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Exact solutions of Schwarz KdV and the (2+1)-dimensional Schwarz-KdV equations via the improved tanh-coth method

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Abstract

In this paper, we obtain exact traveling wave solutions to the Schwarz Korteweg-de Vries equation (SKdV) and the (2+1)-Dimensional Schwarz-Korteweg-de Vries equation ((2+1)-SKdV) using the improved tanh-coth method. The advantage of the method used is that it give us solutions in a more general form that other known methods. Moreover, we apply the method directly to a resultant system, so that, the technique used here is different to other approach which, in the case of the ((2+1)-SKdV), reduces the system to one only equation.

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1. Introduction

The Korteweg-de Vries equation (KdV)

$$u_t + auu_x + bu_{xxx} = 0, \tag{1.1}$$

is one of the most important integrable models of the nonlinear science, and it have been studied in a extensive form by many researches. Solutions for it have been derived using several analytical and computational methods [1],[2]. The integrability of the KdV, have been used to derive other integrable equations, using some special techniques, see for instance [1] and references therein. One of the most important integrable models related to the KdV equation, is the well known Schwarzian Korteweg-de Vries equation (SKdV) [3],

$$\frac{\Phi_t}{\Phi_x} + \{\tilde{\Phi}; x\} = 0, \tag{1.2}$$

or given in the form

$$\frac{\Phi_t}{\Phi_x} - \{\tilde{\Phi}; x\} = 0, \tag{1.3}$$

being $\{\tilde{\Phi}; x\} = \left(\frac{\Phi_{xx}}{\Phi_x}\right)_x - \frac{1}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2$, the Schwarzian derivative (see [3] and references therein). Now, making $\Phi(x, t) = u(x, t)$, we can write (1.3) as

$$2u_t u_x - 2u_{xxx}u_x + 3u_{xx}^2 = 0. \tag{1.4}$$

On the other hand, the authors of [4] have been derived the following equation

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx}W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\int \left(\frac{W_x^2}{W^2}\right)_z dx = 0, \tag{1.5}$$

called by them as (2+1)-dimensional integrable generalization of the SKdV equation. This last equation, after integration with respect to x and after the change of variable $W = \Phi_x$, can be written as

$$\frac{\Phi_t}{\Phi_x} + \{\tilde{\Phi}; x\}_{2+1} = 0, \tag{1.6}$$

where $\{\tilde{\Phi}; x\}_{2+1} = \left(\frac{\Phi_{xx}}{\Phi_x}\right)_z - \frac{1}{2}\int \left(\frac{\Phi_{xx}}{\Phi_x}\right)^2_z dx$, is known as the (2+1)-dimensional Schwarzian derivative [5]. Exact solutions for this model, have been derived in [6], using the exponential-function method and the G'/G-expansión method, and by the authors in [7] using Lie Symmetries analysis. One of the facts more relevant respect to this model, is that, as in the SKdV equation, it is invariant by the Möbius Transformation and it reduces to SKdV equation for solutions of the form $W(x, z, t) = W(\overline{x + z}, t)$, [3].

Now, if we take $W = \phi_x$, $\phi = e^{\psi}$, $\psi_x = u$, $\psi_t = v$, then (1.5) reduces to following system

$$\begin{cases} 4u^2 v_x - 4uu_x v + u^2 u_{xxx} - uu_{xx}u_z - 3uu_x u_{zz} + 3u_x^2 u_z - 4u^4 u_z = 0, \\ u_t - v_x = 0. \end{cases} \tag{1.7}$$

which have been studied using Lie Symmetries by the authors in [7].

It is not easy to obtain solutions for arbitrary nonlinear partial differential equation (NLPDE's) or differential system, so that, in the last decades, several computational methods to obtain exact solutions to (NLPDE's) have been implemented. For instance, the tanh method [8], the tanh-coth method [9], and many others as those mentioned early. Many of them use the the standard Riccati equation and its solutions

$$\phi'(\xi) = \phi^2(\xi) + k, \tag{1.8}$$

with k an arbitrary constant. In this paper, we will use the improved tanh-coth method [10], to solve Eq.(1.4) and system Eq.(1.7). This last mentioned method, use the solutions of the most general Riccati equation

$$\phi'(\xi) = \gamma\phi^2(\xi) + \beta\phi(\xi) + \alpha, \tag{1.9}$$

where α, β and $\gamma \neq 0$ are arbitrary parameters.

This paper is organized as follows: In Sec. 2, we give the idea of the improved tanh-coth method; In Sec. 3, we obtain exact solutions to SKdV equation (Eq. (2.2)) using the improved tanh-coth method; In Sec. 4, we obtain exact solutions to system given by Eq. (3.1); Finally, some conclusions are given.

2. The method

Consider a system of two coupled PDE's in the variables x and t

$$\begin{cases} P(u, w, u_x, w_x, u_t, w_t, u_{xt}, w_{x,t}, u_{xx}, w_{xx}, \ldots) = 0, \\ Q(u, w, u_x, w_x, u_t, w_t, u_{xt}, w_{x,t}, u_{xx}, w_{xx}, \ldots) = 0. \end{cases} \tag{2.1}$$

Using the traveling wave transformation

$$\xi = \mu(x + \lambda t + \xi_0),$$

the system (2.1) reduces to a system of ordinary differential equations in the unknowns $u(\xi)$, $w(\xi)$ of the following form

$$\begin{cases} P_1(u, w, u', w', u'', w'', \ldots) = 0, \\ Q_1(u, w, u', w', u'', w'', \ldots) = 0. \end{cases} \tag{2.2}$$

Then, we seek solutions to (2.2) by using the expansions

$$\begin{cases} u(\xi) = \sum_{i=0}^{M} a_i\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i\phi(\xi)^{M-i}, \\ \\ w(\xi) = \sum_{i=0}^{N} b_i\phi(\xi)^i + \sum_{i=N+1}^{2N} b_i\phi(\xi)^{N-i}, \end{cases} \tag{2.3}$$

where M, N are a positive integer that will be determined later and $\phi = \phi(\xi)$ satisfies the Riccati equation (1.9). In this case, the method is called *improved generalized tanh-coth method to coupled systems* (IGTCMCS). Substituting (2.3) into (2.2) and using (1.9) results in an algebraic system of two equations in powers of $\phi(\xi)$. Balancing the linear terms of highest order in the resulting equations with the highest order nonlinear term to obtain M, N, will yields a set of algebraic equations for μ, α, β, γ, λ, $a_0, \ldots, a_{2M}$, $b_0, \ldots, b_{2N}$ because all coefficients of $\phi(\xi)^i$ $(i = 1, 2, \ldots)$ have to vanish. Solving the algebraic system, and reversing, we obtain exact solutions to (2.1) in the original variables.

3. Exact solutions to SKdV equation (1.4)

We seek solutions to (1.4) in the form $u = v(\xi)$, $\xi = x + \lambda t + \xi_0$. In this case, (1.4) converts to ordinary differential equation

$$2\lambda[v'(\xi)]^2 - 2v'''(\xi)v'(\xi) + 3[v''(\xi)]^2 = 0. \tag{3.1}$$

According with the improved tanh-coth method, we consider that (3.1) have solution in the form

$$v(\xi) = \sum_{i=0}^{M} a_i\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i\phi(\xi)^{M-i}, \tag{3.2}$$

where $\phi(\xi)$ satisfies the generalized Ricati equation (1.9). Balancing $v'''v'$ with $[v'']^2$ we obtain that M can be any positive number. However, due the calculations are very big, for sake of simplicity, we take $M = 2$, so that Eq.(3.2) reduces to

$$v(\xi) = a_0 + a_1\phi(\xi) + a_2\phi(\xi)^2 + a_3\phi(\xi)^{-1} + a_4\phi(\xi)^{-2}, \tag{3.3}$$

where

$$\phi(\xi) = -\frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh[\frac{1}{2}\sqrt{\beta^2 - 4\alpha\gamma}\,\xi] + \beta}{2\gamma}, \tag{3.4}$$

is the general solutions of Eq.(1.9). Now, substituting (3.3) into (3.1) and using (1.9), we obtain an algebraic system in the unknowns $a_0, a_1, a_2, a_3, a_4, \alpha, \beta, \gamma, \lambda$. For simplicity, we consider only the following solution (other obtained solutions are particular cases of this).

$$a_0 = a_0, \quad a_1 = a_2 = 0, \quad a_3 = a_3, \quad a_4 = \frac{a_3\,\beta}{2\gamma}, \quad \alpha = 0, \quad \lambda = -2\beta^2. \tag{3.5}$$

Respect to this set of solutions, and using (3.4),(3.3) we have

$$u(x,t) = v(\xi) = a_0 - a_3\left[\frac{2\gamma}{\sqrt{\beta^2}\tanh\left[\frac{1}{2}\sqrt{\beta^2}\xi\right] + \beta}\right] + \frac{a_3\beta}{2\gamma}\left[\frac{2\gamma}{\sqrt{\beta^2}\tanh\left[\frac{1}{2}\sqrt{\beta^2}\xi\right] + \beta}\right]^2, \tag{3.6}$$

with $\xi = x - 2\beta^2 t + \xi_0$, a_0, a_3, β, γ and ξ_0 arbitrary constants.

4. Exact traveling solutions to Eq. (1.7)

As in the previous section, we consider solutions to (1.7) in the form

$$\begin{cases} u(x,z,t) = u(\xi), \\ v(x,z,t) = v(\xi), \end{cases} \tag{4.1}$$

where $\xi = x + z + \lambda t + \xi_0$, being ξ_0 a constant. The system (1.7) reduces to

$$\begin{cases} 4u^2v' - 4uu'v + u^2u''' - uu''u' - 3uu'u'' + 3u'^2u' - 4u^4u' = 0, \\ \lambda u' - v' = 0. \end{cases} \tag{4.2}$$

As we mentioned in Sec.2, we seek solutions in the form

$$\begin{cases} u(\xi) = \sum_{i=0}^{M} a_i\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i\phi(\xi)^{M-i}, \\ v(\xi) = \sum_{i=0}^{N} b_i\phi(\xi)^i + \sum_{i=N+1}^{2N} b_i\phi(\xi)^{N-i}. \end{cases} \tag{4.3}$$

From second equation of (4.2) we have $M = N$. As in previous section, by simplicity, we take $M = 2$, so that, (4.3) take the form

$$\begin{cases} u(\xi) = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^{-1} + a_4\phi^{-2}, \\ v(\xi) = b_0 + b_1\phi + b_2\phi^2 + b_3\phi^{-1} + b_4\phi^{-2}. \end{cases} \tag{4.4}$$

Substituting the expressions of (4.4) into (4.2) we obtain an algebraic system in the unknowns $\alpha, \beta, \gamma, \lambda, a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4$. Solving it with aid of *Mathematica*, we obtain a lot of solutions. By simplicity, we consider only the following solution:

$$\begin{cases} a_2 = a_4 = b_2 = b_4 = 0, \quad \alpha = a_3, \quad \beta = a_0, \quad \gamma = a_1, \quad \lambda = \dfrac{a_3 b_1}{a_1}. \end{cases} \tag{4.5}$$

Proceeding as before, we have

$$\begin{cases} u(x,z,t) = u(\xi) = a_0 + a_1 \left[\dfrac{-\sqrt{a_0^2 - 4a_1a_3}\tanh\left[\frac{1}{2}\sqrt{a_0^2 - 4a_1a_3}\xi\right] - a_0}{2a_1} \right] \\ +a_3 \left[\dfrac{-\sqrt{a_0^2 - 4a_1a_3}\tanh\left[\frac{1}{2}\sqrt{a_0^2 - 4a_1a_3}\xi\right] - a_0}{2a_1} \right]^{-1}, \\ v(x,z,t) = v(\xi) = \dfrac{a_0b_1}{a_1} + b_1 \left[\dfrac{-\sqrt{a_0^2 - 4a_1a_3}\tanh\left[\frac{1}{2}\sqrt{a_0^2 - 4a_1a_3}\xi\right] - a_0}{2a_1} \right] \\ +\dfrac{a_3b_1}{a_1} \left[\dfrac{-\sqrt{a_0^2 - 4a_1a_3}\tanh\left[\frac{1}{2}\sqrt{a_0^2 - 4a_1a_3}\xi\right] - a_0}{2a_1} \right]^{-1}, \end{cases}$$

where $\xi = x + z + \dfrac{b_1}{a_1}t + \xi_0$, a_0, a_1, a_3, b_1 arbitrary constants. Many other solutions can obtained, however, by space reasons we omit here.

5. Conclusions

We illustrate the use of the improved tanh-coth solving two important models: the SKdV equation and the (2+1)-dimensional SKdV equation. The effectiveness of the method is clear, however, the main idea was the use of the method directly for solving the system (1.7). In other works, the authors transform the system in one equation and they applied other computational methods. The advantage of the technique used here is that the method can be applied to solve other systems which cannot be reduced to one equation. Clearly, the solutions obtained here are in a most general form that those obtained in other works, the reason of that is the use of the Eq. (1.2).

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