Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 3 (2017), pp. 957-966

http://www.ripublication.com

Fractional Calculus Associated with General Polynomials and Aleph Functions

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Abstract

This paper is devoted to study a number of Eulerian integrals and the fractional calculus associated with general class of polynomial given by Shrivastava [4, p.1, eqn.(1)], generalized polynomials given by Shrivastava [8, p.185, eqn.(7)] and Aleph function given by Südland et al[4] [14,15]. Finally, importance of main result also recorded herein.

Keywords: Aleph function, Lauricella function, Fractional Calculus.

MSC Classification code(2010): 33EXX, 33E30, 31A10

1. INTRODUCTION

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the last three decades Fractional calculus has been applied to almost every field of science, engineering and mathematics. Many applications of fractional calculus can be found in turbulence and fluid dynamics, Stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, non-linear biological systems, astrophysics

In recent years, several authors namely Saigo and Saxena [5], Srivastava and Hussain [11], Saxena and Saigo [5], Saxena and Nishimoto [7], Südland and Baumann [15] have established certain fractional integral formulae, deduced from Eulerian integral. The Aleph function introduced by Südland et al[4] is defined in terms of Mellin-Barnes type contour integral as below:

$$\aleph(z) = \aleph^{M,N}_{p_i,q_i,c_i;r}\left[z \left| \begin{matrix} (a_j,A_j)_{1,n}, [c_i(a_{ji},A_{ji})]_{N+1,p_i;r} \\ (b_j,B_j)_{1,M}, [c_i(b_{ji},B_{ji})]_{M+1,q_i;r} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{p_i,q_i,c_i;r}(s)\, z^{(s)} ds \qquad \ldots(1.1)$$

for all z ≠0, where $\omega = \sqrt{-1}$ and

$$\Omega^{M,N}_{p_i,q_i,c_i;r}(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j + B_j s) \prod_{j=1}^{N} \Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} c_i \prod_{j=N+1}^{p_i} \Gamma a_{ji} + A_{ji} s \prod_{j=M+1}^{q_i} \Gamma(1 - b_j - B_j s)} \qquad \ldots(1.2)$$

where integration path $L = L_{\gamma\omega\infty}, \gamma \in R$ extends from $\gamma - \omega\infty$ to $\gamma + \omega\infty$ and is such that poles, assumed to be simple, of $\Gamma 1 - a_j - A_j s,\ j = 1,\ldots, n$ do not coincide with pole of $\Gamma(b_j + B_j s), j = 1,\ldots, m,$ the parameters p_i, q_i are non-negative integers satisfying

$$0 \le m \le p_i, 1 \le n \le q_i, c_i > 0;\ i = 1,\ldots, r.$$

The parameters $A_j, B_j, A_{ij}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in$ C.

The conditions for defining above integers are

$$\phi_\ell > 0; |\arg(z)| < \frac{\pi}{2}\phi_\ell,\ \ell = 1,\ldots, r$$

$$\phi_\ell > 0 \,;\, |\arg(z)| < \frac{\pi}{2}\phi_\ell, \text{and } R\{\xi_\ell\} + 1 < 0,$$

where

$$\phi_\ell = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - c_i\left(\sum_{j=N+1}^{p_i} A_{j\ell} + \sum_{j=M+1}^{q_i}\right) + \frac{1}{2}(p_i - q_i), \ell = 1, \ldots, r.$$

An equivalent form of Beta function is [3, p.10. eq. (10)]

$$\int_x^y (u-x)^{m-1}(y-u)^{n-1}du = (y-x)^{m+n-1}B(m,n)$$

where $x, y \in R$, $(x < y)$, $Re(m) > 0$, $Re(n) > 0$.

Using [3, p.62, eq.(15)], we have

$$(\alpha u + \beta)^v = (x\alpha + \beta)^v \left[1 + \frac{\alpha(u-x)}{x\alpha + \beta}\right]^v$$

$$= \frac{(x\alpha+\beta)^v}{\Gamma-(v)} \frac{1}{2\pi i}\int_{-i\omega}^{+i\omega} \Gamma(-\gamma)\Gamma(\gamma - v)\left[\frac{\alpha(u-x)}{x\alpha+\beta}\right]^v d\gamma \qquad \ldots(1.4)$$

where α, β, $v \in c$; x, $u \in R$ $\left|\arg\left(\frac{\alpha}{x-\alpha-\beta}\right)\right| < \pi$, path of integration is indented.

The general class of polynomials introduced by Srivastava [ ] is given by

$$S_{N'}^{M'}[w] = \sum_{k=0}^{[N'/M']} \frac{(-N')_{M'k}}{k!} B'_{N',k},\ k = 0,1,2,\ldots$$
$\ldots(1.5)$

where M is an arbitrary positive integer and the coefficient $B'_{N',k}(N', k \geq 0)$ are arbitrary constant, real or complex.

The generalized polynomials defined by Srivastava and Garg [8, p.686, eq. (1.4)], is defined in following manner

$$S_{N_1,\ldots,N_t}^{M_1,\ldots,M_t}[w_1,\ldots,w_t]=\sum_{k_1=0}^{[N_1/M_1]}\ldots\sum_{k_s=0}^{[N_t/M_t]}\frac{(-N_1)_{M_1k_1}}{k_1!}\ldots$$

$$\frac{(-N_s)_{M_tk_t}}{k_t!}A(N_1k_1;\ldots;N_tk_t)w_1^{k_1}\ldots w_t^{k_t},$$

where $M_1,\ldots,M_t$ are arbitrary positive integers and $A(N_1k_1,\ldots,N_tk_t)$ are constants, real or complex.

The Lauricelli function $F_D^{(n)}$ is defined in integral form as

$$\frac{\Gamma(a)\Gamma(b_1)\ldots\Gamma b_h}{c!}F_D^{(h)}[a_1b_1,\ldots,b_h;c;x_1,\ldots,x_h]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+\xi_1+\ldots+\xi_n)\Gamma(b_1+\xi_1)\ldots\Gamma(b_h+\xi_h)}{\Gamma(c+\xi_1+\ldots+\xi_h)}$$

$$\times\Gamma(-\xi_1)\ldots\Gamma(-\xi_h)(-x_1)^{\xi_1}\ldots(-x_h)^{\xi_h}d\xi_1\ldots d\xi_h.$$

…(1.7)

where $\max[|\arg(-x_1)|,\ldots,|(x_h)\}]<\pi;\ c\neq 0,-1,-2,\ldots$

The following result is used in establishing Eulerian integral

$$\int_x^y (u-x)^{m-1}(y-u)^{n-1}(\alpha_1u+\beta_1)^{v_1}\ldots(\alpha_hu+\beta_h)^{v_h}du=(y-x)^{m+n-1}B(m,n)$$

.

$$(\alpha_1x+\beta_1)^{v_1}\ldots(\alpha_1x+\beta_1)^{v_h}F_D^{(\mu)}\left[m,-v_1,\ldots,v_h;m+n;-\frac{(y-x)\alpha_1}{x\alpha_1+\beta_1},\ldots,\frac{(y-x)\alpha_h}{x\alpha_h+\beta_h}\right],$$

…(1.8)

where $x,y\in R\ (x<y);\alpha_j,\beta_j,v_j(C\ (h=1,\ldots,h).$

2. EULERIAN INTEGRAL

The main integral to be evaluated here is

$$\int_x^y (u-x)^{m-1}(y-u)^{n-1}\left\{\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{v_j}\right\}.$$

$$S_{N_1,\ldots,N_t}^{M_1,\ldots,M_t}\left[\begin{array}{l}\omega_1(u-x)^{a_1}(y-u)^{b_1}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu'_j}\\ \vdots \\ \omega_t(u-x)^{a_t}(y-u)^{b_t}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu^t_j}\end{array}\right]$$

$$.S_{N'}^{M'}\left[\omega(u-x)^{a}(y-u)^{b}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu_j}\right]^k$$

$$.\aleph_{p_i,q_i,c_i;r}^{M,N}\left[(u-x)^{\sigma}(y-u)^{\rho}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu_j}\right]^s$$

$$=E_1\sum_{k=0}^{[N'/M']}\frac{(-N')_{M'k}}{k!}B'_{N'k}\sum_{k_1=0}^{[N_1/M_1]}\cdots\sum_{k_t=0}^{[N_t/M_t]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(N_t)_{M_tk_t}}{k_t!}$$

$$.\ A(N_1,k_1;\ldots;N_t,k_t)\omega^k\,\omega_1^{k_1}\ldots\omega_t^{k_t}\,E_2\,\aleph_{p_i+1,q_i+1,c_i;r}^{M_1N+h+2} \qquad \ldots(2.1)$$

$$\left[z\left|\begin{array}{l}F_1F_2F_3(a_j,A_j)_{1,n},[c_i(a_{ji},A_{ji})]_{N+1,p_i;r};-;\ldots;-;G_1\\(b_j,B_j)_{1,M},F_4,F_5;[c_i(b_{ji},B_{ji})]_{M+1,q_i;r};[0,1];G_2\end{array}\right.\right]$$

which holds true under following conditions:

(1) $x,y\in R_1(x<y);\sigma,\rho a_i,b_i,a,b,\mu_i\in R^+,v_j\in R,\alpha_j,\beta_j\in c,z\in c$

(i = 1,…,t; j = 1,…,h).

(2) $\max\limits_{1\le j\le h}\left[\left|\dfrac{(y-x)\alpha_j}{\alpha_j x+\beta_j}\right|\right]<1;$

(3) $$\left|\arg\left(z_i \prod_{j=1}^{h} (\alpha_j u + \beta_j)^{z_j^{(i)}}\right)\right| < \frac{T_i \pi}{2} (x \le u \le y; i = 1, ..., r);$$

(4) $M', M_1, ..., M_t$ are arbitrary positive integer and coefficient $B'_{N',k} (N', k \ge 0)$ and $A(N_1 k_1; ...; N_t, k_t)$ are arbitrary constants, real or complex.

(5) $A_j, B_j, A_{ij}, B_{ij} > 0$ and $c_{ij}, b_j, a_{ji}, b_{ji} \in c$.

where

$$E_1 = (y-x)^{m+n-1} \left\{ \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{v_j} \right\}$$

$$E_2 = (y-x)^{\sum_{i=1}^{t} (a_i k_i + b_i k_i) + ak + bk} \left[\prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\sum_{i=1}^{t} (\mu_j^i k_i) + \mu_j k} \right]$$

$$F_1 = \left[1 - m - \sum_{i=1}^{t} a_i k_i - ak; \sigma, 1 \right]$$

$$F_2 = \left[1 = n - \sum_{i=1}^{t} b_i k_i - bk; \rho, 0 \right]$$

$$F_3 = \left[1 + v_j + \sum_{i=1}^{t} \mu_j^{(i)} k_i + \mu_j k; z_j, 0, ..., 1, ..., 0 \right]_{1,h},$$

$$F_4 = \left[1 + v_j + \sum_{i=1}^{t} \mu_j^{(i)} k_i + \mu_j k; z_j, 0, ..., 0 \right]; h$$

$$F_5 = \left[1 - m - n - \sum_{i=1}^{t} (a_i k_i + b_i k_i) - (ak + bk); (\sigma + \rho), 1 \right]$$

$$G_1 = \left[z(y-x)^{\sigma+\rho} / \prod_{j=1}^{h} (\alpha_j x + \beta_j)^{z_j} \right]$$

$$G_2 = \left\{ \frac{(y-x)\alpha_1}{(x\alpha_1+\beta_1)}, \ldots, \frac{(y-x)\alpha_h}{(x\,\alpha_h+\beta_h)} \right\}$$

Proof

To establish the result (2.1), express the general class of polynomials, the generalized polynomials with help of equations (1.3),(1.4), (1.5), (1.6) and Aleph functions by virtue of(1.1) and interchanging the order of summation and integration (which is permissible under the conditions of validity stated). Appealing to the results in (1.7), (1.8), (1.4), we arrive to the right hand side of (2.1).

3. PARTICULAR CASES

1. Here taking $c_i = 1$ in equation (2.1) of Aleph function, the Aleph function coincides with the i-function given by Saxena [10,11].

2. Taking H-function in place of Aleph function in the above result (2.1), right hand side of equation (2.1) reduces to

$$\int_x^y (u-x)^{m-1}(y-u)^{n-1}\left\{\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{v_j}\right\}.$$

$$S_{N_1,\ldots,N_t}^{M_1,\ldots,M_t}\begin{bmatrix} \omega_1(u-x)^{a_1}(y-u)^{b_1}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu'_j} \\ \omega_t(u-x)^{a_t}(y-u)^{b_t}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu^t_j} \end{bmatrix}$$

$$.S_{N'}^{M'}\left[\omega(u-x)^{a}(y-u)^{b}\prod_{j=1}^{h}(\alpha_j u+\beta_j)^{\mu_j}\right]^k$$

$$.\ H\begin{bmatrix} z_1(p-x)^{\sigma_1}(y-p)^{\rho_1}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{-\lambda'_j} \\ z_r(p-x)^{\sigma_r}(y-p)^{\rho_r}\prod_{j=1}^{h}(\alpha_j p+\beta_j)^{\lambda^r_y} \end{bmatrix}dp$$

$$= E_1 \sum_{k=0}^{[N'/M']} \frac{(-N')_{M'k}}{k!} B'_{N',k} \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_t=0}^{[N_t/M_t]} (\omega)^k\, \omega_1^{k_1} \ldots \omega_t^{k_t}\, E_2$$

$$. \; H^{0,\lambda+h+2 \; ;(u',v');...;(u^r,v^r);(1,0);...;(1,0)}_{A+h+2,c+h+1;[B',D'];...;[B^{(r)},D^{(r)}];[0,1];...;[0,1]}$$

$$\left[\begin{matrix} F_1,F_2,F_3,[(a):\theta',...,\theta^{(r)},0,...,0];[(b');\phi'];...;[b^{(r)}:\phi^{(r)}];-;...;-;G_1 \\ [(c);\psi',...,\psi^{(r)};0,...,0],F_4,F_5:[(d'):\delta'];...;[d^{(r)}:\delta^{(r)}];[0,1];...;[0,1]; \end{matrix} \right| G_2$$

which holds under the following conditions

(a) $x, y \in R_1 (x < y); \sigma_i, \rho_i \, c_j^{(i)}, a_i, b_i, \mu_j^{(i)}, a, b, \mu_i \in R^+, \alpha_j, \beta_j \in c, z_i \in c$

$(i = 1,\ldots,r;\; j = 1,\ldots,h)$.

(b) $$\max_{1 \le j \le h} \left[\left| \frac{(y-x)\alpha_j}{\alpha_j x + \beta_j} \right| \right] < 1;$$

(c) $$\mathrm{Re}\left[m + \sum_{i=1}^{r} \frac{\sigma_j d_j^{(i)}}{\delta_j^{(i)}} \right] > 0 \; (j = 1,..., u^{(i)}),$$

$$\mathrm{Re}\left[n + \sum_{i=1}^{r} \frac{\rho_j d_j^{(i)}}{\delta_j^{(i)}} \right] > 0 \; (j = 1,..., u^{(i)})$$

(d) $$\left| \arg\left(z_i \prod_{j=1}^{h} (\alpha_j p + \beta_j)^{-\lambda_j^{(i)}} \right) \right| < \frac{T_i \pi}{2} (x \le p \le y; i = 1,..., r)$$

M and M_i are arbitrary positive integers. Coefficient $B'_{N',k}, A(N_1, k_1;...; N_t, k_t)$ are arbitrary constants, real or complex.

where

$$E_1 = (y-x)^{m+n-1} \left[\prod_{j=1}^{h} (\alpha_j x + \beta_j)^{v_j} \right]$$

$$E_2 = (y-x)^{\sum_{i=1}^{t} (a_i k_i + b_i k_i) + ak + bk} \left[\prod_{j=1}^{h} (\alpha_j x + \beta_j)^{\sum_{i=1}^{t} (\mu_j^i k_i) + \mu_j k} \right]$$

$$G_1 = \left\{ \begin{array}{l} z_1 (y-x)^{\sigma_1+\rho_1} / \prod_{j=1}^{h} (\alpha_j x+\beta_j)^{v_j'} \\ \vdots \\ z_r (y-x)^{\sigma_r+\rho_r} / \prod_{j=1}^{h} (\alpha_j x+\beta_j)^{v_j^r} \end{array} \right\}$$

$$G_2 = \left\{ \begin{array}{l} (y-x)\alpha_1 /(x\, \alpha_1+\beta_1) \\ \vdots \\ (y-x)\alpha_h / x\, \alpha_h+\beta_h \end{array} \right\}$$

CONCLUSION

The results derived in this paper is uncommon and find its utility in various fields of mathematics such as applied mathematics, analytical mathematics, mathematical physics etc. Thus the main result presented in this article would at once yield a very large number of results containing a large variety of simpler special functions occurring scientific and technological fields.

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