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On Inequality for the Non-Local Fractional Differential Equation

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Abstract

In the paper the nonlocal fractional differential equation with boundary conditions will be treated and will form the inequalities like Lyapunov inequality, Hartman and Wintner inequality for the solution of nonlocal fractional differential equation with the help of the greens function involved in the solution of nonlocal fractional differential equation with boundary conditions.

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1. Introduction

In many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc the fractional differential and integral equations represents the processes in a more effective manner than

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by integer order. Because of this the subject of fractional order differential and integral equations became the interest of mathematicians and researchers.

Due to vast application in various fields inequality of differential equation involving differential operators with fractional order is also one of the interesting topic for researchers and mathematicians. As the inequality provides the distance between consecutive zeros of the solution so it is very useful in applications e.g. in oscillation and Sturmian theory, asymptotic theory, disconjugacy, eigenvalue problems e.t.c the Lyapunov inequality and some of its generalizations gives successful results [6, 7, 9, 10, 11], as Lyapunov inequality

$$\int_a^b |p(s)|ds > \frac{4}{b-a} \tag{1.1}$$

studied by Russian mathematician A. M. Liapunov [4] for a nontrivial solution of

$$x''(t) + p(t)x(t) = 0, a < t < b \quad with \quad x(a) = x(b) = 0 \tag{1.2}$$

where $p : [a, b] \to \mathbb{R}$ is continuous function, provides lower bound for the distance between consecutive zeros of x(t). Most of the properties of Liapunov inequality and its generalization studied by [12, 13, 14, 15, 16, 17, 18, 19] and the references therein.

The number of generalization are present, one of them is

$$\int_a^b (b-s)(s-a)p^+(s)ds > b-a,\ p^+(s) = max_{a\leq s\leq b}\{p(s), 0\} \tag{1.3}$$

studied by Hartman and Wintner [5] and also many other extensions to fractional order differential equation with boundary conditions were obtained by [20, 21, 22, 23, 24, 25, 26].

In the present paper we will find such type of inequalities for the non-local fractional differential equations with boundary conditions

$$\begin{aligned} {}^cD_a^\alpha u(t) &= q(t)f(t, u(t), {}^cD_a^\eta u(t)) \\ u(a) = u(a) = u''(a) &= 0, u'''(b) = \gamma u(\tau) \end{aligned} \tag{1.4}$$

where ${}^cD_a^\alpha, {}^cD_a^\eta$ denotes the Caputo fractional derivative of order α and η, $a < t, \tau < b$, $3 < \alpha, \eta \leq 4$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

2. Auxiliary Results

We need the following definitions,lemmas and theorems in the sequel.

The extensively studied fractional derivative and integral is Riemann-Liouville fractional derivative and integral defined by [1, 2, 3, 27, 28, 29, 30].

Definition 2.1. The Riemann-Liouville fractional integral of order α of a real valued function f is defined by

$$I_a^\alpha f(z) = \frac{1}{\Gamma\alpha}\int_a^z (z-t)^{\alpha-1} f(t)dt \tag{2.5}$$

and when $a = 0$ this becomes

$$I^{\alpha} f(z) = \frac{1}{\Gamma\alpha} \int_0^z (z-t)^{\alpha-1} f(t)dt \tag{2.6}$$

The property studied by T.L. Holambe and Mohammed Mazhar-Ul-Haque [8] is,
Let $\alpha > 0$, $\beta > 0$ and f be any function. Then

$$I_k^{\alpha}(I_k^{\beta} f(z)) = I_k^{\beta}(I_k^{\alpha} f(z)) \tag{2.7}$$

$$I_k^{\alpha}(I_k^{\beta} f(z)) = I_k^{\beta}(I_k^{\alpha} f(z)) = I_k^{\alpha+\beta} f(z) = \frac{1}{k\Gamma_k(\alpha+\beta)} \int_0^z (z-t)^{\frac{\alpha+\beta}{k}-1} f(t)dt \tag{2.8}$$

for the $k-$generalized fractional integral.

Definition 2.2. The Riemann-Liouville fractional derivative of order α of a real valued function f is defined by

$$D_a^{\alpha} f(z) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dz}\right)^n \int_a^z (z-t)^{n-\alpha-1} f(t)dt \tag{2.9}$$

where $n = [alpha] + 1$.

Definition 2.3. The Caputo fractional derivative of order α of a real valued function f is defined by

$${}^{c}D_a^{\alpha} f(z) = \frac{1}{\Gamma(n-\alpha)} \int_a^z (z-t)^{n-\alpha-1} f^n(t)dt \tag{2.10}$$

where $n = [\alpha] + 1$.

Definition 2.4. A real valued function f from $I \times \mathbb{R}$ is called Caratheodary function if

1. f is measurable on $\mathbb{R}$.
2. f is continuous on I.
3. There exist Lebesgue function h on I such that h is an Upper bound of f on I.

Theorem 2.5. (Kolmogorov compactness criterion [19]) Let $\Omega \subseteq L^p(0,T)$, $1 \leq p < \infty$ if

(i) Ω is bounded in $L^p(0,T)$and

(ii) $u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$,

then Ω is relatively compact in $L^p(0,T)$ where $u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds$.

The following lemma will be very useful

Lemma 2.6. Assume that $f \in C(a,b) \cap L^1(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L^1(a,b)$ Then

$$I_a^\alpha({}^cD_a^\alpha f)(t) = c_0+c_1(t-a)+c_2(t-a)^2+c_3(t-a)^3+\cdots\cdots+c_{n-1}(t-a)^{n-1}+c_n(t-a)^n \tag{2.11}$$

for $t \in [a,b]$, $c_i \in \mathbb{R}$ and $n = [\alpha]+1$.

3. Main Results

Theorem 3.1. For any $4 < \alpha \leq 5$and $u \in C[a,b]$ the nonlocal fractional boundary value problem

$$\begin{aligned} {}^cD_a^\alpha u(t) &= q(t)f(t,u(t),{}^cD_a^\eta u(t)) \\ u(a) = u(a) = u''(a) &= 0, u'''(b) = \gamma u(\tau) \end{aligned} \tag{3.12}$$

has unique solution given by

$$\begin{aligned} u(t) = &-\int_a^b G(t,s)q(s)f(s,u(s),{}^cD_a^\eta u(s))ds \\ &- \frac{((t-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\int_a^b G(\tau,s)q(s)f(s,u(s),{}^cD_a^\eta u(s))ds \end{aligned} \tag{3.13}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6} - (t-s)^{\alpha-1}, a \leq s \leq t \leq b \\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6}, a \leq t \leq s \leq b \end{cases}$$

Proof. For the given non-local fractional boundary value problem the solution by lemma 2.6 is

$$\begin{aligned} u(t) &= c_0 + c_1(t-a) + c_2(t-a)^2 + c_3(t-a)^3 \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)f(s,u(s),{}^cD_a^\eta u(s))ds \end{aligned} \tag{3.14}$$

$u(a) = u'(a) = u''(a) = 0$ implies $c_0 = c_1 = c_2 = 0$

$$u(t) = c_3(t-a)^3 + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)f(s,u(s),{}^cD_a^\eta u(s))ds$$

$$u'(t) = 3c_3(t-a)^2 + \frac{1}{\Gamma(\alpha)}\int_a^t (\alpha-1)(t-s)^{\alpha-2}q(s)f(s,u(s),{}^cD_a^\eta u(s))ds$$

$$u''(t) = 6c_3(t-a) + \frac{1}{\Gamma(\alpha)}\int_a^t (\alpha-1)(\alpha-2)(t-s)^{\alpha-3}q(s)f(s,u(s),{}^cD_a^\eta u(s))ds$$

$$u'''(t) = 6c_3 + \frac{1}{\Gamma(\alpha)}\int_a^t (\alpha-1)(\alpha-2)(\alpha-3)(t-s)^{\alpha-4}q(s)f(s,u(s),{}^cD_a^\eta u(s))ds \tag{3.15}$$

$u'''(b) = \gamma u(\tau)$

$$\begin{aligned}
6c_3 + \frac{1}{\Gamma(\alpha)} \int_a^b (\alpha-1)(\alpha-2)(\alpha-3)(b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
= \gamma \left(c_3(\tau-a)^3 + \frac{1}{\Gamma(\alpha)} \int_a^\tau (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \right) \\
c_3[6-\gamma(\tau-a)^3] = \frac{\gamma}{\Gamma(\alpha)} \int_a^\tau (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
- \frac{1}{\Gamma(\alpha)} \int_a^b (\alpha-1)(\alpha-2)(\alpha-3)(b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
c_3 = \frac{\gamma}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^\tau (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
- \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^b (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds
\end{aligned} \tag{3.16}$$

from

$$\begin{aligned}
u(t) = \frac{\gamma(t-a)^3}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^\tau (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
- \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^b (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\frac{1}{[6-\gamma(\tau-a)^3]} &= \frac{1}{6}\left[\frac{6}{[6-\gamma(\tau-a)^3]}\right] \\
&= \frac{1}{6}\left[\frac{6-\gamma(\tau-a)^3+\gamma(\tau-a)^3}{[6-\gamma(\tau-a)^3]}\right] \\
&= \frac{1}{6}\left[1+\frac{\gamma(\tau-a)^3}{[6-\gamma(\tau-a)^3]}\right]
\end{aligned}$$

$$\begin{aligned}
u(t) = \frac{\gamma(t-a)^3}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^\tau (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
- \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{6\Gamma(\alpha)} \left[1+\frac{\gamma(\tau-a)^3}{[6-\gamma(\tau-a)^3]}\right] \\
\times \int_a^b (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds
\end{aligned} \tag{3.18}$$

thus we have

$$
\begin{aligned}
u(t) = {} & \frac{\gamma(t-a)^3}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_a^{\tau} (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{6\Gamma(\alpha)} \int_a^{t} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{6\Gamma(\alpha)} \int_t^{b} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3 \gamma(\tau-a)^3}{6\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \\
& \times \int_a^{b} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_a^{t} (t-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
u(t) = {} & \frac{\gamma(t-a)^3}{\Gamma(\alpha)[\gamma(\tau-a)^3-6]} \int_a^{\tau} (\tau-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{6\Gamma(\alpha)} \int_a^{t} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3}{6\Gamma(\alpha)} \int_t^{b} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3 \gamma(\tau-a)^3}{6\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \\
& \times \int_a^{\tau} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& - \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3 \gamma(\tau-a)^3}{6\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \\
& \times \int_{\tau}^{b} (b-s)^{\alpha-4} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_a^{t} (t-s)^{\alpha-1} q(s) f(s, u(s), {}^c D_a^{\eta} u(s)) ds
\end{aligned}
\tag{3.20}
$$

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_a^t \left[\frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6} - (t-s)^{\alpha-1} \right]$$
$$\times q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds$$
$$- \frac{1}{\Gamma(\alpha)} \int_t^b \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6}$$
$$\times q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds$$
$$- \frac{((t-a)^3\gamma}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]}$$
$$\times \int_a^\tau \left[\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\tau-a)^3(b-s)^{\alpha-4}}{6} - (\tau-s)^{\alpha-1} \right]$$
$$\times q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds$$
$$- \frac{(t-a)^3\gamma}{\Gamma(\alpha)[6-\gamma(\tau-a)^3]} \int_\tau^b \left[\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\tau-a)^3(b-s)^{\alpha-4}}{6} \right]$$
$$\times q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \tag{3.21}$$

$$u(t) = -\int_a^b G(t,s) q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds$$
$$- \frac{((t-a)^3\gamma}{[6-\gamma(\tau-a)^3]} \int_a^b G(\tau,s) q(s) f(s, u(s), {}^c D_a^\eta u(s)) ds \tag{3.22}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6} - (t-s)^{\alpha-1}, a \le s \le t \le b \\ \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6}, a \le t \le s \le b \end{cases}$$

■

Theorem 3.2. Greens function from theorem 3.1 satisfies the following conditions

1. $G(t,s)$ is non-negative for $t, s \epsilon [a,b]$.
2. $G(t,s)$ is non-decreasing with respect to the first variable.
3. $G(t,s) \le G(b,s)$ for $t, s \epsilon [a,b]$.

Proof. 1. Since $\alpha \ge 4$ so for $a \le t \le s \le b$

$$G(t,s) = \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6\Gamma(\alpha)} \ge 0$$

and for $a \leq s \leq t \leq b$

$$
\begin{aligned}
G(t,s) &= \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\geq \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-s)^3(t-s)^{\alpha-4}}{6\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\{ \frac{(\alpha-1)(\alpha-2)(\alpha-3)-6}{6} \right\} \geq 0
\end{aligned}
$$

2. Again for $a \leq t \leq s \leq b$

$$
\begin{aligned}
G(t,s) &= \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6\Gamma(\alpha)} \\
\frac{\partial G(t,s)}{\partial t} &= \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^2(b-s)^{\alpha-4}}{2\Gamma(\alpha)} \geq 0
\end{aligned}
$$

and for $a \leq s \leq t \leq b$

$$
\begin{aligned}
G(t,s) &= \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^3(b-s)^{\alpha-4}}{6\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
\frac{\partial G(t,s)}{\partial t} &= \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-a)^2(b-s)^{\alpha-4}}{2\Gamma(\alpha)} - \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} \\
&\geq \frac{(\alpha-1)(\alpha-2)(\alpha-3)(t-s)^2(t-s)^{\alpha-4}}{2\Gamma(\alpha)} - \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)} \\
&= \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)} \left\{ \frac{(\alpha-1)(\alpha-2)(\alpha-3)-2(\alpha-1)}{2} \right\} \geq 0
\end{aligned}
$$

this shows that the Greens function is non decreasing and immediately satisfies.

3. $G(t,s) \leq G(b,s)$ for $t, s \epsilon [a,b]$. ■

Theorem 3.3. Assume that $|f(t,x,y)| \leq c_1|x|^m + c_2|x|^n$, $t \in [a,b]$, $c_1, c_2 > 0$, $m, n \geq b$, Suppose the non-local fractional boundary value problem

$$
\begin{aligned}
{}^cD_a^\alpha u(t) &= q(t) f(t, u(t), {}^c D_a^\eta u(t)) \\
u(a) = u(a) &= u''(a) = 0, u'''(b) = \gamma u(\tau)
\end{aligned}
$$

has nontrivial continuous solution then

$$
\begin{aligned}
&6\Gamma(\alpha) \left(1 + \frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]} \right)^{-1} \\
&\quad \leq \int_a^b \left\{ (\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3(b-s)^{\alpha-4} - 6(b-s)^{\alpha-1} \right\} |q(s)| ds
\end{aligned} \tag{3.23}
$$

Proof. Here we are taking Banach space $C[a,b] = \{u : [a,b] \to \mathbb{R} : u\, is\, continuous\}$ with the standard supremum norm $\|u\| = max\,\{u(t) : a \le t \le b\}$ we have the solution for the given non-local fractional boundary value problem

$$
\begin{aligned}
u(t) = &-\int_a^b G(t,s)q(s)f(s,u(s),{}^c D_a^\eta u(s))ds \\
&- \frac{((t-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\int_a^b G(\tau,s)q(s)f(s,u(s),{}^c D_a^\eta u(s))ds
\end{aligned}
$$

now for $t \in [a,b]$

$$
\begin{aligned}
|u(t)| &\le \int_a^b |G(t,s)||q(s)||f(s,u(s),{}^c D_a^\eta u(s))|ds \\
&\quad + \frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\int_a^b |G(\tau,s)||q(s)||f(s,u(s),{}^c D_a^\eta u(s))|ds \\
|u(t)| &\le \int_a^b |G(b,s)||q(s)||f(s,u(s),{}^c D_a^\eta u(s))|ds \\
&\quad + \frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\int_a^b |G(b,s)||q(s)||f(s,u(s),{}^c D_a^\eta u(s))|ds \\
|u(t)| &\le \left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)\int_a^b |G(b,s)||q(s)|\left\{c_1|u|^m + c_2|{}^c D_a^\eta u|\right\}ds \\
\|u(t)\| &\le \left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)\int_a^b |G(b,s)||q(s)|\left\{c_1\|u\| + c_2\|{}^c D_a^\eta u\|\right\}ds
\end{aligned}
$$

for $U \in \mathbb{R}$ such that $max\left\{\|u\|, c_1\|u\| + c_2\|{}^c D_a^\eta u\|\right\} \le U$ implies

$$
1 \le \left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)\int_a^b G(b,s)|q(s)|ds \tag{3.24}
$$

$$
\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \le \int_a^b G(b,s)|q(s)|ds \tag{3.25}
$$

now from the obtained Greens function

$$
\begin{aligned}
&\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \\
&\quad \le \frac{1}{6\Gamma(\alpha)}\int_a^b \left\{(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3(b-s)^{\alpha-4} - 6(b-s)^{\alpha-1}\right\}|q(s)|ds
\end{aligned}
$$

$$6\Gamma(\alpha)\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1}$$
$$\leq \int_a^b \left\{(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3(b-s)^{\alpha-4}-6(b-s)^{\alpha-1}\right\}|q(s)|ds \tag{3.26}$$

■

Theorem 3.4. Let the non-local fractional boundary value problem

$$^cD_a^\alpha u(t) = q(t)f(t,u(t),^c D_a^\eta u(t))$$
$$u(a) = u(a) = u''(a) = 0, u'''(b) = \gamma u(\tau)$$

has nontrivial continuous solution then

$$\frac{6\Gamma(\alpha-3)}{(b-a)^\alpha}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b |q(s)|ds \tag{3.27}$$

Proof. Since

$$\left\{(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3(b-s)^{\alpha-4}-6(b-s)^{\alpha-1}\right\}$$
$$\leq \left\{(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3(b-s)^{\alpha-4}\right\} \tag{3.28}$$

so the inequality 3.26 becomes

$$\frac{6\Gamma(\alpha)}{(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^3}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b (b-s)^{\alpha-4}|q(s)|ds$$

$$\frac{6\Gamma(\alpha-3)}{(b-a)^3}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b (b-s)^{\alpha-4}|q(s)|ds$$

by taking $\psi(s) = (b-s)^{\alpha-4}$ where $s \in [a,b]$
we have $\psi'(s) = -(\alpha-4)(b-s)^{\alpha-5} \leq 0$ now it is obvious that $\psi(s) = (b-s)^{\alpha-4} \leq (b-a)^{\alpha-4}$ so

$$\frac{6\Gamma(\alpha-3)}{(b-a)^3}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b (b-a)^{\alpha-4}|q(s)|ds$$

$$\frac{6\Gamma(\alpha-3)}{(b-a)^3}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b (b-a)^{\alpha-3}|q(s)|ds$$

$$\frac{6\Gamma(\alpha-3)}{(b-a)^\alpha}\left(1+\frac{((b-a)^3\gamma}{[6-\gamma(\tau-a)^3]}\right)^{-1} \leq \int_a^b |q(s)|ds \tag{3.29}$$

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