^n-B_n=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases} \tag{1.2}$$

In [2], Carlitz defined the recurrence relation for the ultra Bernoulli numbers as

$$\gamma_{0,q}=1, \qquad (q\gamma_q+1)^n-\gamma_{n,q}=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n>1.\end{cases} \tag{1.3}$$

But we find that $\gamma_{2,q}=-\frac{1}{q^2-1}$, which means that $\lim_{q\to 1}\gamma_{2,q}=\frac{1}{0}=\infty$. In [1], L. Carlitz considered the q-analogue of Bernoulli numbers which are given by the recurrence relation

$$\beta_{0,q}=1, \qquad q(q\beta_q+1)^n-\beta_{n,q}=\begin{cases}1, & \text{if } n=1,\\ 0 & \text{if } n>1,\end{cases} \tag{1.4}$$

with the usual convention of replacing β_q^n by $\beta_{n,q}$. He defined q-Bernoulli polynomials as

$$\beta_{n,q}(x)=\sum_{l=0}^{n}\binom{n}{l}q^{lx}\beta_{l,q}[x]_q^{n-l}, \quad \text{see [1, 14]).} \tag{1.5}$$

When $x = 0$, $\beta_{n,q}(0) = \beta_{n,q}$, we note that such numbers $\beta_{n,q}$ are called the Carlitz q-Bernoulli numbers.

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$ respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$.

Let $q \in \mathbb{C}_p$ be an indeterminate such that $|1-1|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$\begin{aligned} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [14])}. \end{aligned} \tag{1.6}$$

In [14], T. Kim proved that the Carlitz's q-Bernoulli polynomials are represented as the p-adic q-integral on $\mathbb{Z}_p$, which are given by

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \tag{1.7}$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz q-Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}^*(x)\frac{t^n}{n!}. \tag{1.8}$$

Note that $\lim_{\lambda\to 0} B_{n,\lambda}^*(x) = B_n^*(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 11]). When $x = 0$, $B_{n,\lambda}^* = B_{n,\lambda}^*(0)$ are calle the degenerate Bernoulli numbers.

In [17], Kim considered the degenerate q-Bernoulli polynomials, which are given by the p-adic q-integral on $\mathbb{Z}_p$ in [6] as

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called the degenerate q-Bernoulli numbers. Note that $\lim_{\lambda\to 0} \beta_{n,\lambda,q}(x) = \beta_{n,q}(x)$, $n \geq 0$.

Kim and Kim (see [4]) gave some identities of symmetry for the degenerate q-Bernoulli polynomials under symmetry group of degree n arising from the p-adic q-integral on $\mathbb{Z}_p$.

Motivated from Kim's Carlitz q-Bernoulli numbers by using p-adic q-integral, Lee and Jang defined the modified Carlitz q-Bernoulli numbers by using p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} q^{-x}[x]_q^n d\mu_q(x) = B_{n,q}. \tag{1.10}$$

Also we are able to define the modified Carlitz's q-Bernoulli polynomials as follows:

$$\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}. \tag{1.11}$$

For (1.11), we can have the following relation:

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n d\mu_q(y). \tag{1.12}$$

We note that modified Carlitz q-Bernoulli numbers can recover Carlitz ultra Bernoulli numbers' recurrence relation except $B_{0,q}$, which is

$$B_{0,q} = \frac{q-1}{\log q}, \qquad (qB_q+1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.13}$$

Here, we assume that $\lambda, t \in \mathbb{C}_p$, $0 < |\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$.

In terms of (1.11), Lee and Jang in [20] define the modified degenerate Carlitz q-Bernoulli polynomials as

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{1.14}$$

When $x = 0$, $B_{n,\lambda,q} = B_{n,\lambda,q}(0)$ are called the modified degenerate Carlitz q-Bernoulli numbers.

At first, we introduce some known results on modified degenerate Carlitz q-Bernoulli numbers and polynomials.

Theorem 1.1. [Lee and Jang [20], Thm 3.1] For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} q^{-y}[x+y]_{q,n,\lambda} d\mu_q(y) = B_{n,\lambda,q}(x),$$

where

$$\left(\frac{[x+y]}{\lambda}\right)_n = \frac{[x+y]}{\lambda} \times \left(\frac{[x+y]}{\lambda} - 1\right) \times \cdots \times \left(\frac{[x+y]}{\lambda} - n + 1\right).$$

Note that$[x+y]_{q,n,\lambda} = [x+y]_q([x+y]_q - \lambda)\cdots([x+y]_q - (n-1)\lambda)$ $(n \geq 1)$.

Theorem 1.2. [Lee and Jang [20], Thm 3.2] For $n \geq 0$, we have

$$B_{n,\lambda,q}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}B_{l,q}(x),$$

where $S_1(n,m)$ are the Stirling numbers of the first kind.

Note that $S_1(n,l)$'s are defined by $(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l$ $(n \geq 0)$. Also note that $\lim_{\lambda\to 0} B_{n,\lambda,q}(x) = B_{n,q}(x)$.

In this paper, we give some identities of symmetry for the degenerate modified Carlitz q-Bernoulli polynomials under symmetry group of degree n arising from the p-adic q-integral on $\mathbb{Z}_p$.

2. Identities of symmetry for the degenerate q-Bernoulli polynomials

We assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$. In this section, we let $w_1, w_2, \ldots, w_n$ be positive integers.

For $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-w_1w_2\cdots w_{n-1}y}(1+\lambda t)^{\frac{1}{\lambda}[WW1]_q}d\mu_{q^{w_1w_2\cdots w_{n-1}}}(y)\\
&\quad= \lim_{N\to\infty}\frac{1}{[w_np^N]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{y=0}^{w_nP^N-1}(1+\lambda t)^{\frac{1}{\lambda}[WW1]_q}\\
&\quad= \lim_{N\to\infty}\frac{1}{[w_np^N]_{q^{w_1w_2\cdots w_{n-1}}}}\sum_{k_n=0}^{w_n-1}\sum_{y=0}^{p^N-1}q^{w_1w_2\cdots w_{n-1}}(k_n+w_ny)\\
&\qquad\times(1+\lambda t)^{\frac{1}{\lambda}[WW2]_q}.
\end{aligned} \tag{2.15}$$

where

$$WW1 = w_1w_2\cdots w_{n-1}y + w_1w_2\cdots w_nx + w_n\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i)k_j, \quad \text{and}$$

$$WW2 = (\prod_{j=1}^{n-1}w_j)(k_n+w_ny) + \prod_{j=1}^{n}w_jx + w_n\prod_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1})k_j.$$

From (2.15), we note that

$$\frac{1}{[w_1w_2\cdots w_{n-1}]_q}\prod_{l=1}^{n-1}\prod_{k_l=0}^{w_l-1}\int_{\mathbb{Z}_p} q^{-w_1w_2\cdots w_{n-1}y}(1+\lambda t)^{\frac{1}{\lambda}[WW1]_q} d\mu_{q^{w_1w_2\cdots w_{n-1}}}(y)$$
$$= \lim_{N\to\infty}\frac{1}{[w_1w_2\cdots w_n p^N]_q}\prod_{l=1}^{n-1}\prod_{k_l=0}^{w_l-1}\prod_{k_n=0}^{w_n-1}\prod_{y=0}^{p^N-1}(1+\lambda t)^{\frac{1}{\lambda}[WW2]_q}. \tag{2.16}$$

We can see that (2.16) is invariant under any permutation in the symmetry group of degree n. Therefore, by (2.16), we obtain the following theorem.

Theorem 2.1. Let $w_1,\ldots,w_n$ be positive integers. Then the following expressions

$$\frac{1}{[W_\sigma]_q}\prod_{l=1}^{n-1}\prod_{k_l=0}^{w_{\sigma(l)}-1}\int_{\mathbb{Z}_p} q^{-W_\sigma y}(1+\lambda t)^{\frac{1}{\lambda}[W_\sigma y+\sum_{j=1}^{n} w_j x + w_{\sigma(n)}\sum_{j=1}^{n-1}(\prod_{i=1, i\neq j}^{n-1} w_i)k_j]_q} d\mu_{q^{W_\sigma}}(y),$$

where $W_\sigma = w_{\sigma(1)}\cdots w_{\sigma(n-1)}$, are the same for any permutation σ in the symmetry group of degree n.

Remark 2.2. We have the following two observations from (12), (16) of Kim and Kim [4].

$$\Big[w_1w_2\cdots w_{n-1}y + w_1w_2\cdots w_n x + w_n\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_q$$
$$= [w_1w_2\cdots w_{n-1}]_q\Big[y + w_n x + \frac{w_n}{w_1}k_1 + \cdots + \frac{w_n}{w_{n-1}}k_{n-1}\Big]_{q^{w_1w_2\cdots w_{n-1}}} \tag{2.17}$$

$$\Big[y + w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big]_{q^{w_1w_2\cdots w_{n-1}}} = \frac{[w_n]_q}{[w_1w_2\cdots w_{n-1}]_q}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}$$
$$+ q^{w_n\sum_{j=1}^{n-1}\left(\prod_{i=1, i\neq j}^{n-1} w_i\right)k_j}[y + w_n x]_{q^{w_1w_2\cdots w_{n-1}}} \tag{2.18}$$

From Remark 2.2.(2.17), we note that, by letting $W = w_1 w_2 \cdots w_{n-1}$,

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-Wy}(1+\lambda t)^{\frac{1}{\lambda}\left[Wy+Ww_nx+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q} d\mu_{q^W}(y) \\
&= \int_{\mathbb{Z}_p} q^{-Wy}(1+\lambda t)^{\frac{[W]_q\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^W}}{\lambda}} d\mu_{q^W}(y) \\
&= \int_{\mathbb{Z}_p} q^{-Wy}\left(1+\frac{\lambda}{[W]_q}[W]_q t\right)^{\frac{[W]_q}{\lambda}\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^W}} d\mu_{q^W}(y) \\
&= \sum_{m=0}^{\infty}[W]_q^m B_{m,\frac{\lambda}{[W]_q},q^W}\left(w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right)\frac{t^m}{m!}.
\end{aligned}
\tag{2.19}
$$

Henceforth, we will use the abbreviations

$$
\begin{aligned}
W &= w_1 w_2 \cdots w_{n-1}, \\
W_\sigma &= w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(n-1)}, \\
YW &= y + w_n x + w_n \sum_{j=1}^{n-1} \frac{k_j}{w_j}.
\end{aligned}
$$

Therefore, by Theorem 2.1 and (2.19), we obtain the following theorem.

Theorem 2.3. For $m \geq 0$, $w_1, w_2, \ldots, w_n \in \mathbb{N}$, the following expressions

$$
\begin{aligned}
&[W_\sigma]_q^{m-1} \prod_{l=1}^{n-1} \prod_{k_l=0}^{w_{\sigma(l)}-1} q^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j w_{\sigma(n)}} B_{m,\frac{\lambda}{[w_\sigma]_q},q^{W_\sigma}} \\
&\times \left(w_{\sigma(n)}x + \frac{w_{\sigma(n)}}{w_{\sigma(1)}} + \cdots + \frac{w_{\sigma(n)}}{w_{\sigma(n-1)}}\right)
\end{aligned}
$$

are the same for any permutation σ in the symmetry group of degree n.

By Theorem 1.2, we get

$$
\begin{aligned}
&B_{m,\frac{\lambda}{[W]_q},q^W}\left(w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right) \\
&= \left(\frac{\lambda}{[W]_q}\right)^m \int_{\mathbb{Z}_p} q^{-Wy}\left(\left(\frac{\lambda}{[W]_q}\right)^{-1}[YW]_q^l\right) d\mu_{q^W}(y) \\
&= \left(\frac{\lambda}{[W]_q}\right)^m \sum_{l=0}^{m} S_l(m,l)[W]_q^l \lambda^{-l} \int_{\mathbb{Z}_p} q^{-Wy}[YW]_q^l d\mu_{q^W}(y).
\end{aligned}
\tag{2.20}
$$

From Remark 2.2.(2.18), we can derive the following equation

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-Wy}[YW]_q^l d\mu_{q^W}(y) \\
&\quad = \sum_{s=0}^{l} \binom{l}{s} \left(\frac{[w_n]_q}{[W]_q}\right)^{l-s} \Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{l-s} q^{w_n s\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} \\
&\qquad \times \int_{\mathbb{Z}_p} q^{-Wy}[y+w_n x]_{q^W}^s d\mu_{q^W}(y) \\
&\quad = \sum_{s=0}^{l} \binom{l}{s} \left(\frac{[w_n]_q}{[W]_q}\right)^{l-s} \Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{l-s} q^{w_n s} B_{s,q^W}(w_n x).
\end{aligned} \tag{2.21}$$

By (2.20) and (2.21), we get

$$\begin{aligned}
&B_{m,\frac{\lambda}{[W]_q},q^W}\left(w_n x + \frac{w_n}{w_1}k_1 + \cdots + \frac{w_n}{w_{n-1}}k_{n-1}\right) \\
&\quad = \sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s} S_1(m,p)\lambda^{m-p}[W]_q^{s-m}[w_n]_q^{p-s}\Big[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\Big]_{q^{w_n}}^{p-s} \\
&\qquad \times q^{w_n s\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} B_{s,q^W}(w_n x).
\end{aligned} \tag{2.22}$$

From (2.22), we note that

$$\begin{aligned}
&[W]_q^{m-1}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} B_{m,\frac{\lambda}{[W]_q},q^W}\left(w_n x + w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right) \\
&\quad = \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s} S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s} B_{s,q^W}(w_n x) \\
&\qquad \times q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j} B_{s,q^W}(w_n x)
\end{aligned}$$

$$= \sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s}B_{s,q^W}(w_nx)$$
$$\times \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\big)k_j\Big]_{q^{w_n}}^{p-s} \tag{2.23}$$
$$= \sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[W]_q^{s-1}[w_n]_q^{p-s}B_{s,q^W}(w_nx)$$
$$\times K_{n,q^{w_n}}(w_1,\cdots,w_{n-1}\mid p-s,s),$$

where

$$K_{n,q^{w_n}}(w_1,\cdots,w_{n-1}\mid i,t) = \prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{(t+1)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\right)k_j}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\i\neq j}}^{n-1} w_i\big)k_j\Big]_q^{i}. \tag{2.24}$$

Therefore, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.4. Let $m \geq 0$ and $w_1, w_2, \ldots, w_n \in \mathbb{N}$. Then the following expressions

$$\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[W_\sigma]_q^{s-1}[w_{\sigma(n)}]_q^{p-s}B_{s,q^{W_\sigma}}(w_{\sigma(n)}x)$$
$$\times K_{n,q^{w_{\sigma(n)}}}(w_\sigma(1),\cdots,w_{\sigma(n-1)}\mid p-s,s)$$

are the same for any permutation σ in the symmetry group of degree n.

References

[1] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., **76** (1954), 332–350.

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88.

[3] Y. He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Adv. Difference Equ., **2013** 2013:246, 10 pages.

[4] D. S. Kim, T. Kim, *Some identities of symmetry for q-Bernoulli polynomials under symmetric sroup of degree n*, Ars. Comb., **126** (2016), 435–441.

[5] D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (I)*, Adv. Stud. Contemp. Math., **23** (2013), no. 3, 461–482.

[6] D. S. Kim, N. Lee, J. Na, K. H. Park, *Identities of symmetry for higher-order Euler polynomials in three variables (I)*, Adv. Stud. Contemp. Math., **22** (2012), no. 1, 51–74.

[7] T. Kim, D. S. Kim, J. J. Seo, *Fully degenerate poly-Bernoulli numbers and polynomials*, Open Math., **2016**; 14: 545–556.

[8] T. Kim, D. V. Dolgy, J. J. Seo, *Identities of symmetry for degenerate q-Euler polynomials*, Adv. Stud. Contemp. Math., **25** (2015), 577-582.

[9] T.Kim, D. V. Dolgy, D. S. Kim, *Symmetric identities for degenerate generalized Bernoulli polynomials*, J. Nonlinear Sci. Appl., **9** (2016), no. 2, 677–683.

[10] T. Kim, D. Dolgy, L. C. Jang, H. I. Kwon, J. J. Seo, *Some identities of symmetry for the denegerate q-Bernoulli polynomials under symmetry group of degree n*, Global J. Pure and Appl. Math., **12** (2016), no. 5, 4385–4394.

[11] T. Kim, *Symmetric identities of degenerate Bernoulli polynomials*, Proc. Jangjeon Math. Soc., **18** (2015) no. 4, 593–599.

[12] T. Kim, D. S. Kim, H. I. Kwon, D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, J. Nonlinear Sci. Appl., **9** (2016), no. 3, 1077–1082.

[13] T. Kim, *An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$*, Rocky Mountain J. Math., **41** (2011), no. 1, 239–247.

[14] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9**, (2002), no. 3, 288-299.

[15] T. Kim, H. I. Kwon, J. J. Seo, *Identities of symmetry for degenerate q-Bernoulli polynomials*, Proc. Jangjeon Math. Soc., **18** (2015), no. 4, 495–499.

[16] T. Kim, *Some identities of the q-Euler polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), no. 4, 484–491.

[17] T. Kim, *On degenerate q-Bernoulli polynomials*, Bull. Korean Math. Soc., **53** (2016), no. 4, 1149–1156.

[18] Y. H. Kim, K. W. Hwang, *Symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math., **18** (2009), no. 2, 127–133.

[19] H. I. Kwon, T. Kim, J. J. Seo, *Some identities of symmetry for modified degenerate Frobenius-Euler polynomials*, Adv. Stud. Contemp. Math., **26** (2016), 299–305.

[20] J. G. Lee, L. C. Jang, *On modified degenerate Carlitz q-Bernoulli numbers and polynomials*, Adv. Difference Equ., (2017) 2017:22 DOI 10.1186/s13662-016-1060-7.

[21] E.-J. Moon, S.-H. Rim, J.-H. Jin, S.-J. Lee, *On the symmetric properties of higher-order twisted q-Euler numbers and polynomials*, Adv. Difference Equ., **2010** Article ID 765259, 8 pages.