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The Moment Generating Function of the Four-Parameter Generalized F Distribution and Related Generalized Distributions

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Abstract

The main objective of this article is to show the derivation of the moment generating function of the four-parameter of generalized (G4F) F distribution. Through parameterization of its moment generating function, the behavior in relation to several well-known generalized distributions is presented. By utilizing MacLaurin series expansion and Stirling formula, it is shown that with parameterization of its moment generating function, the generalized F distribution might have special relationship to several well-known generalized distributions, such as generalized beta of the second kind (GB2), generalized log-logistic (G4LL), and generalized gamma (G3G) distributions.

Keywords: Moment generating function; Generalized beta of the second kind distribution; Generalized log-logistic distribution; Generalized gamma; MacLaurin series; Stirling formula

1. INTRODUCTION

As stated by several authors, the generalized F is one of the most well-known generalized distributions in probability modeling. In fitting survival data of carcinoma patients, Ciampi, et al. (1986) discussed the properties and maximum likelihood inference of the family of the generalized F distribution. Peng, et al. (1998) investigated the application of the generalized *F* distribution to a mixture model from lymphoma patients. Kalbfleisch and Prentice (2002) discussed the use of the generalized F distribution.

The generalized F distribution is parameterization of other well-known generalized distributions, such as the generalized beta and generalized log-logistic distributions (Pham-Ghia and Duong 1989, Singh 1989). According to Ciampi, et al. (1986) when a random variable X has four-parameter generalized F distribution with degrees of freedom $2m_1$ and $2m_2$, as denoted by X ~ G4F(μ,σ,m_1,m_2) or G4F(m_1,m_2), the corresponding probability density function (PDF) of the G4F(μ,σ,m_1,m_2) can be written in the following form;

$$f_{G4F}(x)=\frac{e^{-\frac{\mu m_1}{\sigma}}\, x^{\left(\frac{m_1}{\sigma}\right)-1}\left(\frac{m_1}{m_2}\right)^{m_1}\left[1+\left(\frac{m_1}{m_2}\right)\left(e^{-\mu}x\right)^{1/\sigma}\right]^{-(m_1+m_2)}}{\sigma\frac{\Gamma(m_1)\cdot\Gamma(m_2)}{\Gamma(m_1+m_2)}}\;;\; x\geq 0 \qquad (1)$$

where $\mu>0$, $\sigma>0$, $m_1>0$, $m_2>0$, and Γ is gamma function.

It can be shown that if we let $\mu=\ln b+(1/a)\ln(m_1/m_2)$, $\sigma=1/a$, and $B(m_1,m_2)$ that is beta function, then the PDF of the G4F distribution becomes

$$f_{GB2}(x)=\frac{a}{xB(m_1,m_2)}\frac{\left[\left(\frac{x}{b}\right)^a\right]^{m_1}}{\left[1+\left(\frac{x}{b}\right)^a\right]^{m_1+m_2}}$$

which is the PDF of the GB2 distribution as deliberated by McDonald (1984), and McDonald and Richards (1987).

If we let $\mu=(-\beta-\ln(m_1/m_2))(1/\alpha)$ and $\sigma=1/\alpha$*,* where and, then the PDF of the G4F distribution can be written in the form

$$f_{G4LL}(x)=\frac{\alpha}{xB(m_1,m_2)}\left[\frac{1}{1+e^{-(\beta+\alpha\ln(x))}}\right]^{m_1}\left[1-\frac{1}{1+e^{-(\beta+\alpha\ln(x))}}\right]^{m_2}$$

which is the PDF of the G4LL distribution considered by Singh, et al.(1997).

Moreover, as noted by several authors (Ciampi, et al. 1986; Peng et al. 1998; Kalbfleisch and Prentice 2002; and Zhou et al. 2011), the generalized F contains several commonly used distributions as special cases or limiting distributions, such as exponential, gamma, lognormal, and Weibull distributions. Recently, Cox (2008) has linked the G4F distribution to the three-parameter generalized (G3G) and the three-parameter generalized log-logistic (G3LL) distributions. To link the G4F distribution to other generalized distributions, he has parameterized the probability density function (PDF) of the G4F distribution.

However, in term of a moment generating function (MGF), flexibilities of the G4F distribution and relation to other generalized distributions are apparently not too readily accesible in all of above literatures. Based on the MGF, an alternative method could be generated for computing moments and characterizing distributions from their MGFs. Based on MGFs, Warsono (2009) demonstrated the relationship between GB2 and G3G distributions, and Warsono (2010) linked the G4LL distribution to the GB2 and G3G distrbution. In developing other aspects of the G4F distribution, it seems worthwhile to extend all of above endeavors and to record them in this paper. Therefore, the main objective of this paper is to derive the MGF of the generalized F distribution and relate other generalized distributions, such as generalized beta, generalized log-logistic, and generalized gamma.

The rest of the article is outlined as follows. In Section 2, the explicit moment generating function of the G4F distribution is derived. By using parameterization of the moment generating function of the the G4F, Section 3 provides discussions of the relation between G4F distribution and GB2 and G4LL distributions. Section 4 contains a description of limiting behavior of the moment generating function of the G4F distribution as a general case of the moment generating function of G3G and G3F distributions. Finally, some concluding remarks are noted in section 5.

2. MOMENT GENERATING FUNCTION OF THE GENERALIZED F DISTRIBUTION

In term of a beta function the pdf of GF distribution in equation (1) can be rewritten as:

$$g(x) = \frac{1}{\sigma x B(m_1, m_2)} \frac{\left(e^{-\frac{\mu}{\sigma}} \left(\frac{m_1}{m_2} \right) x^{1/\sigma} \right)^{m_1}}{\left[1 + \left(e^{-\frac{\mu}{\sigma}} \right) x^{1/\sigma} \left(\frac{m_1}{m_2} \right) \right]^{m_1+m_2}} \; ; \quad x \geq 0 \qquad (2)$$

where $\mu > 0$, $\sigma > 0$, $m_1 > 0$, and , Γ is gamma function.

Theorem 2.1 Let *X* be a random variable of the G4F(μ,σ,m_1,m_2) distribution, then moment generating function (MGF) of X is given by

$$M_{G4F}(t) = e^{te^{\mu}}$$

Proof:

$$M_{G4F}(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \left(\frac{1}{\sigma x B(m_1, m_2)} \right) \frac{e^{-\frac{\mu m_1}{\sigma}} \left(\frac{m_1}{m_2} \right)^{m_1} x^{m_1/\sigma}}{\left[1 + \left(e^{-\mu} x\right)^{1/\sigma} \frac{m_1}{m_2} \right]^{m_1+m_2}} dx$$

By algebra manipulation we may find the following equation

$$= \frac{1}{\sigma B(m_1, m_2)} \int_{0}^{\infty} \left(\frac{e^{tx}}{x} \right) \frac{\left[e^{-\frac{\mu}{\sigma}} \left(\frac{m_1}{m_2} \right) x^{\frac{1}{\sigma}} \right]^{m_1}}{\left[1 + e^{-\frac{\mu}{\sigma}} \left(\frac{m_1}{m_2} \right) x^{\frac{1}{\sigma}} \right]^{m_1+m_2}} dx \tag{3}$$

By letting $y = e^{-\frac{\mu}{\sigma}} \left(\frac{m_1}{m_2} \right) x^{\frac{1}{\sigma}}$ we may rewrite the equation (3) in the following form

$$M_{G4F}(t) = \frac{1}{\sigma B(m_1, m_2)} \int_{0}^{\infty} \left(\frac{\sigma e^{tx}}{y} \right) \frac{y^{m_1}}{(1+y)^{m_1+m_2}} dy$$

$$= \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} e^{tx} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy \tag{4}$$

Making use of a well-known property of MacLaurin series of the e^{tx} function, then equation (4) is given by

$$M_{G4F}(t) = \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \right) \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy$$

$$= \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots \right) \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy$$

$$=\frac{1}{B(m_1,m_2)}\int_0^\infty\left(\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}+tx\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}+\frac{(tx)^2}{2!}\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}+\frac{(tx)^3}{3!}\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}+\ldots\right)dy$$

$$=\frac{1}{B(m_1,m_2)}\left[\int_0^\infty\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\int_0^\infty tx\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\right.$$

$$\left.\int_0^\infty\frac{(tx)^2}{2!}\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\int_0^\infty\frac{(tx)^3}{3!}\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\ldots\ldots\right]$$

$$=\frac{1}{B(m_1,m_2)}\left[\int_0^\infty\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\int_0^\infty y^\sigma\cdot\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\right.$$

$$\left.\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^2}{2!}\int_0^\infty y^{2\sigma}\cdot\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^3}{3!}\int_0^\infty y^{3\sigma}\cdot\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+\ldots\right]$$

$$=\frac{1}{B(m_1,m_2)}\left[\int_0^\infty\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}dy+t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\int_0^\infty\frac{y^{m_1+\sigma-1}}{(1+y)^{m_1+\sigma+m_2-\sigma}}dy+\right.$$

$$\left.\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^2}{2!}\int_0^\infty\frac{y^{m_1+2\sigma-1}}{(1+y)^{m_1+2\sigma+m_2-2\sigma}}dy+\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^3}{3!}\int_0^\infty\frac{y^{m_1+3\sigma-1}}{(1+y)^{m_1+3\sigma+m_2-3\sigma}}dy+\ldots\right]$$

$$=\frac{1}{B(m_1,m_2)}\left[B(m_1,m_2)+t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\cdot B(m_1+\sigma,m_2-\sigma)+\right.$$

$$\left.\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^2}{2!}\cdot B(m_1+2\sigma,m_2-2\sigma)+\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^\sigma\right)^3}{3!}\cdot B(m_1+3\sigma,m_2-3\sigma)+\ldots\right]$$

$$= 1 + t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma} \cdot \frac{B(m_1+\sigma, m_2-\sigma)}{B(m_1,m_2)} + \frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^2}{2!} \cdot \frac{B(m_1+2\sigma, m_2-2\sigma)}{B(m_1,m_2)} +$$

$$\frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^3}{3!} \cdot \frac{B(m_1+3\sigma, m_2-3\sigma)}{B(m_1,m_2)} + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^n}{n!} \cdot \frac{B(m_1+n\sigma, m_2-n\sigma)}{B(m_1,m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^n}{n!} \cdot \frac{\dfrac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1+n\sigma+m_2-n\sigma)}}{\dfrac{\Gamma(m_1)\cdot\Gamma(m_2)}{\Gamma(m_1+m_2)}}$$

$$= \sum_{n=0}^{\infty} \frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1+m_2)} \cdot \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(t\left(e^{\frac{\mu}{\sigma}}\left(\frac{m_2}{m_1}\right)\right)^{\sigma}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

Therefore, the MGF of the G4F distribution is

$$M_{G4F}(t) = \sum_{n=0}^{\infty} \frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)} \tag{5}$$

It is well-known that the Stirling's approximation formula of the gamma function (Spiegel 1968) is given by

$$\Gamma(az+b) \sim \sqrt{2\pi} \cdot e^{-az}(az)^{az+b-\frac{1}{2}}$$

By Stirling's approximation, equation (5) can be expressed as

$$M_{G4F}(t)=\sum_{n=0}^{\infty}\frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!}\cdot\frac{\sqrt{2\pi}.e^{-m_1}(m_1)^{m_1+n\sigma-\frac{1}{2}}\cdot\sqrt{2\pi}.e^{-m_2}(m_2)^{m_2+n\sigma-\frac{1}{2}}}{\sqrt{2\pi}.e^{-m_1}(m_1)^{m_1-\frac{1}{2}}\cdot\sqrt{2\pi}.e^{-m_2}(m_2)^{m_2-\frac{1}{2}}}$$

$$M_{G4F}(t)=\sum_{n=0}^{\infty}\frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!}\cdot\left(m_1^{\sigma}\right)^n\left(\left(\frac{1}{m_2}\right)^{\sigma}\right)^n$$

$$M_{G4F}(t)=\sum_{n=0}^{\infty}\frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!}\cdot\left(\left(\frac{m_1}{m_2}\right)^{\sigma}\right)^n \tag{6}$$

By MacLaurin series, equation (6) can be written as

$$M_{G4F}(t)=e^{te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\left(\frac{m_1}{m_2}\right)^{\sigma}}$$

$$M_{G4F}(t)=e^{te^{\mu}}$$

2.2. Moment Generating Function (MGF) of Three-Parameter G3F Distribution

Based on the gamma distribution (Malik 1967, and Dyer 1982) and generalized beta with three- parameter distribution (Pham-Ghia and Duong 1989) obtained three-parameters of generalized F distribution with the following pdf

$$f_{G3F}(x)=\frac{\alpha^{m_1}}{B(m_1,m_2)}\frac{x^{m_1-1}}{[1+\alpha x]^{m_1+m_2}}\,dx;\,x>0,m_1,m_2,\alpha>0$$

Theorem 2.2 Let *X* be a random variable of the G3F (α,m_1,m_2) distribution, then moment generating function (MGF) of X is given by

$$M_{G3F}(t)=\sum_{n=0}^{\infty}\frac{\left(\frac{t}{\alpha}\right)^n}{n!}\cdot\frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

Proof:

$$M_{G3F}(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \left(\frac{\alpha^{m_1}}{B(m_1, m_2)} \right) \frac{x^{m_1-1}}{[1+\alpha x]^{m_1+m_2}}\, dx$$

By algebra manipulation we may find the following equation

$$= \frac{\alpha^{m_1}}{B(m_1, m_2)} \int_{0}^{\infty} \left(e^{tx}\right) \frac{x^{m_1-1}}{[1+\alpha x]^{m_1+m_2}}\, dx \tag{7}$$

By letting $y = \alpha x$ we may rewrite the equation (7) in the following form

$$M_X(t) = \frac{\alpha^{m_1}}{B(m_1, m_2)} \int_{0}^{\infty} \left(e^{tx}\right) \frac{\left(\frac{y}{\alpha}\right)^{m_1-1}}{(1+y)^{m_1+m_2}} \frac{1}{\alpha}\, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(e^{tx}\right) \frac{(y)^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy \tag{8}$$

Making use of a well-known property of MacLaurin series of the e^{tx} function, then equation (8) can be written as

$$M_x(t) = \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(tx)^n}{n!} \right) \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \ldots \right) \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy$$

$$= \frac{1}{B(m_1, m_2)} \int_{0}^{\infty} \left(\frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} + tx \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} + \frac{(tx)^2}{2!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} + \frac{(tx)^3}{3!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} + \ldots \right) dy$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_{0}^{\infty} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy + \int_{0}^{\infty} tx \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy + \right.$$

$$\left. \int_{0}^{\infty} \frac{(tx)^2}{2!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy + \int_{0}^{\infty} \frac{(tx)^3}{3!} \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}}\, dy + \ldots\ldots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^\infty \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + t\left(\frac{1}{\alpha}\right) \int_0^\infty y \cdot \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \right.$$

$$\left. \frac{\left(t\left(\frac{1}{\alpha}\right)\right)^2}{2!} \int_0^\infty y^2 \cdot \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \frac{\left(t\left(\frac{1}{\alpha}\right)\right)^3}{3!} \int_0^\infty y^3 \cdot \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \ldots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[\int_0^\infty \frac{y^{m_1-1}}{(1+y)^{m_1+m_2}} dy + \left(\frac{t}{\alpha}\right) \int_0^\infty \frac{y^{m_1+1-1}}{(1+y)^{m_1+1+m_2-1}} dy + \right.$$

$$\left. \frac{\left(\frac{t}{\alpha}\right)^2}{2!} \int_0^\infty \frac{y^{m_1+2-1}}{(1+y)^{m_1+2+m_2-2}} dy + \frac{\left(\frac{t}{\alpha}\right)^3}{3!} \int_0^\infty \frac{y^{m_1+3-1}}{(1+y)^{m_1+3+m_2-3}} dy + \ldots \right]$$

$$= \frac{1}{B(m_1, m_2)} \left[B(m_1, m_2) + \left(\frac{t}{\alpha}\right) \cdot B(m_1+1, m_2-1) + \right.$$

$$\left. \frac{\left(\frac{t}{\alpha}\right)^2}{2!} \cdot B(m_1+2, m_2-2) + \frac{\left(\frac{t}{\alpha}\right)^3}{3!} \cdot B(m_1+3, m_2-3) + \ldots \right]$$

$$= 1 + \frac{t}{\alpha} \cdot \frac{B(m_1+1, m_2-1)}{B(m_1, m_2)} + \frac{\left(\frac{t}{\alpha}\right)^2}{2!} \cdot \frac{B(m_1+2, m_2-2)}{B(m_1, m_2)} + \frac{\left(\frac{t}{\alpha}\right)^3}{3!} \cdot \frac{B(m_1+3, m_2-3)}{B(m_1, m_2)} + \ldots.$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \cdot \frac{B(m_1+n, m_2-n)}{B(m_1, m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \cdot \frac{\frac{\Gamma(m_1+n) \cdot \Gamma(m_2-n)}{\Gamma(m_1+n+m_2-n)}}{\frac{\Gamma(m_1) \cdot \Gamma(m_2)}{\Gamma(m_1+m_2)}}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1+m_2)} \cdot \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

Therefore, the MGF of the G3F distribution is

$$M_x(t) = \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)} \qquad (9)$$

3. THE RELATION OF THE GF DISTRIBUTION WITH GB2 DISTRIBUTION

Warsono (2010) has mathematically derived the MGF of the GB2 distribution using MGF defintion. Based on reparameterization of the MGF of the GLF distribution, use passive sentence- The MGF of the GB2 distribution is provided in this section. The reparameterization proposition is stated and proved.

Proposition 3.1 Let X be a random variable having the G4F (μ, σ, m_1, m_2) moment and

$\mu = \ln b + \frac{1}{a}\ln\left(\frac{m_1}{m_2}\right)$ and $\sigma = \frac{1}{a}$, then X has the GB2(a, b, m_1, m_2) moment.

Proof:

$$M_{G4F}(t) = \sum_{n=0}^{\infty} \frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!} \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{\ln b + \frac{1}{a}\ln\left(\frac{m_1}{m_2}\right)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!} \frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$=\sum_{n=0}^{\infty}\frac{\left(te^{\ln b}\,e^{\frac{1}{a}\ln\left(\frac{m_1}{m_2}\right)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!}\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$=\sum_{n=0}^{\infty}\frac{\left(tb\left(\frac{m_1}{m_2}\right)^{\frac{1}{a}}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!}\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$=\sum_{n=0}^{\infty}\frac{(tb)^n}{n!}\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

This is the moment generating function of the GB2 (a,b, m_1, m_2) stated by Warsono (2010).

4. THE RELATION OF THE GF DISTRIBUTION WITH GLL DISTRIBUTION

Warsono (2010) derived mathematically the MGF of the GLL distribution using definition of MGF's. Based on reparameterization of the MGF of the GF distribution, the MGF of the GLL distribution is provided in this section.

Proposition 4.1 Let X be a random variable having the GF (μ, σ, m_1, m_2) moment and $\mu=\left(-\beta-\ln\left(\frac{m_2}{m_1}\right)\right)\frac{1}{\alpha}$ and $\sigma=\frac{1}{\alpha}$, then X has the GLL(α, β, m_1, m_2) moment.

Proof:

$$M_{G4F}(t)=\sum_{n=0}^{\infty}\frac{\left(te^{\mu}\left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!}\frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$=\sum_{n=0}^{\infty}\frac{\left(te^{\left(-\beta-\ln\left(\frac{m_2}{m_1}\right)\right)\frac{1}{\alpha}}\left(\frac{m_2}{m_1}\right)^{\frac{1}{\alpha}}\right)^n}{n!}\frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\beta}{\alpha}-\frac{1}{\alpha}\ln\left(\frac{m_2}{m_1}\right)} \left(\frac{m_2}{m_1}\right)^{\frac{1}{\alpha}} \right)^n}{n!} \frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\beta}{\alpha}} e^{-\frac{1}{\alpha}\ln\left(\frac{m_1}{m_2}\right)} \left(\frac{m_2}{m_1}\right)^{\frac{1}{\alpha}} \right)^n}{n!} \frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\beta}{\alpha}} \left(\frac{m_2}{m_1}\right)^{-\frac{1}{\alpha}} \left(\frac{m_2}{m_1}\right)^{\frac{1}{\alpha}} \right)^n}{n!} \frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\frac{\beta}{\alpha}} \right)^n}{n!} \frac{\Gamma\left(m_1+\frac{n}{\alpha}\right)\cdot\Gamma\left(m_2-\frac{n}{\alpha}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

5. THE RELATION OF THE G4F DISTRIBUTION WITH G3F DISTRIBUTION

Equation (8) has mathematically derived the MGF of the G3F distribution. Based on reparameterization of the MGF of the GF distribution, the MGF of the G3F distribution is provided in this section.

Proposition 5.1 Let X be a random variable having the G4F (μ, σ, m_1, m_2) moment and

$$\mu = \left(-\ln\left(\frac{m_2}{m_1}\right) - \ln\alpha \right)$$ and $\sigma = 1$, then X has the G3F(α, m_1, m_2) moment.

Proof:

$$M_{G4F}(t) = \sum_{n=0}^{\infty} \frac{\left(te^{\mu} \left(\frac{m_2}{m_1}\right)^{\sigma} \right)^n}{n!} \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{\left(-\ln\left(\frac{m_2}{m_1}\right) - \ln\alpha\right)} \left(\frac{m_2}{m_1}\right)\right)^n}{n!} \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{-\ln\left(\frac{m_1}{m_2}\right)} e^{-\ln\alpha} \left(\frac{m_2}{m_1}\right)\right)^n}{n!} \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(te^{\ln\frac{1}{\alpha}} \left(\frac{m_2}{m_1}\right)^{-1} \left(\frac{m_2}{m_1}\right)\right)^n}{n!} \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{t}{\alpha}\right)^n}{n!} \frac{\Gamma(m_1+n)\cdot\Gamma(m_2-n)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

This is the moment generating function of the G3F(α, m_1, m_2) stated in equation (9).

6. THE MOMENT OF THE GG AS A LIMITING MOMENT OF THE GF DISTRIBUTIONS

In this section, propositions of limiting moment properties of the GF distribution are stated and proved.

Proposition 6.1 The GF(μ,σ,m_1,m_2) distribution converges to the GG distribution as m_2 tends to ∞ and $\mu = \ln\gamma + \frac{1}{a}\ln(m_1)$, and $\sigma = \frac{1}{a}$.

Proof:

$$\lim_{m_2\to\infty} M_{G4F}(t) = \lim_{m_2\to\infty} \sum_{n=0}^{\infty} \frac{\left(te^{\mu} \left(\frac{m_2}{m_1}\right)^{\sigma}\right)^n}{n!} \cdot \frac{\Gamma(m_1+n\sigma)\cdot\Gamma(m_2-n\sigma)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty}\sum_{n=0}^{\infty}\frac{\left(te^{\ln\gamma+\frac{1}{a}\ln(m_1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty}\sum_{n=0}^{\infty}\frac{\left(te^{\ln\gamma}e^{\frac{1}{a}\ln(m_1)}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty}\sum_{n=0}^{\infty}\frac{\left(t\gamma(m_1)^{\frac{1}{a}}\left(\frac{m_2}{m_1}\right)^{\frac{1}{a}}\right)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty}\sum_{n=0}^{\infty}\frac{\left(t\gamma(m_2)^{\frac{1}{a}}\right)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)\cdot\Gamma\left(m_2-\frac{n}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}$$

$$= \lim_{m_2\to\infty}1+\lim_{m_2\to\infty}\left(t\gamma(m_2)^{\frac{1}{a}}\right)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)\cdot\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+$$

$$\lim_{m_2\to\infty}\frac{\left(t\gamma(m_2)^{\frac{1}{a}}\right)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)\cdot\Gamma\left(m_2-\frac{2}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+$$

$$\lim_{m_2\to\infty}\frac{\left(t\gamma(m_2)^{\frac{1}{a}}\right)^3}{3!}\cdot\frac{\Gamma\left(m_1+\frac{3}{a}\right)\cdot\Gamma\left(m_2-\frac{3}{a}\right)}{\Gamma(m_1)\cdot\Gamma(m_2)}+\ldots$$

$$= \lim_{m_2\to\infty}1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}\cdot(m_2)^{\frac{1}{a}}\frac{\Gamma\left(m_2-\frac{1}{a}\right)}{\Gamma(m_2)}+$$

$$\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}\cdot(m_2)^{\frac{2}{a}}\frac{\Gamma\left(m_2-\frac{2}{a}\right)}{\Gamma(m_2)}+$$

$$\frac{(t\gamma)^3}{3!}\cdot\frac{\Gamma\left(m_1+\frac{3}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}\cdot(m_2)^{\frac{3}{a}}\frac{\Gamma\left(m_2-\frac{3}{a}\right)}{\Gamma(m_2)}+\dots$$

Using Stirling's approximation formula of the gamma function, the limiting moment property of the G4F(μ,σ,m_1,m_2) distribution can be written as:

$$\lim_{m_2\to\infty} M_X(t)GF\left(\mu=\ln\gamma+\frac{1}{a}\ln(m_1),\sigma=\frac{1}{a},m_1,m_2\right)$$

$$=\lim_{m_2\to\infty}1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{1}{a}}\cdot\frac{1}{(m_2)^{\frac{1}{a}}}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}\lim_{m_2\to\infty}(m_2)^{\frac{2}{a}}\cdot\frac{1}{(m_2)^{\frac{2}{a}}}+\dots$$

$$=1+(t\gamma)\cdot\frac{\Gamma\left(m_1+\frac{1}{a}\right)}{\Gamma(m_1)}+\frac{(t\gamma)^2}{2!}\cdot\frac{\Gamma\left(m_1+\frac{2}{a}\right)}{\Gamma(m_1)}+$$

$$=\sum_{n=0}^{\infty}\frac{(t\gamma)^n}{n!}\cdot\frac{\Gamma\left(m_1+\frac{n}{a}\right)}{\Gamma(m_1)}$$

This result is the MGF of the GG stated by Warsono(2010). Thus, the G4F distribution converges to the GG distribution as m_2 tends to ∞ and $\alpha=a$, and $\beta=-a\ln\left(\gamma(m_2)^{\frac{1}{a}}\right)$

.

7. CONCLUSION

The moment of the generalized F distribution is parameterization of the generalized beta of the second kind (GB2) and the generalized log-logistic (GLL). The moment of the generalized gamma (GG) distribution is the limiting moment of (GF) distribution. Moreover, since the moments of the gamma and exponential distributions are special cases of the moment of the generalized gamma distribution (Warsono 2009), the moments of both special distributions are also special cases of the moments of the moment of the GF distribution.

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