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# Syntactic Semihypergroup

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**Abstract**

The objective of this paper is the study of Syntactic regular relation on a semihypergroup with identity and Syntactic semihypergroup with identity as a generalization of Syntactic congruence on a semigroup with identity. In the first part we study homomorphism between two semihypergroups with identities and some extended properties of it. Then we introduce concept of $L$-subset of a semihypergroup with identity. In the last section we characterize a finite syntactic semihypergroup $Syn(L) = \Sigma_\circ^\star/\sigma_L$ with identity corresponding to a $L$-subset of the semihypergroup $\Sigma_\circ^\star$, where $\Sigma^\star$ is the free monoid derived from $\Sigma$, the set of finite alphabets.

**AMS subject classification:**
**Keywords:** Regular relation, Syntactic regular relation, $L$-subset, Maximal regular relation Syntactic, semihypergroup.

## 1. Introduction

Hyperstructure theory was surfaced up in 1934 when F. Marty [9] defined hypergroups. Since then many researchers have studied in this field and published several papers on it.

In this paper we introduce the notion of syntactic regular relation and syntactic semihypergroup with identity on a semihypergroup $S$ with identity.

We recall some definitions and theorems from [2] which are needed to develop this paper.

**Definition 1.1.** Let $S(\neq \emptyset)$ be a set and $P^\star(S)$ be the family of all nonempty subsets of $S$ and $\circ$ a *hyperoperation* or a *join operation*, that is, $\circ$ is a mapping from $S \times S$ to $P^\star(S)$. If $(a, b) \in S \times S$, its image under $\circ$ is denoted by $a \circ b$ or $ab$.

**Remark 1.2.** The hyperoperation is extended to subsets of $S$ in a natural way, so that $A \circ B$ or $AB$ is given by

$$AB = \bigcup \{ab| \ \ a \in A, b \in B\}.$$

The notations $aA$ and $Aa$ are used for $\{a\} \circ A$ and $A \circ \{a\}$, respectively. Generally, the singleton $\{a\}$ is identified with its element $a$.

**Definition 1.3.** If $\circ$ is a hyperoperation on a set $S$ then $(S, \circ)$ is called a hypergroupoid.

**Definition 1.4.** A hypergroupoid $(S, \circ)$ which is associative, that is

$$x \circ (y \circ z) = (x \circ y) \circ z \ \ for \ \ all \ \ x, y, z \in S,$$

is called a semihypergroup.

**Definition 1.5.** A semihypergroup $(S, \circ)$ is said to be a semihypergroup with an identity if there exists an element $1 \in S$ such that

$$x \in x \circ 1 \cap 1 \circ x \ \ \text{for all} \ \ x \in S.$$

**Definition 1.6.** An equivalence relation $\rho$ on a semihypergroup $(S, \circ)$ is said to be a regular relation on $S$ if

$$a\rho b \ \text{ and } \ c\rho d \Longrightarrow \ \text{ for each } \ x \in a \circ c \ \text{ there exists } \ y \in b \circ d \ \text{ such that } \ x\rho y.$$

**Theorem 1.7.** If $\rho$ be a regular relation on a semihypergroup $(S, \circ)$ then the quotient set $S/\rho = \{a\rho : a \in S\}$ of all equivalence classes forms a semihypergroup with respect to the hyperoperation $\otimes$ on $S/\rho = \{a\rho : a \in S\}$ defined by $a\rho \otimes b\rho = \{c\rho : c \in a \circ b\}$.

If $(S, \circ)$ is a semihypergroup with identity then $(S/\rho, \otimes)$ is a semihypergroup with identity.

**Definition 1.8.** Let $(S, \circ)$ and $(T, \star)$ be two semihypergroups. A mapping $\phi : S \mapsto T$ is said to be a homomorphism if

$$\phi(x \circ y) = \phi(x) \star \phi(y) \ \ \text{for all} \ \ x, y \in S, \quad \text{where} \ \ \phi(x \circ y) = \{\phi(z) : z \in x \circ y\}.$$

If $(S, \circ)$ and $(T, \star)$ are two semihypergroups with identities then $\phi : S \mapsto T$ is said to be a homomorphism if

$$\phi(x \circ y) = \phi(x) \star \phi(y) \ \ \text{for all} \ \ x, y \in S$$

and

$$\phi(1_S) = 1_T.$$

where $1_S$ and $1_T$ are respective identities of $(S, \circ)$ and $(T, \star)$.

$\phi$ is called an isomorphism if $(i)$ $\phi$ is bijective and
$(ii)$ $\phi$ is an homomorphism.

Two semihypergroups $(S, \circ)$ and $(T, \star)$ are said to be isomorphic if there exist an isomorphism $\phi : S \mapsto T$ written as $(S, \circ) \cong (T, \star)$.

**Definition 1.9.** Let $\phi : (S, \circ) \mapsto (T, \star)$ be a semihypergroup (with identities) homomorphism the kernel of $\phi$ is denoted by ker$\phi$ and is defined by $ker\phi = \{(a, b) \in S \times S : \phi(a) = \phi(b)\}$.

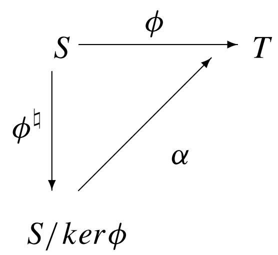
**Theorem 1.10.** If $\phi : (S, \circ) \mapsto (T, \star)$ be a semihypergroup (with identities) homomorphism then ker$\phi$ is a regular relation on $(S, \circ)$.

**Theorem 1.11.** If $\rho$ be a regular relation on a semihypergroup $(S, \circ)$ with identity then $(S/\rho, \otimes)$ is a semihypergroup with identity. Then the mapping $\rho^\natural : (S, \circ) \mapsto (S/\rho, \otimes)$ defined by

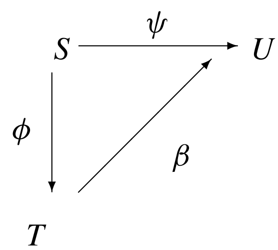
$$\rho^\natural(a) = a\rho \quad \text{for all} \quad a \in S$$

is the natural surjective homomorphism and $ker\rho^\natural = \rho$.

**Theorem 1.12.** Let $(S, \circ)$, $(T, \star)$ be two semihypergroups (with identities) and $\phi : S \mapsto T$ be a homomorphism. Then there exists an injective homomorphism $\alpha : S/ker\phi \mapsto T$ such that the following diagram commutes




**Theorem 1.13.** Let $S, T, U$ be semihypergroups (with identities) and let $\phi : S \mapsto T$, $\psi : S \mapsto U$ be homomorphisms such that $\phi$ is surjective and $ker\phi \subseteq ker\psi$, then there exists a homomorphism $\beta : T \mapsto U$ such that $im(\beta) = im(\psi)$ and the following diagram commutes




**Theorem 1.14.** Let $\rho$ and $\sigma$ be regular relations on a semihypergroup (with identity) $S$ such that $\rho \subseteq \sigma$. Then there is a regular relation $\sigma/\rho$ on the semihypergroup (with identity) $S/\rho$ such that $S/\sigma \cong (S/\rho)/(\sigma/\rho)$, where $\sigma/\rho = \{(a\rho, b\rho) \in S/\rho \times S/\rho : (a, b) \in \sigma\}$.

## 2. L-subset

In this section we define $L$-subset of a semihypergroup with identity and study some properties of it.

**Definition 2.1.** Let $(S, \circ)$ be a semihypergroup with identity 1. A subset $P$ of $S$ is said to be a left $L-subset$ of $(S, \circ)$ if $a \circ 1 \cap P \neq \emptyset \Longrightarrow a \in P$ for all $a \in S$.

**Example 2.2.** Let $S = \{1, 2, 3\}$ then
$S_3 = \{e, (12), (13), (23), (123), (132)\}$ is the symmetric group of permutations defined on $S$. Let $H = \{e, (12)\}$ then $H$ is a subgroup of $S_3$. Define a hyper operation $\circ$ on $S_3$ by

$$a \circ b = aHb \quad \text{for all} \quad a, b \in S_3.$$

It can be easily verify that $(S_3, \circ)$ forms a semihypergroup with identity $e$ and is denoted by $\overset{\circ}{S_3}$.

**Example 2.3.** Consider the example 2.2. Let $P = \{e, (13), (12), (132)\}$ then $P$ is a subset of $\overset{\circ}{S_3}$. We have $a \circ e = aH = \{a, a(12)\}$ for all $a \in \overset{\circ}{S_3}$.

Let $a \circ e \cap P \neq \emptyset$ that is $\{a, a(12)\} \cap \{e, (13), (12), (132)\} \neq \emptyset$. Then the following cases may arise:

(i) $a = e \in P \;\; or \;\; a = (13) \in P$ or $a = (12) \in P$ or $a = (132) \in P$

(ii) $a(12) = e \;\; or \;\; a(12) = (13)$ or $a(12) = (12)$ or $a(12) = (132)$
then $a = (12) \in P \;\; or \;\; a = (132) \in P$ or $a = e \in P$ or $a = (13)$.
It follows that $P = \{e, (13), (12), (132)\}$ is a left $L - subset$ of $\overset{\circ}{S_3}$.

**Definition 2.4.** Let $(S, \circ)$ be a semihypergroup with identity 1. A subset $P$ of $S$ is said to be a right $L - subset$ of $(S, \circ)$ if $1 \circ a \cap P \neq \emptyset \Longrightarrow a \in P$ for all $a \in S$.

**Example 2.5.** Consider the example 2.2. Let $P = \{e, (13), (12), (123)\}$ then $P$ is a subset of $\overset{\circ}{S_3}$. We have $e \circ a = aH = \{a, (12)a\}$ for all $a \in \overset{\circ}{S_3}$.
Let $e \circ a \cap P \neq \emptyset$ that is $\{a, (12)a\} \cap \{e, (13), (12), (132)\} \neq \emptyset$. Then the following cases may arise:

(i) $a = e \in P \;\; or \;\; a = (13) \in P \;\; or \;\; a = (12) \in P \;\; or \;\; a = (123) \in P$

(ii) $(12)a = e \;\; or \;\; (12)a = (13) \;\; or \;\; (12)a = (12) \;\; or \;\; (12)a = (123)$
then $a = (12) \in P \;\; or \;\; a = (123) \in P \;\; or \;\; a = e \in P \;\; or \;\; a = (13)$.
It follows that $P = \{e, (13), (12), (123)\}$ is a right $L - subset$ of $\overset{\circ}{S_3}$.

**Example 2.6.** Consider the example 2.3. Here $P = \{e, (13), (12), (132)\}$ is a left $L - subset$ of $\overset{\circ}{S_3}$. If $a = (23)$ then $e \circ a \cap P = \{a, (12)a\} \cap P = \{(23), (132)\} \cap \{e, (13), (12), (132)\} = \{(132)\} \neq \emptyset$ but $(23) \notin P$. Therefore $P$ is not a right $L-subset$ of $\overset{\circ}{S_3}$.

**Example 2.7.** Consider the example 2.5. Here $P = \{e, (13), (12), (123)\}$ is a right $L - subset$ of $\overset{\circ}{S_3}$. If $a = (23)$ then $a \circ e \cap P = \{a, a(12)\} \cap P = \{(23), (123)\} \cap \{e, (13), (12), (123)\} = \{(123)\} \neq \emptyset$ but $(23) \notin P$. Therefore $P$ is not a left $L - subset$ of $\overset{\circ}{S_3}$.

**Definition 2.8.** Let $(S, \circ)$ be a semihypergroup with identity 1. A subset $P$ of $S$ is said to be a $L - subset$ of $(S, \circ)$ if it is a left $L - subset$ of $(S, \circ)$ and a right $L - subset$ of $(S, \circ)$.

**Theorem 2.9.** Let $(S, \circ)$ be a semihypergroup with identity 1. A subset $P$ of $S$ is a $L$-subset of $(S, \circ)$ if and only if $1 \circ a \circ 1 \cap P \neq \emptyset \Longrightarrow a \in P$ for all $a \in S$.

*Proof.* Let $P$ be a $L$-subset of $(S, \circ)$. Therefore $P$ is a left as well as right $L$-subset of $(S, \circ)$.

Let $1 \circ a \circ 1 \cap P \neq \emptyset$ then $x \circ 1 \cap P \neq \emptyset$ for some $x \in 1 \circ a$. $P$ is a left $L$-subset of $(S, \circ)$. Then $x \circ 1 \cap P \neq \emptyset$ implies $x \in P$. Again $P$ is a right $L$-subset of $(S, \circ)$. Then $1 \circ a \cap P \neq \emptyset$ implies $a \in P$.

Therefore $1 \circ a \circ 1 \cap P \neq \emptyset \Longrightarrow a \in P$ for all $a \in S$. Conversely assume that $1 \circ a \circ 1 \cap P \neq \emptyset \Longrightarrow a \in P$ for all $a \in S$. Let $x \circ 1 \cap P \neq \emptyset$. Then there exists $y \in x \circ 1$ such that $y \in P$. We have $y \in 1 \circ y$. Then $1 \circ y \cap P \neq \emptyset$. This implies that $1 \circ x \circ 1 \cap P \neq \emptyset$. Therefore by assumption $x \in P$.

Therefore $x \circ 1 \cap P \neq \emptyset \Longrightarrow x \in P$ implies $P$ is a left $L$-subset of $(S, \circ)$. Similarly we can show that $P$ is a right $L$-subset of $(S, \circ)$. ■

**Example 2.10.** Let $S = \{1, 2, 3, 4\}$ then $S_4 = \{e, (12), (13), (14), (23), (24), (34), (123), (132), (124), (142), (134), (143), (234), (243), (1234), (1243), (1324), (1342), (1423), (1432), (12)(34), (13)(24), (14)(23)\}$ is the symmetric group of permutations defined on $S$. Let $H = \{e, (12)\}$ then $H$ is a subgroup of $S_4$. Define a hyper operation $\circ$ on $S_4$ by

$$a \circ b = aHb \quad \text{for all} \quad a, b \in S_4.$$

It can be easily verify that $(S_4, \circ)$ forms a semihypergroup with identity $e$ and is denoted by $\overset{\circ}{S_4}$.

**Example 2.11.** $\overset{\circ}{S_4}$ is a semihypergroup with identity $e$. For any $a \in \overset{\circ}{S_4}$ we have $e \circ a \circ e = HaH = \{a, a(12), (12)a, (12)a(12)\}$. Let $P = \{(13), (23), (123), (132)\}$ then $P$ is a subset of $\overset{\circ}{S_4}$.

Let $e \circ a \circ e \cap P \neq \emptyset$ that is $\{a, a(12), (12)a, (12)a(12)\} \cap \{(13), (23), (123), (132)\} \neq \emptyset$. Then the following cases may arise:

(i) $a = (13)$ or $(23)$ or $(123)$ or $(132)$

(ii) $a(12) = (13)$ or $a(12) = (23)$ or $a(12) = (123)$ or $a(12) = (132)$ then $a = (132)$ or $a = (123)$ or $a = (23)$ or $a = (13)$

(iii) $(12)a = (13)$ or $(12)a = (23)$ or $(12)a = (123)$ or $(12)a = (132)$ then $a = (123)$ or $a = (132)$ or $a = (13)$ or $a = (23)$

(iv) $(12)a(12) = (13)$ or $(12)a(12) = (23)$ or $(12)a(12) = (123)$ or $(12)a(12) = (132)$
then $a = (23)$ or $a = (13)$ or $a = (132)$ or $a = (123)$
Therefore we see that $a \circ e \circ a \cap P \neq \emptyset \Longrightarrow a \in P$. It follows that $P$ is a $L - subset$ of $S_4^\circ$.

**Theorem 2.12.** Let $P$, $Q$ be two $L - subsets$ of a semihypergroup $(S, \circ)$ with identity 1. Then $P \cup Q$ and $P \cap Q$ are also the $L - subsets$.

*Proof.* $1 \circ x \circ 1 \cap (P \cup Q) \neq \emptyset$
$\Longrightarrow (1 \circ x \circ 1 \cap P) \cup (1 \circ x \circ 1 \cap Q) \neq \emptyset$
$\Longrightarrow$ either $1 \circ x \circ 1 \cap P \neq \emptyset$ or $1 \circ x \circ 1 \cap Q \neq \emptyset$
$\Longrightarrow x \in P$ or $x \in Q$
$\Longrightarrow x \in P \cup Q \Longrightarrow P \cup Q$ is a $L$-subset of $(S, \circ)$.
$1 \circ x \circ 1 \cap (P \cap Q) \neq \emptyset$
$\Longrightarrow 1 \circ x \circ 1 \cap P \neq \emptyset$ and $1 \circ x \circ 1 \cap Q \neq \emptyset$
$\Longrightarrow x \in P$ and $x \in Q$
$\Longrightarrow x \in P \cap Q \Longrightarrow P \cap Q$ is a $L$-subset of $(S, \circ)$. ■

**Theorem 2.13.** Let $G$ be a group and $H$ be a subgroup of $G$. Define a hyperoperation $\circ$ on $G$ by

$$a \circ b = aHb \quad \text{for all} \quad a, b \in G$$

Then $(G, \circ)$ is a semihypergroup with identity 1. Then any subset $P$ of $G$ is a $L - subset$ of $(G, \circ)$ if and only if $1 \circ P \circ 1 = P$.

*Proof.* Let $P$ be a $L - subset$ of $(G, \circ)$. We have $P \subseteq 1 \circ P \circ 1$.

Let $x \in 1 \circ P \circ 1$. Then $x = h_1 p h_2$ for some $h_1, h_2 \in H$ and $p \in P$. Then $p = h_1^{-1} x h_2^{-1} \in HxH = 1 \circ x \circ 1$. Therefore $1 \circ x \circ 1 \cap P \neq \emptyset$. $P$ being $L - subset$ we have $x \in P$. Therefore $1 \circ P \circ 1 \subseteq P$. It follows that $1 \circ P \circ 1 = P$.

Conversely assume that $1 \circ P \circ 1 = P$. Let $1 \circ x \circ 1 \cap P \neq \emptyset$ then $h_1 x h_2 \in P$ for some $h_1, h_2 \in H$
$\Longrightarrow x \in h_1^{-1} P h_2^{-1} \subseteq HPH = 1 \circ P \circ 1 = P$.
Therefore $1 \circ x \circ 1 \cap P \neq \emptyset \Longrightarrow x \in P$ and so $P$ is a $L - subset$ of $(G, \circ)$. ■

**Theorem 2.14.** In the Theorem 2.13, the semihypergroup $(G, \circ)$ with identity 1 product of two $L - subsets$ of $(G, \circ)$ is a $L - subset$ of $(G, \circ)$.

*Proof.* Let $P$ and $Q$ be two $L - subsets$ of $(G, \circ)$. Then $1 \circ P \circ 1 = P$ and $1 \circ Q \circ 1 = P$. $P \circ Q \subseteq 1 \circ P \circ Q \circ 1 \subseteq 1 \circ P \circ 1 \circ 1 \circ Q \circ 1 = P \circ Q$. This implies $1 \circ P \circ Q \circ 1 = P \circ Q$. Therefore $P \circ Q$ is a $L - subset$ of $(G, \circ)$. ■

**Theorem 2.15.** Let $P$ be a subset of a semihypergroup $(S, \circ)$. Define a relation $\sigma_P$ on $S$ by

$x \sigma_P y$ if and only if $u \circ x \circ v \cap P \neq \emptyset \Longleftrightarrow u \circ y \circ v \cap P \neq \emptyset$ for all $u, v \in S$ .

Then $\sigma_P$ is a regular relation on $(S, \circ)$. $\sigma_P$ is called Syntactic regular relation on $(S, \circ)$.

*Proof.* It is clear that $\sigma_P$ is an equivalence relation on $S$.

Let $a\sigma_P b$ and $c\sigma_P d$ and $x \in a \circ c$. Now $u \circ x \circ v \cap P \neq \emptyset$
$\Longrightarrow u \circ a \circ c \circ v \cap P \neq \emptyset$
$\Longrightarrow u \circ a \circ w \cap P \neq \emptyset$ for some $w \in c \circ v$
$\Longrightarrow u \circ b \circ w \cap P \neq \emptyset$ for some $w \in c \circ v$ (by the definition of $\sigma_P$)
$\Longrightarrow u \circ b \circ c \circ v \cap P \neq \emptyset$
$\Longrightarrow z \circ c \circ v \cap P \neq \emptyset$ for some $z \in u \circ b$
$\Longrightarrow z \circ d \circ v \cap P \neq \emptyset$ for some $z \in u \circ b$ (by the definition of $\sigma_P$)
$\Longrightarrow u \circ b \circ d \circ v \cap P \neq \emptyset$
$\Longrightarrow u \circ y \circ v \cap P \neq \emptyset$ for some $y \in b \circ d$.

Therefore for each $x \in a \circ c$ there exists $y \in b \circ d$ such that $x\sigma_P y$. In other wards $\sigma_P$ is a regular relation on $(S, \circ)$. ■

**Definition 2.16.** Let $(S, \circ)$ be a semihypergroup with identity and $P$ be a subset of $S$. By the Theorem 2.15, $\sigma_P$ is a regular relation on $(S, \circ)$. By the Theorem 1.7, $S/\sigma_P = \{a\sigma_P : a \in S\}$ of all equivalence classes forms a semihypergroup with respect to the hyperoperation $\otimes$ on $S/\sigma_P$ defined by $a\sigma_P \otimes b\sigma_P = \{c\sigma_P : c \in a \circ b\}$ is a semihypergroup with identity. $(S/\sigma_P, \otimes)$ is called Syntactic semihypergroup with identity and is denoted by $Syn(P)$.

**Definition 2.17.** A relation $\rho$ on a set $S$ is said to saturates a subset $A$ of $S$ if the following property holds: $x\rho y$ and $x \in A \Longrightarrow y \in A$ for all $x, y \in S$.

**Theorem 2.18.** If $P$ be a $L - subset$ of a semihypergroup $(S, \circ)$ with identity 1 then the syntactic regular relation $\sigma_P$ on $(S, \circ)$ saturates $P$.

*Proof.* Let $x\sigma_P y$ and $x \in P$.
Now $x\sigma_P y$
$\Longrightarrow u \circ x \circ v \cap P \neq \emptyset \Longleftrightarrow u \circ y \circ v \cap P \neq \emptyset$ for all $u, v \in S$
$\Longrightarrow 1 \circ x \circ 1 \cap P \neq \emptyset \Longleftrightarrow 1 \circ y \circ 1 \cap P \neq \emptyset$
Now $x \in 1 \circ x \circ 1$ and $x \in P \Longrightarrow 1 \circ x \circ 1 \cap P \neq \emptyset$
$\Longrightarrow 1 \circ y \circ 1 \cap P \neq \emptyset$. By the Theorem 2.9, $y \in P \Longrightarrow \sigma_P$ saturates $P$. ■

**Definition 2.19.** A regular relation $\sigma$ on a semihypergroup $(S, \circ)$ with identity is said to be maximal regular relation if $\rho \subseteq \sigma$ for all regular relation $\rho$ on $(S, \circ)$.

**Theorem 2.20.** Let $\rho$ be a regular relation on a semihypergroup $(S, \circ)$. Let $x\rho y$ and $u, v \in S$. Then $a \in u \circ x \circ v \Longrightarrow$ there exists $b \in u \circ y \circ v$ such that $a\rho b$.

*Proof.* $a \in u \circ x \circ v \Longrightarrow a \in u \circ s$ for some $s \in x \circ v$. $\rho$ being a regular relation on the semihypergroup $(S, \circ)$ and $x\rho y$, $s \in x \circ v$ there exists $t \in y \circ v$ such that $s\rho t$. Again $\rho$ being a regular relation and $s\rho t$, $a \in u \circ s$ there exists $b \in u \circ t \subseteq u \circ y \circ v$ such that $a\rho b$. It follows that $a \in u \circ x \circ v \Longrightarrow$ there exists $b \in u \circ y \circ v$ such that $a\rho b$. ■

**Theorem 2.21.** If $P$ be a $L - subset$ of a semihypergroup $(S, \circ)$ with identity 1 then the relation $\sigma_P$ is the maximal regular relation on $(S, \circ)$ saturating $P$.

*Proof.* By the Theorem 2.18, $\sigma_P$ is a regular relation on $(S, \circ)$ saturating $P$. Let $\rho$ be a regular relation on $(S, \circ)$ such that $\rho$ saturates $P$.
Let $x\rho y$ and $u, v \in S$. $Now$ $u \circ x \circ v \cap P \neq \emptyset \Longrightarrow$ there exists $a \in u \circ x \circ v$ and $a \in P$. By the Theorem 2.20, $x\rho y$ and $a \in u \circ x \circ v$ implies that there exists $b \in u \circ y \circ v$ such that $a\rho b$. Since $\rho$ saturates $P$ and $a\rho b$, $a \in P$ then $b \in P$. Therefore $b \in u \circ y \circ v \cap P$ i.e. $u \circ y \circ v \cap P \neq \emptyset$.

Therefore $u \circ x \circ v \cap P \neq \emptyset \Longrightarrow u \circ y \circ v \cap P \neq \emptyset$. Similarly we can show that $u \circ y \circ v \cap P \neq \emptyset \Longrightarrow u \circ x \circ v \cap P \neq \emptyset$. Therefore $x\sigma_P y$ and so $\rho \subseteq \sigma_P$. This proves the theorem. ■

**Definition 2.22.** Let $\Sigma$ be a finite alphabet and $\Sigma^\star$ be the set of all words of finite length determined by the set $\Sigma$. Define a hyperoperation $\circ$ on $\Sigma^\star$ by

$$x \circ y = \{xy\} \cup \{xay : a \in \Sigma\}.$$

It can be verify that $(\Sigma^\star, \circ)$ is a semihypergroup with identity $\lambda$ (empty word) and is denoted by $\Sigma_\circ^\star$.

**Example 2.23.** Let $\Sigma = \{a, b\}$. $\Sigma_\circ^\star$ is a semihypergroup with identity $\lambda$. For any $x \in \Sigma_\circ^\star$ we have $x \circ \lambda = \{x\lambda\} \cup \{xy\lambda : y \in \Sigma\} = \{x\} \cup \{xy : y \in \Sigma\} = \{x, xa, xb\}$ and $\lambda \circ x = \{\lambda x\} \cup \{\lambda yx : y \in \Sigma\} = \{x, ax, bx\}$.

Let $P = \{\lambda, a, aa, ab\}$ then $P$ is a subset of $\Sigma_\circ^\star$. Let $x \circ \lambda \cap P \neq \emptyset$ that is $\{x, xa, xb\} \cap \{\lambda, a, aa, ab\} \neq \emptyset$. Then the following cases may arise:

(i) $x = \lambda$ *or* $a$ *or* $aa$ *or* $ab$

(ii) $xa = a$ *or* $xa = aa$ *or* $xa = ab$
then $x = \lambda$ *or* $x = a$ *or* $x = b$

(iii) $xb = ab$ then $x = a$.
Therefore we see that $x \circ \lambda \cap P \neq \emptyset \Longrightarrow x \in P$.

It follows that $P$ is a left $L - subset$ of $\Sigma_\circ^\star$.
$\lambda \circ b \cap P = \{b, ab, bb\} \cap \{\lambda, a, aa, ab\} = \{ab\} \neq \emptyset$ but $b \notin P$.

It follows that $P$ is not a right $L - subset$ of $\Sigma_\circ^\star$.

Let $P = \{\lambda, b, ab, bb\}$ then $P$ is a subset of $\Sigma_\circ^\star$. Let $\lambda \circ x \cap P \neq \emptyset$ that is $\{x, ax, bx\} \cap \{\lambda, b, ab, bb\} \neq \emptyset$. Then the following cases may arise:

(i) $x = \lambda$ *or* $b$ *or* $ab$ *or* $bb$
$(ii)$ $ax = ab$ then $x = b$.

(iii) $bx = b \;\; or \;\; bx = bb$
then $x = \lambda \;\; or \;\; x = b$
Therefore we see that $\lambda \circ x \cap P \neq \emptyset \Longrightarrow x \in P$.

It follows that $P$ is a right $L - subset$ of $\Sigma_{\circ}^{\star}$.
$a \circ \lambda \cap P = \{a, aa, ab\} \cap \{\lambda, b, ab, bb\} = \{ab\} \neq \emptyset$ but $a \notin P$. It follows that $P$ is not a left $L - subset$ of $\Sigma_{\circ}^{\star}$.

Let $P = \{\lambda, a, b, ab, ba, aa, bb\}$ then $P$ is a subset of $\Sigma_{\circ}^{\star}$. We have $\lambda \circ x \circ \lambda = \{x, ax, bx\} \circ \lambda = x \circ \lambda \cup ax \circ \lambda \cup bx \circ \lambda = \{x, xa, xb\} \cup \{ax, axa, axb\} \cup \{bx, bxa, bxb\} = \{x, xa, xb, ax, axa, axb, bx, bxa, bxb\}$.

Let $\lambda \circ x \circ \lambda \cap P \neq \emptyset$ that is $\{x, xa, xb, ax, axa, axb, bx, bxa, bxb\} \cap \{\lambda, a, b, ab, ba, aa, bb\} \neq \emptyset$. Then the following cases may arise:

(i) $x = \lambda \;\; or \;\; a \;\; or \;\; b \;\; or \;\; ab \;\; or \;\; ba \;\; or \;\; aa \;\; or \;\; bb$

(ii) $xa = a \;\; or \;\; ba \;\; or \;\; aa$ then $x = \lambda \;\; or \;\; b \;\; or \;\; a$

(iii) $xb = b \;\; or \;\; ab \;\; or \;\; bb$ then $x = \lambda \;\; or \;\; a \;\; or \;\; b$

(iv) $ax = a \;\; or \;\; ab \;\; or \;\; aa$ then $x = \lambda \;\; or \;\; b \;\; or \;\; a$

(v) $bx = b \;\; or \;\; ba \;\; or \;\; bb$ then $x = \lambda \;\; or \;\; a \;\; or \;\; b$

(vi) $axa = aa, axb = ab, bxa = ba, bxb = bb$ then $x = \lambda$

In all cases $\lambda \circ x \circ \lambda \cap P \neq \emptyset \Longrightarrow x \in P$. Therefore $P$ is a $L - subset$ of $\Sigma_{\circ}^{\star}$.

**Definition 2.24. [2]** A five-tuple $\mathcal{A} = (\Sigma, Q, F, q_0, T)$ is said to be a Non-Deterministic Finite Automata (NDFA). Where

(i) $\Sigma$ is a finite set, called the set of inputs.

(ii) $Q$ is a finite set, called the set of states.

(iii) $F : \Sigma \times Q \mapsto P^{\star}(Q)$ is a mapping extended to $F^{\star} : \Sigma^{\star} \times Q \mapsto P^{\star}(Q)$ by

$$F^{\star}(\lambda, q) = \{q\} \;\; for \;\; all \;\; q \in Q.$$
$$F^{\star}(x_1 x_2 \cdots x_n, q) = \cup\{F^{\star}(x_1 x_2 \cdots x_{(n-1)}, r) : r \in F(x_n, q)\}$$
$$\text{for all } \; x_1, x_2, \ldots, x_n \in \Sigma \; \text{ and } \; q \in Q.$$

(iv) $q_0 \in Q$ is called the initial state.

(v) $T \subseteq Q$ is called the set of final states.

**Definition 2.25. [2]** Let $\mathcal{A} = (\Sigma, Q, F, q_0, T)$ be a $NDFA$. The behaviour of $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$ and is defined by

$$\mathcal{L}(\mathcal{A}) = \{x \in \Sigma^\star : F(x, q_0) \cap T \neq \emptyset\}.$$

**Definition 2.26. [2]** Let $\Sigma$ be a finite alphabet. A subset $L$ of $\Sigma^\star$ is said to be recognized by an $NDFA$ $\mathcal{A} = (\Sigma, Q, F, q_0, T)$ if

$$\mathcal{L}(\mathcal{A}) = L.$$

**Definition 2.27.** Let $\Sigma$ be a finite alphabet. By the Definition 2.22, $\Sigma_\circ^\star$ is a semihypergroup with identity. A subset $L$ of $\Sigma_\circ^\star$ is said to be recognized by a semihypergroup $(S, \circ)$ with identity 1 if there exists a homomorphism $\phi : \Sigma_\circ^\star \mapsto S$ and a subset $P$ of $S$ such that $\phi^{-1}(P) = L$.

**Theorem 2.28.** Let $\Sigma$ be a finite alphabet and $R$ be a $L - subset$ of $\Sigma_\circ^\star$. Then the following statements are equivalent:

(i) $R$ is recognized by an $NDFA$.

(ii) $Syn(R)$ is finite.

(iii) $R$ is recognized by a finite semihypergroup with identity.

*Proof.*

(i) Let $R$ be recognized by an $NDFA$. Then there exists an $NDFA$, $\mathcal{A} = (\Sigma, Q, F, q_0, T)$ such that $\mathcal{L}(\mathcal{A}) = R$. i.e. $R = \{x \in \Sigma_\circ^\star : F(x, q_0) \cap T \neq \emptyset\}$.
We define a relation $\tau$ on $\Sigma_\circ^\star$ by

$$x\tau y \Longleftrightarrow F(x, q) = F(y, q) \;\; for \;\; all \;\; q \in Q.$$

clearly $\tau$ is an equivalence relation on $\Sigma_\circ^\star$.
Let $a\tau b$ and $c\tau d$. Let $x \in a \circ c$ then $x = a\sigma c$, where $\sigma = \lambda$ or $\sigma \in \Sigma$. Let $y = b\sigma d$ then $y \in b \circ d$.
$F(x, q) = F(a\sigma c, q) = \cup\{F(a, r) : r \in F(\sigma c, q)\} = \cup\{F(b, r) : r \in F(\sigma c, q)\} = F(b\sigma c, q) = \cup\{F(b\sigma, r) : r \in F(c, q)\} = \cup\{F(b\sigma, r) : r \in F(d, q)\} = F(b\sigma d, q) = F(y, q)$. This implies that for each $x \in a \circ c$ there exists $y \in b \circ d$ such that $x\tau y$. By the definition 1.6, $\tau$ is a regular relation on the semihypergroup $\Sigma_\circ^\star$ with identity. By the Theorem 1.7, $\Sigma_\circ^\star/\tau$ is a semihypergroup with identity.
Let $Q = \{q_1, q_2, \ldots, q_n\}$. Then a $\tau$ class $x\tau (x \in \Sigma_\circ^\star)$ is determined by the subsets of $Q$ taken by $F(x, q_1), F(x, q_2), \ldots, F(x, q_n)$. Since the set of subsets of $Q$ is finite. It follows that $\Sigma_\circ^\star/\tau$ is finite.

Let $x\tau y$ and $u, v \in \Sigma_\circ^\star$. Let $u \circ x \circ v \cap R \neq \emptyset$. Then there exists $a \in u \circ x \circ v$ and $a \in R$. Now $a \in R$ implies $F(a, q_0) \cap T \neq \emptyset$. Since $x\tau y$ and $\tau$ is regular then

$a \in u \circ x \circ v$ implies there exists $b \in u \circ y \circ v$ such that $a\tau b$. i.e. $F(a, q) = F(b, q)$ for all $q \in Q$. In particular $F(a, q_0) = F(b, q_0)$. Therefore $F(b, q_0) \cap T \neq \emptyset$ and so $b \in R$. Therefore $u \circ y \circ v \cap R \neq \emptyset$.

Therefore $u \circ x \circ v \cap R \neq \emptyset \Longrightarrow u \circ y \circ v \cap R \neq \emptyset$. Similarly we can show that $u \circ y \circ v \cap R \neq \emptyset \Longrightarrow u \circ x \circ v \cap R \neq \emptyset$. By the Theorem 2.15, $x\sigma_R y$. This implies that $\tau \subseteq \sigma_R$. By the Theorem 1.14, $\sigma_R/\tau$ is a regular relation on the semihypergroup $\Sigma_\circ^\star/\tau$ with identity such that $\Sigma_\circ^\star/\sigma_R \cong (\Sigma_\circ^\star/\tau)/(\sigma_R/\tau)$, where $\sigma_R/\tau = \{(a\tau, b\tau) \in S/\tau \times S/\tau : (a, b) \in \sigma_R\}$. $\Sigma_\circ^\star/\tau$ being finite, $(\Sigma_\circ^\star/\tau)/(\sigma_R/\tau)$ is finite. It follows that $\Sigma_\circ^\star/\sigma_R = Syn(R)$ is finite.
Therefore $(i) \Longrightarrow (ii)$.

(ii) Let $Syn(R)$ be finite. Then $Syn(R) = \Sigma_\circ^\star/\sigma_R$ is a finite Semihypergroup with identity. Let $\sigma_R^\natural : \Sigma_\circ^\star \mapsto \Sigma_\circ^\star/\sigma_R$ be the natural surjective homomorphism. Let $P = \sigma_R^\natural(R)$. Then $P \subseteq \Sigma_\circ^\star/\sigma_R$.
Let $x \in (\sigma_R^\natural)^{-1}(P)$ $then$ $\sigma_R^\natural(x) \in P$
$\Longrightarrow x\sigma_R \in P = \sigma_R^\natural(R)$
$\Longrightarrow$ $there$ $exists$ $y \in R$ $such$ $that$ $x\sigma_R = y\sigma_R$
$\Longrightarrow x\sigma_R y$.
$R$ being a $L$-subset of $\Sigma_\circ^\star$ and $x\sigma_R y$, by the Theorem 2.18, $x \in R$. This implies that $(\sigma_R^\natural)^{-1}(P) \subseteq R$.
Let $x \in R$. Then $\sigma_R^\natural(x) \in \sigma_R^\natural(R) = P \Longrightarrow x \in (\sigma_R^\natural)^{-1}(P) \Longrightarrow R \subseteq (\sigma_R^\natural)^{-1}(P)$.
It follows that $R = (\sigma_R^\natural)^{-1}(P)$. Therefore $R$ is recognized by a finite Semihypergroup $\Sigma_\circ^\star/\sigma_R$ with identity.
Therefore $(ii) \Longrightarrow (iii)$.

(iii) Let $R$ be recognized by a finite Semihypergroup $(S, \circ)$ with identity 1. Then there exists a homomorphism $\phi : \Sigma_\circ^\star \mapsto S$ and a subset $P$ of $S$ such that $\phi^{-1}(P) = R$. Define a mapping $F : \Sigma \times S \mapsto P^\star(S)$ by

$$F(a, q) = \phi(a) \circ q \quad \text{for all} \quad a \in \Sigma \quad and \quad q \in S.$$

We extend the mapping $F : \Sigma \times S \mapsto P^\star(S)$ to the mapping $F : \Sigma^\star \times S \mapsto P^\star(S)$ by

$$F(\lambda, q) = \{q\} \quad \text{for all} \quad q \in S.$$

and

$$F(x_1x_2......x_n, q) = \cup\{F(x_1x_2......x_{n-1}, r) : r \in F(x_n, q)\}$$
$$\text{for all} \quad x_1, x_2, ......, x_n \in \Sigma \quad \text{and} \quad q \in S.$$

Consider the five-tuple $\mathcal{A} = (\Sigma, S, F, 1, P)$. It is clear that $\mathcal{A} = (\Sigma, S, F, 1, P)$ is an $NDFA$ and $\mathcal{L}(\mathcal{A}) = \{x \in \Sigma^\star : F(x, 1) \cap P \neq \emptyset\}$.

Let $x \in \mathcal{L}(\mathcal{A})$ then $F(x, 1) \cap P \neq \emptyset$
$\Longrightarrow \phi(x) \circ 1 \cap P \neq \emptyset$.
$(S, \circ)$ being a semihypergroup with identity 1 and $\phi(x) \in S$, $\phi(x) \in 1 \circ \phi(x)$. It follows that
$1 \circ \phi(x) \circ 1 \cap P \neq \emptyset$
$\Longrightarrow \phi(\lambda) \circ \phi(x) \circ \phi(\lambda) \cap P \neq \emptyset$
$\Longrightarrow \phi(\lambda \circ x \circ \lambda) \cap P \neq \emptyset$
$\Longrightarrow$ *there exists an element* $u \in \lambda \circ x \circ \lambda$ *such that* $\phi(u) \in P$
$\Longrightarrow$ *there exists an element* $u \in \lambda \circ x \circ \lambda$ *such that* $u \in \phi^{-1}(P) = R$
$\Longrightarrow \lambda \circ x \circ \lambda \cap R \neq \emptyset$.
$R$ being a $L$-subset of $\Sigma_\circ^\star$ and $\lambda \circ x \circ \lambda \cap R \neq \emptyset$, by the Theorem 2.9, $x \in R$.
Therefore $\mathcal{L}(\mathcal{A}) \subseteq R$.
$x \in R \Longrightarrow x \circ \lambda \cap R \neq \emptyset \Longrightarrow \phi(x \circ \lambda) \cap \phi(R) \neq \emptyset \Longrightarrow \phi(x) \circ 1 \cap P \neq \emptyset \Longrightarrow F(x, 1) \cap P \neq \emptyset \Longrightarrow x \in \mathcal{L}(\mathcal{A}) \Longrightarrow R \subseteq \mathcal{L}(\mathcal{A})$.
Therefore $\mathcal{L}(\mathcal{A}) = R$. Therefore $R$ is recognized by an $NDFA$.
Therefore $(iii) \Longrightarrow (i)$. ■

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