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Coincidences and Common Fixed-Point Theorems in Intuitionistic Fuzzy Metric Space

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Abstract

Using the idea of intuitionistic fuzzy set Atanassov [2], Turkoglu, Alaca, Cho and Yildiz [18] defined the notion of intuitionistic fuzzy metric space and proved intuitionistic fuzzy version of Pant's Theorem [13]. In this paper we prove an improved version of the results of Turkoglu, Alaca, Cho and Yildiz in intuitionistic fuzzy metric space by not using the continuity of the function.

INTRODUCTION

Atanassov [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [4] introduced the concept of the intuitionistic fuzzy topological spaces. In 2004, using the idea intuitionistic fuzzy sets, Park [14] introduced and studied a notion of intuitionistic fuzzy metric space as a natural generalization of fuzzy metric space.

In 2006, Ciric, Jesiic and Ume [3], Alaca, Turkoglu and Yildiz [1] and Turkoglu, Alaca, Cho and Yildiz [18] studied intuitionistic fuzzy metric space. In [3] Ciric, Jesiic and

Ume introduced and investigated a class of asymptotically nonexpansive mappings and proved existence theorems for fixed and periodic points of these mappings. Alaca, Turkoglu and Yildiz [1] extended the well known fixed point theorems of Banach and Edelstein to intuitionistic fuzzy metric space. There- after Turkoglu, Alaca, Cho and Yildiz [18] introduced the definition of weakly commuting and R–weakly mappings in intuitionistic fuzzy metric space and proved intuitionistic fuzzy version of Pant's theorem [13]. They also proved the common fixed point theorems for commuting mappings in intuitionistic fuzzy metric space by using the continuity of functions which are generalization Jungck's common fixed point theorem [7] . In 2010 and 2016, Tripathi, Mishra and Gupta [15], [16] proved some fixed point theorems by using the properties of contraction mapping. The result of Tripathi et al. prompted us to go for further generalization of some results of fixed point proved by Turkoglu, Alaca, Cho and Yildiz [18] by using coincidently commuting mappings and without using continuity of mappings.

DEFINITION 1.1 [17]. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t–norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative.

(ii) $*$ is continuous.

(iii) $a * 1 = a$ for all $a \in [0,1]$.

(iv) $a * b \le c * d$, whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

DEFINITION 1.2 [17]. A binary operation $\Diamond:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t–conorm if $\Diamond$ is satisfies the following conditions:

(i) $\Diamond$ is commutative and associative.

(ii) $\Diamond$ is continuous.

(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$.

(iv) $a \Diamond b \le c \Diamond d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

The concept of triangular norms (t–norms) and triangular conorms (t–conorms) are introduced by Menger [12] in his study of statistical metric space. Several examples for these concepts were given by many authors ([6], [9]).

The following definition of intuitionistic fuzzy metric spaces along with their fundamental properties was given by Alaca, Turkoglu and Yildiz [1].

DEFINITION 1.3[1] A 5–tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space (IF space) if X is an arbitrary set, $*$ is a continuous t–norm, $\Diamond$ is a continuous t–conorm and M, N are fuzzy sets on $X^2 \times [0,\infty)$ satisfying the following conditions.

(i) $M(x,y,t)+N(x,y,t)\le 1 \ \forall x,y\in X, \forall t>0,$

(ii) $M(x,y,0)=0 \ \forall x,y\in X,$

(iii) $M(x,y,t)=1 \ \forall x,y\in X, \forall t>0,$ if and only if $x=y$,

(iv) $M(x,y,t)=M(y,x,t) \ \forall x,y\in X, \forall t>0,$

(v) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s) \ \forall x,y\in X, \forall s,t>0,$

(vi) $M(x,y,.):[0,\infty)\to[0,1]$ is left continuous, $\forall x,y\in X,$

(vii) $\lim_{t\to\infty} M(x,y,t)=1 \ \forall x,y\in X,$

and,

(viii) $N(x,y,0)=1 \ \forall x,y\in X,$

(ix) $N(x,y,t)=0 \ \forall x,y\in X, \forall t>0,$ if and only if $x=y$,

(x) $N(x,y,t)=N(y,x,t) \ \forall x,y\in X, \forall t>0,$

(xi) $N(x,y,t)\Diamond N(y,z,s)\ge N(x,z,t+s) \ \forall x,y\in X, \forall s,t>0,$

(xii) $N(x,y,.):[0,\infty)\to[0,1]$ is right continuous, $\forall x,y\in X,$

(xiii) $\lim_{t\to\infty} N(x,y,t)=0 \ \forall x,y\in X.$

REMARK 1.1 [11]. The function $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and degree of non–nearness between x and y with respect to t, respectively.

REMARK 1.2 [10] Every fuzzy metric space (X, M, $*$) is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t–norm $*$ and t–conorm $\Diamond$ are associated [10], as $x \Diamond y = 1-((1-x)*(1-y)) \forall x, y \in X$.

REMARK 1.3 [10] In intuitionistic fuzzy metric space X, M(x, y, t) is non-decreasing and $N(x, y, t)$ is non-increasing for all $x, y \in X$.

DEFINITION 1.4 [1]. Let (X, M, N, $*$, $\Diamond$) be an intuitionistic fuzzy metric space. Then

(a) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$

$$\lim_{n\to\infty} M(x_{n+p,} x_n, t) = 1, \text{ and } \lim_{n\to\infty} N(x_{n+p,} x_n, t) = 0.$$

(b) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$

$$\lim_{n\to\infty} M(x_{n,} x, t) = 1, \text{and } \lim_{n\to\infty} N(x_{n,} x, t) = 0.$$

Since $*$ and $\Diamond$ are continuous, the limit is uniquely determined from (v) and (xi) , respectvily.

DEFINITION 1.5 [1]. An intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) is said to be complete if every Cauchy sequence in X is convergent.

DEFINITION 1.6 [1]. An intuitionistic fuzzy metric space (X, M, N, *, $\Diamond$) is said to be compact if every sequence in X has a convergent subsequence.

The following theorem was proved by Jungck [7] as a generalization of Banach's contraction principle in metric space.

THEOREM 1.1 [7]. Let f be continuous mapping on a complete metric space

(X, d) and $g : X \to X$ be a mapping satisfying the conditions,

(i) $g(X) \subset f(X)$,

(ii) g commutes with f,

and (iii) $\exists\, k \in (0,1)$ such that $d(g(x), g(y)) \le kd(f(x), f(y)) \;\; \forall x, y \in X$.

Then f and g have unique common fixed point.

Turkoglu, Alaca, Cho and Yildiz [18] proved the theorem in intuitionistic fuzzy metric space (X, M, N, $*$, $\Diamond$) as follows.

THEOREM 1.2. Let (X, M, N, $*$, ◊) be a complete intuitionistic fuzzy metric space and $f, g : X \to X$ be mappings satisfying the following conditions.

(i) $g(X) \subset f(X)$,

(ii) g commutes with f and f is continuous,

and (iii) $\exists\, k \in (0,1)$ such that $M(g(x), g(y), kt) \geq M(f(x), f(y), t)$, $N(g(x), g(y), kt) \leq N(f(x), f(y), t)$

$\forall x, y \in X, \forall t > 0.$

Then f and g have unique common fixed point in X.

REMARK 1.4. In a subsequent paper, Jungck [8] weakened the hypothesis by withdrawing the condition of commutativity of f and g. Using an obvious intuitionistic fuzzy metric analogue of Jungck's notion of compatibility of mappings, Turkoglu, Alaca, Cho and Yildiz [18] replaced the commutativity of f and g in Theorem (1.2) by their compatibility.

S. Sessa [5] initiated the concept of weakly commutative mappings and obtained some fixed point theorems in a metric space.

DEFINITION 1.7 [5]. The mappings f and g on a metric space (X, d), are said to be weakly commuting if

$$d(fgx, gfx) \leq d(f(x), gx) \quad \forall x \in X.$$

Turkoglu, Alaca, Cho and Yildiz [18] defined weakly commuting mappings in intuitionistic fuzzy metric space.

DEFINITION 1.8 [18]. Let (X, M, N, *, ◊) be an intuitionist fuzzy metric space. Then the mappings $f, g : X \to X$ are said to be weakly commuting if

$$(fgx, gfx, t) \geq M(fx, gx, t), \quad N(fgx, gfx, t) \leq N(fx, gx, t) \quad \forall x \in X.$$

The following definition was given by Pant [13].

DEFINITION 1.9 [13]. The mappings f and g on a metric space (X, d), are said to be R-weakly commuting if there exist a positive real number R such that,

$$d(fgx, gfx) \leq Rd(fx, gx) \quad \forall x \in X.$$

Turkoglu, Alaca, Cho and Yildiz [18] defined R-weakly commuting mappings in intuitionistic fuzzy metric space as.

DEFINITION 1.10 [18]. Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then mapping $f, g : X \to X$ are said to be R-weakly commuting if there exists a positive real number R such that

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}), \text{ and } N(fgx, gfx, t) \leq N(fx, gx, \frac{t}{R}) \;\; \forall x \in X.$$

REMARK 1.5 [18]. Weak commutativity implies R–weak commutativity in intuitionistic fuzzy metric space. However, *R–weak* commutativity implies Weak commutativity only when $R \leq 1$.

EXAMPLE 1.1 [18]. Let $X = R$ be set of all real numbers.

Define $a * b = ab, \quad a \Diamond b = \min\{1, a+b\}$,

$$M(x, y, t) = \left(\exp(\frac{|x-y|}{t}) \right)^{-1},$$

$$N(x, y, t) = \frac{\left(\exp(\frac{|x-y|}{t}) \right) - 1}{\exp(\frac{|x-y|}{t})},$$

and $M(x, y, 0) = 0, N(x, y, 0) = 1 \;\; \forall x, y \in X, \forall t > 0.$

Then (X, M, N, $*$, ◊) is an intuitionistic fuzzy metric space.

Define $f(x) = 2x - 1$ and $g(x) = x$.

Then

$$M(fgx, gfx, t) = \left(\exp\left(2\frac{|x-1|^2}{t} \right) \right)^{-1},$$

$$N(fgx, gfx, t) = \frac{\exp\left(2\frac{|x-1|^2}{t} \right) - 1}{\exp\left(2\frac{|x-1|^2}{t} \right)},$$

and

$$M(fx, gx, \frac{t}{2}) = \left(\exp\left(2\frac{|x-1|^2}{t} \right) \right)^{-1},$$

$$N(fx, gx, \frac{t}{2}) = \frac{\exp\left(2\frac{|x-1|^2}{t} \right) - 1}{\exp\left(2\frac{|x-1|^2}{t} \right)}.$$

Therefore for $R = 2$, the mappings f and g are R–weakly commuting, but they are not weakly commuting because exponential function is strictly increasing function.

The following theorem was proved by Pant [13].

THEOREM 1.3 [13]. Let *(X, d)* be a complete metric space and $f, g : X \to X$ be R–weakly commuting mappings satisfying the condition,

$$d(fx, fy) \le r(d(gx, gy)) \quad \forall x, y \in X,$$

where $r : \mathrm{R}^+ \to \mathrm{R}^+$ is a continuous function such that $r(t) < t$ for all $t > 0$. Suppose range of g contains the range of f and either f or g is continuous.

Then f and g have a unique common fixed point in X.

Turkoglu, Alaca, Cho and Yildiz [18] proved the intuitionistic fuzzy version of Pant's theorem.

THEOREM 1.4 [18]. Let $(X, M, N, *, \lozenge)$ be a complete intuitionistic fuzzy metric space and $f, g : X \to X$ be R–weakly commuting mappings satisfying the following conditions,

(i) $f(X) \subset g(X)$

(ii) f or g is continuous,

(iii) $M(fx, fy, t) \ge c\left(M(gx, gy, t)\right),\ N(fx, fy, t) \le c'\left(N(gx, gy, t)\right)\ \forall x, y \in X, 0 < t < 1,$

where $c:[0,1]\to[0,1]$ and $c':[0,1]\to[0,1]$ are continuous functions such that $c(t)>t$ and $c'(t)<t$.

(iv) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ implies,

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t) \text{ and } \lim_{n\to\infty} N(x_n, y_n, t) = N(x, y, t).$$

Then f and g have a unique common fixed point in *X.*

2. MAIN RESULT

Now we prove coincidence and common fixed point theorems in intuitionistic fuzzy metric spaces.

THEOREM 2.1. Let (*X*, *M, N*, **,* ◊) be an intuitionistic fuzzy metric space and Y be a nonempty set. Suppose $f, g: Y \to X$ are two mappings satisfying the following conditions.

(i) $f(Y)\subset g(Y)$ and either $f(Y)$ or $g(Y)$ are complete,

(ii) $\exists\, k\in(0,1)$ such that $M(fx, fy, kt)\geq M(gx, gy, t)$ and

$$N(fx, fy, kt)\leq N(gx, gy, t)\ \forall x, y\in Y, \forall t>0.$$

Then f and g have coincidence point.

PROOF: Let $x_0\in Y$. Since $f(Y)\subset g(Y)$ so there exist $x_1\in Y$ such that $f(x_0)=g(x_1)$. Inductively we can construct a sequence $\{x_n\}$ such that $f(x_n)=g(x_{n+1})$. Now from (ii),

$$M(gx_n, gx_{n+1}, t) = M(fx_{n-1}, fx_n, t)$$

$$\geq M(gx_{n-1}, gx_n, \frac{t}{k})$$

$$\geq M(gx_{n-2}, gx_{n-1}, \frac{t}{k^2})$$

$$\geq \ldots$$

$$\geq M(gx_0, gx_1, \frac{t}{k^n}).$$

Hence

$$M(gx_n, gx_{n+1}, t) \geq M(gx_0, gx_1, \frac{t}{k^n}) \quad \forall \quad n.$$

Similarly,

$$N(gx_n, gx_{n+1}, t) \leq N(gx_0, gx_1, \frac{t}{k^n}) \quad \forall \quad n..$$

Hence for any positive integer p,

$$M(gx_n, gx_{n+p}, t) \geq M(gx_n, gx_{n+1}, \frac{t}{k}) * \ldots * (\text{p times}) \ldots * M(gx_{n+p-1}, gx_{n+p}, \frac{t}{k})$$

$$\text{i.e. } M(gx_n, gx_{n+p}, t) \geq M(gx_0, gx_1, \frac{t}{pk^n}) * \ldots * (\text{p times}) \ldots * M(gx_0, gx_1, \frac{t}{pk^{n+p-1}}).$$

Similarly,

$$N(gx_n, gx_{n+p}, t) \leq M(gx_0, gx_1, \frac{t}{pk^n}) \Diamond \ldots \Diamond (\text{p times}) \ldots \Diamond N(gx_0, gx_1, \frac{t}{pk^{n+p-1}}).$$

By (xii), (xiii) of definition (B), since

$$\lim_{n\to\infty} M(gx_0, gx_1, \frac{t}{pk^n}) = 1, \quad \lim_{n\to\infty} N(gx_0, gx_1, \frac{t}{pk^n}) = 0,$$

it follows that,

$$\lim_{n\to\infty} M(gx_n, gx_{n+p}, t) \geq 1*....*1 = 1$$

and

$$\lim_{n\to\infty} N(gx_n, gx_{n+p}, t) \leq 0\Diamond....\Diamond 0 = 0,$$

i.e. $$\lim_{n\to\infty} M(gx_n, gx_{n+p}, t) = 1$$

and

$$\lim_{n\to\infty} N(gx_n, gx_{n+p}, t) = 0.$$

Thus $\{gx_n\} = \{fx_{n-1}\}$ is a Cauchy sequence.

Suppose $g(Y)$ is complete. Then there exists $w \in g(Y)$ such that $\{gx_n\} \to w$ so there exist $z \in Y$ such that $gz = w$.

Putting $x = x_n$, $y = z$ in (ii), we have

$$M(fx_n, fz, kt) \geq M(gx_n, gz, t)$$
and
$$N(fx_n, fz, kt) \leq N(gx_n, gz, t).$$

Making $n \to \infty$, we get,

$$M(w, fz, kt) \geq M(w, w, t)$$
and
$$N(w, fz, kt) \leq N(w, w, t),$$

i.e.

$$M(w, fz, kt) \geq 1$$
and $\forall t > 0.$
$$N(w, fz, kt) \leq 0$$

Thus $w = fz = gz$.

Again if $f(Y)$ is complete then $\{gx_n\} \to w \in f(Y) \subset g(Y)$, and then same argument repeats. Therefore w is coincidence point of f and g.

THEOREM 2.2. Let (*X*, *M*, *N*, $*$, $\Diamond$) be an intuitionistic fuzzy metric space and $f, g : X \to X$ be two mappings satisfying the following conditions,

(i) $f(X) \subset g(X)$ and either $f(X)$ or $g(X)$ are complete,

(ii) $\exists\, k \in (0,1)$ such that $M(fx, fy, kt) \geq M(gx, gy, t)$ and

$N(fx, fy, kt) \leq N(gx, gy, t)\ \forall x, y \in Y, \forall t > 0,$

(iii) f and g are commuting at there coincidence point.

Then *f* and *g* have unique common fixed point.

PROOF: In the theorem 2.1, if we take $Y = X$ then we can construct a Cauchy sequence $\{gx_n\} = \{fx_{n-1}\}$ both converging to $p \in g(X)$ and then we get a point $z \in X$ such that $p = fz = gz$, *z* is a coincidence point of *f* and *g*. Since *f* and *g* are commuting at *z*, hence

$$fgz = gfz = fp = gp \text{ and } fgp = gfp.$$

Putting $x = z,\ y = fz$ in (ii), we have,

$$M(fz, ffz, t) \geq M(gz, gfz, \frac{t}{k}) = M(fz, ffz, \frac{t}{k})$$

i.e. $$M(fz, ffz, t) \geq M(fz, ffz, \frac{t}{k}) \geq M(gz, gfz, \frac{t}{k^2})$$
$$\geq \ldots$$

$$\geq M(fz, ffz, \frac{t}{k^n}).$$

Therefore, $M(fz, ffz, t) \geq M(fz, ffz, \frac{t}{k^n}).$

Similarly,

$$N(fz, ffz, t) \leq N(fz, ffz, \frac{t}{k^n}).$$

Making $n \to \infty$, and using (iii), (iv),(vii) and (xiii) of definition 7.1.3 we get,

$$fz = gz = ffz = gfz \text{ hence } p = fp = gp.$$

Therefore f and g have common fixed point. For uniqueness, let p and q be common fixed points of f and g.

Then by putting $x = p,\ y = q$ in (ii), we get

$$M(p,q,t) = M(fp,fq,t) \geq M(gp,gq,\frac{t}{k}) = M(fp,fq,\frac{t}{k})$$

$$\geq \ldots$$

$$\geq M(p,q,t) \geq M(p,q,\frac{t}{k^n}).$$

Hence $M(p,q,t) \geq M(p,q,\frac{t}{k^n})$.

Similarly,

$$N(p,q,t) \leq N(p,q,\frac{t}{k^n}).$$

Making $n \to \infty$, we have

$$M(p,q,t) = 1,\quad N(p,q,t) = 0,$$

Now from (iii) and (ix) of Definition1.3 we get $p = q$.

This proves the uniqueness.

REFERENCES

[1] **Alaca C., Turkoglu D. and Yildiz C.** *Fixed points in intuitionistic fuzzy metric spaces,* Chaos, Solitons and Fractals 29 (2006), 1073-1078.

[2] **Atanassov K.** *Intiuitionistic fuzzy sets,* fuzzy Sets and Systems, 20 (1986), 87-96.

[3] **Ćirić Lj. B., Ješić S. N., Ume J. S.** *The existence theorems for fixed and periodic points of nonexpansive mappings in intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals Vol 37 (3), 2008, 781-791.

[4] **Coker D.** *An introduction intuitionistic to fuzzy topological spaces*, Fuzzy Sets and Systems 88 (1997), 81-89. **2. Pant R. P.** *Fixed points of weakly commuting mappings*, Math. Student, 62 (1993), 97-102.

[5] **Fisher B. and Sessa S.** *Two common fixed point theorems for weakly commuting mappings,* Period. Math. Hungar. 20 (1989), 207-218. MR 91c:54055

[6] **Hadžić O. and Endre Pap.** *Fixed point theory in probabilistic metric spaces*, Mathematics and its Appl.vol 536, Kluwer Acad. Publ. Dordrecht (2001).

[7] **Jungck G.** *Commuting mappings and fixed points*, Amer. Math. Monthly 83 (1976), 261-263. MR 53#4031

[8] **Jungck G.** *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. 9 (1986), 771-779. MR 87m:54122

[9] **Klement E. P., Mesiar R. and Pap E.** *A characterization of the ordering of continuous t-norms,* fuzzy Sets and Systems 86 (1997), 189-195.

[10] **Lee B. S. and Cho S. J.** *A fixed point theorem for contractive type fuzzy mappings*, Fuzzy Sets and Systems 61 (1994), 309-312.

[11] **Lowen R.** *Fuzzy Set Theory*, Kluwer Acad. Publ., Dordrecht (1996).

[12] **Menger K.** *Statistical metrices*, Proc. Nat. Acad. Sci. USA 28 (1942), 535-537.

[13] **Pant R. P.** *Fixed points of weakly commuting mappings*, Math. Student, 62 (1993), 97-102.

[14] **Park J. H.** *Intuitionistic fuzzy metric spaces*, Chaos, Solitons and fractals 22 (2004), 1039-1046.

[15] **Tripathi P.K., Mishra S. N. and Gupta M**. *Application of Contraction Mapping in Menger Spaces,* Global Journal of Pure and Applied Mathematics, Vol 12, (2016) pp. 1629-1634.

[16] **Tripathi P.K. and Gupta M**. *Jungck Type Contractions and its Applications*, Int. Journal of Math Analysis, Vol 4, (2010) no. 37, pp. 1837-1850.

[17] **Schweizer B., Sherwood H. and Tardiff R. M.** *Contractions on probabilistic metric spaces: examples and counter examples*, Stochastica XII-I (1988), 5-17.

[18] **Turkoglu D., Alaca C., Cho Y. J. and Yildiz C.** *Common fixed point theorems in intuitionistic fuzzy metric spaces*, J. Appl. Math. & Commputing 22 (2006), 411-424.