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Generalized RK Integrators for Solving Ordinary Differential Equations: A Survey & Comparison Study

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Abstract

In this work, we have studied the generalized Runge-Kutta integrators for solving first-, second-, third-, fourth- & fifth-order ordinary differential equations. In several recent papers, the idea of generalized Runge-Kutta integrators for solving ordinary differential equations have been proposed. Among them are some methods specially tuned to integrate ordinary differential problems of second-, third-, fourth-, fifth-order. The main contributions of these papers are the development of direct explicit integrators of Runge-Kutta type for solving ordinary differential equations (ODEs). For this purpose, they generalized the integrators of Runge-Kutta type for solving special first-, second-, third-, fourth- and fifth-order ODEs (RK, RKN, RKD, RKT, RKFD & RKM). Using Taylor expansion and rooted trees approach, they have derived the order conditions for the proposed integrators. Based on these conditions, direct numerical methods with different stages are derived. Also they have tested the methods on the computation of some implementation which shows that the new methods agree well with existing RK methods but require less function evaluations. This is so due to the fact that the new methods are direct; hence, they save considerable amount of computational time.

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1. Introduction

The most important mathematical model for physical phenomena is the differential equation. Motion of objects, fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. The mathematical modeling of many real-life problems in physics, engineering, and economics can be expressed in terms of higher order differential equations (DEs), ordinary or partial. Typical examples can be found in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory.

Finding the solutions to these differential equations had been challenged the ingenuity of mathematicians since the time of Newton. Therefore, many theoretical and numerical studies dealing with the solution of such differential equations of different order have appeared in the literature. There are many analytical and numerical methods for solving some types of the differential equations. Unfortunately analytical tools frequently are inadequate for the solution of such systems. However, in applications it is not necessary to find the solution to more than a finite number of decimal places. For this reason numerical and analytical methods were developed for the solution of ordinary differential equations since the age of Newton, Taylor and Euler. There are also many examples of particular linear variable coefficient or nonlinear systems for which exact solutions are known, but in general for such systems we must resort to either an approximate or a numerical method. These methods of solutions are not able to solve many types of differential equations or they can solved some types of differential equations indirectly. This reason make us to study and derive more direct numerical methods. For this propose some of direct methods: RKD, RKT, RKFD & RKM have been derived. In order to apply indirect numerical method to solve a differential equation of higher than order one, the equation should be transformed into a system of first order differential equations.

2. History of Runge-Kutta Integrators

In numerical analysis, the RK methods comprise an important family of implicit and explicit methods to approximate the solutions of ODEs. These techniques were developed circa 1900 by the German mathematicians C. Runge and M.W. Kutta. [4] introduced the history of RK methods in his paper. Further contributions were made by Huen in 1900, who completely characterized the set of fourth-, fifth- & sixth-order RK methods for the initial value problem. Coefficients of these RK methods are indicated in the Butcher tableau in Tables (3-5). Considerable research has been conducted on the numerical integrator of the RK type for first-order ODEs. The first systematic work on numerical methods for first-order ODEs was that of F. Bashforth and J.C. Adams in 1883 where the idea of multistep methods was introduced. A number of researchers have studied one-step numerical methods. The first one-step method was introduced by Runge in 1895. Heun also constructed one-step methods in 1900, and Kutta formulated the general scheme of RK methods in 1901. The theoretical basis of these methods can be traced back to the paper of Merson in 1957 and the work of [5] in a long series of papers starting in 1963. Additional related papers in developing the RK method were published by researchers such as Fehlberg, Verner, Dormand and Prince, Hairer, Nörsett and Wanner, and Lambert. RK methods became very popular among scientists because these are robust and easily implemented.

The numerical integrator of the RK type for special second-order ODE, is known in the literature as the Runge-Kutta-Nyström (RKN) formula designed by E. J. Nyström in 1925. Studies done on the RKN method, such as those of [15], [8] and [21], discussed the theory of direct finite difference method for solving this equation. Hairer and Wanner proposed the Nyström-type method wherein order conditions for the determination of the parameters of the method were discussed. Henrici, Gear, Chawla and Sharma, and Hairer developed independently explicit and implicit RKN methods for the numerical solution of equation, [40] and [36] derived a singly diagonally implicit RKN method for solving oscillatory problems while [24] has studied solution of special second-order delay differential equations using Runge-Kutta-Nyström Method. Generally, a special third-order ODE is frequently found in many physical problems, such as electromagnetic waves, gravity-driven flows, and thin film flow ([39]). Researchers, scientists, and engineers used to solve the third-order ODE by reducing it to an equivalent first-order system three times the dimension, then solved using a standard RK method or multistep methods. This method developed by [2], and [20] also proposed a class of hybrid collocation methods for the direct solution of higher-order ODEs. [33] developed an embedded hybrid method for solving special second-order ODEs; [41] developed a block multistep method which can solve general third-order equations directly; and [42] developed a multistep method which can solve stiff third-order differential equations directly. All methods discussed previously are multistep methods that need the starting values when used to solve ODEs.

For review of RK type methods, the second-order ODEs, [38]-[40] have derived direct numerical integrators with constant step-size while [7] has derived direct numerical methods with variable step-size for solving second-order ODEs while for third-order, [26] and [43] have derived direct integrators of RK type for solving ODEs of third-order while [34] have derived variable step-size direct integrators of RK type of orders $6(5), 5(4)\&4(3)$ for solving third-order ODEs., a third-order ordinary and delay differential equations have been studied by [23], moreover different orders of direct explicit RKD methods for solving special third-order delay and ordinary differential equations with constant step-size have been derived ([22], [25] & [23]) while [27] has derived direct integrators of Runge-Kutta type for special third-order differential equations with thier applications. However, the regions of stability for RKD methods have been derived by [29].

Moreover, for fourth-order ODEs, [28] & [17] have derived direct numerical integrators with constant step-size for solving fourth-order ODEs named as RKM & RKFD methods while [18] has derived embedded direct numerical integrators with variable step-size for solving fourth-order ODEs. However, [30] has derived direct numerical integrators with constant step-size for solving fifth-order ODEs. Considering that the numerical integrators of the RK type for third-, fourth-, & fifth-order ODEs named as RKD, RKT, RKFD & RKM methods are derived based on the derivation of RK and RKN methods, for use in solving ODEs directly.

3. Runge-Kutta Integrators for Solving First-Order Ordinary Differential Equations

The initial value problem of first-order ODE is defined as follow:

$$y^{'} = f(x, y), \qquad a \leq x \leq b, \tag{3.1}$$

with initial condition,

$$y(a) = \alpha,$$

where,

$$\begin{aligned} f &: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \\ y(x) &= [y_1(x), y_2(x), \ldots, y_n(x)], \\ f(x, y) &= [f_1(x, y), f_2(x, y), \ldots, f_n(x, y)], \end{aligned}$$

and

$$\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n].$$

Proposed RK Methods

The general s-stage RK method for the IVP (3.1) is defined by

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i,$$

where,

$$k_1 = f(x_n, y_n),$$

$$k_i = f(x_n + c_i h, y_n + h \sum_{i=1}^{s} a_{ij} k_i),$$

for $i = 2, 3, \ldots, s$ and by assuming the row-sum condition holds,

$$c_i = \sum_{j=1}^{s} a_{ij}.$$

It is convenient to display the coefficients of the general RK method as in Butcher tableau as in Table 1 or the simplified as in Table 2. Clearly, an s-stage RK method is completely specified by its Butcher's tableau. We define the s-dimensional vectors c and b and the $s \times s$ matrix A by

$$\begin{aligned} c &= [c_1, c_2, \ldots, c_s]^T, \\ b &= [b_1, b_2, \ldots, b_s]^T, \\ A &= [a_{ij}]. \end{aligned}$$

If $a_{ij} = 0$ for $j \geq i$ and $i = 1, 2, \ldots, s$ then each of k_i is given explicitly in term of previously computed k_j, for $j = 1, 2, \ldots, i-1$ and the method is then an explicit RK method. If this is not the case then the method is implicit, and in general, it is necessary to solve at each step of the computation an implicit system for k_i, summarizing, we have:

(a) Explicit method
$a_{ij} = 0$, for $j > i$, $j = 1, 2, \ldots, s$. $\Leftrightarrow$ A is lower strictly lower triangular matrix.

(b) Semi-explicit method
$a_{ij} = 0$, for $j \geq i$, $j = 1, 2, \ldots, s$. $\Leftrightarrow$ A is triangular matrix.

(c) Implicit method
$a_{ij} \neq 0$, for some $j \geq i$, $j = 1, 2, \ldots, s$. $\Leftrightarrow$ A is not lower triangular matrix.

(d) Diagonally-implicit method
$a_{ii} = \gamma$, for $i = 1, 2, \ldots, s$.

Table 1: The Butcher Tableau for the RK Method.

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & & & \ddots & \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array}$$

Table 2: The Butcher Tableau for the RK Method.

$$\begin{array}{c|c} \mathrm{c} & \mathrm{A} \\ \hline & b^T \end{array}$$

3.1. The Diagonally Implicit Runge-Kutta Methods

Implicit numerical methods have always been popular when solving stiff ODEs. However, most of these methods are expensive to use, hence the search for cost-efficient implementable methods. Diagonally-implicit Runge-Kutta (DIRK) methods are a form of semi-implicit Runge-Kutta methods that have almost the same advantages as the implicit Runge-Kutta, especially as regards to stability criterion. These methods are sometimes referred to as singly diagonally-implicit Runge-Kutta because DIRK methods do not necessarily have equal diagonals.

Table 3: The Butcher Tableau for the RK4 Method

$$\begin{array}{c|cccc} 0 & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

4. Runge-Kutta-Nyström Integrators for Solving Second-Order Ordinary Differential Equations

A special method for second-order differential equations was proposed E. J. Nyström in 1925, who also contributed to the development of methods for first-order ODEs. Sixth-order RK methods were introduced only after the 1957 work of Huta. Generally special second-order ODEs of the form

$$y''(x) = f(x, y(x)), \qquad x \geq x_0, \tag{4.1}$$

with initial conditions,

$$y(x_0) = \alpha \text{ and } y'(x_0) = \beta,$$

where,

$$\begin{aligned} f &: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \\ y(x) &= [y_1(x), y_2(x), \ldots, y_n(x)], \\ f(x, y) &= [f_1(x, y), f_2(x, y), \ldots, f_n(x, y)], \\ \alpha &= [\alpha_1, \alpha_2, \ldots, \alpha_n], \\ \beta &= [\beta_1, \beta_2, \ldots, \beta_s]. \end{aligned}$$

Table 4: The Butcher Tableau for the RK5 Method

0	0						
$\frac{1}{5}$	$\frac{1}{5}$	0					
$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$	0				
$\frac{4}{5}$	$\frac{44}{45}$	$-\frac{56}{15}$	$\frac{32}{9}$	0			
$\frac{8}{9}$	$\frac{19372}{6561}$	$-\frac{25360}{2187}$	$\frac{64448}{6561}$	$-\frac{-212}{729}$	0		
1	$-\frac{9017}{3168}$	$-\frac{355}{33}$	$\frac{46732}{5247}$	$\frac{49}{176}$	$-\frac{5103}{18656}$	0	
1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$-\frac{2187}{6784}$	$\frac{11}{84}$	0
	$\frac{5179}{57600}$	0	$\frac{7571}{16695}$	$\frac{393}{640}$	$-\frac{92097}{339200}$	$\frac{187}{2100}$	$\frac{1}{40}$

Proposed RKN Methods

The general s-stage RK method for the IVP (4.1) is defined by:

$$\begin{aligned} y_{n+1} &= y_n + hy_n^{'} + h^2 \sum_{i=1}^{s} b_i k_i, \\ y_{n+1}^{'} &= y_n^{'} + hy_n^{'} + h \sum_{i=1}^{s} b_i^{'} k_i, \end{aligned}$$

with,

$$\begin{aligned} k_1 &= f(x_n, y_n), \\ k_i &= f(x_n + c_i h, y_n + c_i h y_n^{'} + h^2 \sum_{j=1}^{s} a_{ij} k_j), \end{aligned}$$

for $i = 2, 3, \ldots, s$.

The RKN parameters $a_{ij}, b_j, b_j^{'}$ are assumed to be real and s is the number of stages of the method. The s dimensional vectors c, b and $b^{'}$ and $s \times s$ matrix A are define as

Table 5: The Butcher Tableau for the RK6 Method

$$
\begin{array}{c|ccccccc}
0 & 0 & & & & & & \\
\frac{1}{3} & \frac{1}{3} & 0 & & & & & \\
\frac{2}{3} & 0 & \frac{2}{3} & 0 & & & & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{3} & -\frac{1}{12} & 0 & & & \\
\frac{1}{2} & -\frac{1}{16} & \frac{9}{8} & -\frac{3}{16} & -\frac{3}{8} & 0 & & \\
\frac{1}{2} & 0 & \frac{9}{8} & -\frac{3}{8} & -\frac{3}{4} & \frac{1}{2} & 0 & \\
1 & \frac{9}{44} & -\frac{9}{11} & \frac{63}{44} & \frac{18}{11} & 0 & -\frac{16}{11} & 0 \\
\hline
 & \frac{11}{120} & 0 & \frac{27}{40} & \frac{27}{40} & -\frac{4}{15} & -\frac{4}{15} & \frac{11}{120}
\end{array}
$$

following:

$$
\begin{aligned}
c &= [c_1, c_2, \dots, c_s]^T, \\
b &= [b_1, b_2, \dots, b_s]^T, \\
b^{'} &= [b_1^{'}, b_2^{'}, \dots, b_s^{'}]^T, \\
A &= [a_{ij}].
\end{aligned}
$$

We shall always assume the row-sum condition holds,

$$
c_i = \sum_{j=1}^{s} a_{ij},
$$

for $i = 1, 2, 3, \dots, s$.

The s-stage RKN method above can be expressed in Butcher tableau as depicted in the Table 6 or the simplified as in Table 7.

Table 6: The Butcher Tableau for the RKN Method

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & & & \ddots & \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\ \hline & b_1 & b_2 & \dots & b_s \\ & b_1^{'} & b_2^{'} & \dots & b_s^{'} \end{array}$$

Table 7: The Butcher Tableau for the RKN Method.

$$\begin{array}{c|c} \text{c} & \text{A} \\ \hline & \\ & b^T \\ & b^{'T} \end{array}$$

5. General RKD & RKT Integrators for Solving third-Order Ordinary Differential Equations

The problem of interest is the initial value problem of third-order ODEs of the form,

$$y^{'''}(x) = f(x, y(x)); \qquad x \geq x_0, \tag{5.1}$$

with initial conditions,

$$y(x_0) = \alpha^0,$$

$$y^{'}(x_0) = \alpha^1.$$

and,

$$y^{''}(x_0) = \alpha^2.$$

where,

$$f \;:\; \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N,$$

with,

$$y(x) = [y_1(x), y_2(x), \dots, y_N(x)],$$

$$f(x, y) = [f_1(x, y), f_2(x, y), \dots, f_N(x, y)],$$

$$\begin{aligned} \alpha^0 &= [\alpha_1^0, \alpha_2^0, \dots, \alpha_N^0], \\ \alpha^1 &= [\alpha_1^1, \alpha_2^1, \dots, \alpha_N^1], \\ \alpha^2 &= [\alpha_1^2, \alpha_2^2, \dots, \alpha_N^2], \end{aligned}$$

when the ODE (5.1) in n dimension space, then we can simplified to

$$z^{'''}(x) = g(z(x)), \tag{5.2}$$

using the following assumption,

$$z(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \\ y_N(x) \\ x \end{pmatrix}, g(z) = \begin{pmatrix} f_1(z_1, z_2, \ldots, z_N, z_{N+1}) \\ f_2(z_1, z_2, \ldots, z_N, z_{N+1}) \\ f_3(z_1, z_2, \ldots, z_N, z_{N+1}) \\ \vdots \\ f_N(z_1, z_2, \ldots, z_N, z_{N+1}) \\ 0 \end{pmatrix},$$

with the initial conditions,

$$z(x_0) = \overline{\alpha}^0,$$
$$z^{'}(x_0) = \overline{\alpha}^1,$$
$$z^{''}(x_0) = \overline{\alpha}^2,$$

where,

$$\overline{\alpha}^0 = [\alpha_1^0, \alpha_2^0, \ldots, \alpha_N^0, x_0],$$
$$\overline{\alpha}^1 = [\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1, 1],$$
$$\alpha^2 = [\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2, 0].$$

Throughout the numerical solutions, they assumed that the unique solution to this problem always exists. Special third-order ODE (5.1) or (5.2) can be solved more efficiently by using direct numerical methods than by converting the ODE into a system of first-order equations with three times the dimensions.

Proposed RKM & RKT Methods

[26] & [43] have proposed the general form of RKD & RKT methods with s-stage for solving special third-order ODEs can be written as

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} + h^3 \sum_{i=1}^{s} b_i k_i, \tag{5.3}$$

$$y_{n+1}^{'} = y_n^{'} + hy_n^{''} + h^2 \sum_{i=1}^{s} b_i^{'} k_i, \tag{5.4}$$

$$y_{n+1}^{''} = y_n^{''} + h \sum_{i=1}^{s} b_i^{''} k_i, \tag{5.5}$$

where,

$$k_1 = f(x_n, y_n), \quad (5.6)$$

$$k_i = f\left(x_n + c_i h, y_n + hc_i y_n^{'} + \frac{h^2}{2} c_i^2 y_n^{''} + h^3 \sum_{j=1}^{i-1} a_{ij} k_j\right), \quad (5.7)$$

for $i = 2, 3, \ldots, s$.

The parameters of the proposed method are c_i, a_{ij}, b_i, $b_i^{'}$, and $b_i^{''}$ for $i = 1, 2, \ldots, s$ and, $j = 1, 2, \ldots, s$ are assumed to be real. If $a_{ij} = 0$ for $i \leq j$, it is an explicit method and implicit otherwise, which given bellow the Table 8. The two-stage third-order RKD3, which can be expressed in the Butcher tableau 9 while the Butcher tableau RKD5 method is given in Table 10. The four-stage six-order RKD method is denoted by RKD6 which can be expressed in the Butcher tableau (11).

Table 8: The Butcher Tableau of The RKD Method

c	A
	b^T
	$b^{'T}$
	$b^{''T}$.

Table 9: The Butcher Tableau RKD3 Method of Third-Order

0		
$\frac{2}{3}$	$\frac{11}{200}$	
	$\frac{1}{8}$	$\frac{1}{24}$
	$\frac{1}{4}$	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$

5.1. General Theory for Embedded Method

The review of embedded RK methods researches, [9], [31], [10], [12], [16] and [4] have studied the embedded RK methods for solving first-order ODEs whereas [19], [37], [32],

Table 10: The Butcher Tableau RKD5 Method

0	0		
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$\frac{27}{500}-\frac{19\sqrt{6}}{1000}$	0	
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{33}{2500}+\frac{51\sqrt{6}}{5000}$	$\frac{51}{1250}+\frac{11\sqrt{6}}{1250}$	0
	$\frac{1}{18}$	$\frac{1}{18}+\frac{\sqrt{6}}{48}$	$\frac{1}{18}-\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}+\frac{\sqrt{6}}{18}$	$\frac{7}{36}-\frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{4}{9}+\frac{\sqrt{6}}{36}$	$\frac{4}{9}-\frac{\sqrt{6}}{36}$

Table 11: The Butcher Tableau RKD6 Method

0	0			
$\frac{1}{2}-\frac{\sqrt{15}}{10}$	$\frac{7}{120}-3\frac{\sqrt{15}}{200}$			
$\frac{1}{2}$	$\frac{-1}{96}+\frac{\sqrt{15}}{480}$	$\frac{1}{32}-\frac{\sqrt{15}}{480}$		
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	$-\frac{1}{600}+\frac{\sqrt{15}}{600}$	$\frac{\sqrt{15}}{50}$	$\frac{3}{50}-\frac{\sqrt{15}}{150}$	0
	0	$\frac{1}{18}+\frac{\sqrt{15}}{72}$	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$
	0	$\frac{5}{36}+\frac{\sqrt{15}}{36}$	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$
	0	$\frac{5}{18}$	$\frac{4}{9}$	$\frac{5}{18}$

[38], [7] and [35] have studied the explicit embedded RKN methods for solving solving second-order ODEs.

The general form of the RKD methods with s-stage for solving third-order initial value problem (5.1) or (5.2) is given in the forms (5.3)-(5.5) in section 5. In this subsection, the derivation of the embedded pair for solving third-order ODEs is based on the strategies introduced in ([11]) for deriving explicit embedded RKN methods ([34]). Hence, the following relative errors should be as small as possible.

An embedded $s(\overline{s})$ is generally a pair of explicit RKD method used in variable step-size algorithm, thus providing a cheap error estimation. An embedded $s(\overline{s})$ is based on the RKD method $(c, A, b, b^{'}, b^{''})$ of sth-order and the other RKD method $(c, A, \hat{b}, \hat{b}^{'}, \hat{b}^{''})$ of $\overline{s}th$-order where s is greater or equal to $\overline{s}+1$, that is, $s \geq \overline{s}+1$.

The RKD method $(c, A, b, b^{'}, b^{''})$ of sth-order and the RKD method $(c, A, \hat{b}, \hat{b}^{'}, \hat{b}^{''})$ of $\overline{s}th$-order is as follows: Finally, the new embedded three-stage method is denoted as RKD5(4) which can be expressed in Butcher tableau 12 and as the following:

Table 12: The Butcher Tableau for the Embedded RKD5(4) method

0	0		
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{27}{500}+\frac{19\sqrt{6}}{1000}$	0	
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$\frac{33}{2500}-\frac{51\sqrt{6}}{5000}$	$\frac{51}{1250}-\frac{11\sqrt{6}}{1250}$	0
	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{6}}{48}$	$\frac{1}{18}+\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{18}$	$\frac{7}{36}+\frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{4}{9}-\frac{\sqrt{6}}{36}$	$\frac{4}{9}+\frac{\sqrt{6}}{36}$
	$\frac{1}{15}$	$\frac{1}{20}-\frac{11\sqrt{6}}{720}$	$\frac{1}{20}+\frac{11\sqrt{6}}{720}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{18}$	$\frac{7}{36}+\frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{4}{9}-\frac{\sqrt{6}}{36}$	$\frac{4}{9}+\frac{\sqrt{6}}{36}$

6. General RKM & RKFM Integrators for Solving Fourth-Order Ordinary Differential Equations

The problem of interest is the initial value problem of fourth-order ODEs of the form,

$$y^{(4)}(x) = f(x, y(x)); \qquad x \geq x_0, \tag{6.1}$$

with initial conditions,

$$y(x_0) = \alpha^0,$$
$$y'(x_0) = \alpha^1.$$
$$y''(x_0) = \alpha^2.$$

and,

$$y'''(x_0) = \alpha^3.$$

where,

$$f \ : \ \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N,$$

with,

$$y(x) = [y_1(x), y_2(x), \ldots, y_N(x)],$$
$$f(x, y) = [f_1(x, y), f_2(x, y), ..., f_N(x, y)],$$

$$\begin{aligned} \alpha^0 &= [\alpha_1^0, \alpha_2^0, \ldots, \alpha_N^0], \\ \alpha^1 &= [\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1], \\ \alpha^2 &= [\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2], \\ \alpha^3 &= [\alpha_1^3, \alpha_2^3, \ldots, \alpha_N^3], \end{aligned}$$

when the ODE (6.1) in n dimension space, then we can simplified to

$$z^{(4)}(x) = g(z(x)), \tag{6.2}$$

using the following assumption,

$$z(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ \vdots \\ y_N(x) \\ x \end{pmatrix}, g(z) = \begin{pmatrix} f_1(z_1, z_2, \ldots, z_N, z_{N+1}) \\ f_2(z_1, z_2, \ldots, z_N, z_{N+1}) \\ f_3(z_1, z_2, \ldots, z_N, z_{N+1}) \\ \vdots \\ f_N(z_1, z_2, \ldots, z_N, z_{N+1}) \\ 0 \end{pmatrix},$$

with the initial conditions,

$$
\begin{aligned}
z(x_0) &= \overline{\alpha}^0,\\
z^{'}(x_0) &= \overline{\alpha}^1,\\
z^{''}(x_0) &= \overline{\alpha}^2,\\
z^{'''}(x_0) &= \overline{\alpha}^3,
\end{aligned}
$$

where,

$$\overline{\alpha}^0 = [\alpha_1^0, \alpha_2^0, \ldots, \alpha_N^0, x_0],$$

$$\overline{\alpha}^1 = [\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1, 1],$$

$$\alpha^2 = [\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2, 0].$$

$$\alpha^3 = [\alpha_1^3, \alpha_2^3, \ldots, \alpha_N^3, 0].$$

The solution to Equation (6.1) or (6.2) can be obtained by reducing it to an equivalent first-order system four-times the dimension and be solved using a standard Runge-Kutta method or a multistep method. Most researchers, scientists and engineers used to solve higher order ODEs by converting the nth-order ODE into a system of first-order ODEs n-times the dimensions (see [1]).

Some researchers can solve this ordinary differential equation using multistep methods. However, it would be more efficient if higher order ODEs can be directly solved using special numerical methods. [26] have derived different orders direct integrators of Runge-Kutta type for solving special third-order ODEs with constant step-size while [34] is derived a variable step-size direct integrators for Runge-Kutta type of orders 6(5), 5(4) and 4(3) for solving third-order ODEs. Accordingly, we can use RKD methods of different orders companying with the method of lines to solve third-order PDEs (see [22]). However, the the regions of stability for RKD methods have been studied by [29].

In this subsection, we are introduced one-step method particularly Runge-Kutta integrator for directly solving fourth-order ODEs. Accordingly, [28], [17] developed the order conditions for direct Runge-Kutta methods, so that based on the order conditions RKM method can be derived.

Proposed RKM methods

[28], [17] have proposed the general form of RKM & RKFD methods with s-stage for solving fourth-order ODE as the following:

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2}y_n^{''} + \frac{h^3}{6}y_n^{'''} + h^4\sum_{i=1}^{s} b_i k_i \tag{6.3}$$

$$y_{n+1}^{'} = y_n^{'} + hy_n^{''} + \frac{h^2}{2}y_n^{'''} + h^3\sum_{i=1}^{s} b_i^{'} k_i \tag{6.4}$$

$$y_{n+1}^{''} = y_n^{''} + hy_n^{'''} + h^2\sum_{i=1}^{s} b_i^{''} k_i \tag{6.5}$$

$$y_{n+1}^{'''} = y_n^{'''} + h\sum_{i=1}^{s} b_i^{'''} k_i \tag{6.6}$$

where

$$k_1 = f(x_n, y_n) \tag{6.7}$$

$$k_i = f(x_n + c_i h, y_n + hc_i y_n^{'} + \frac{h^2}{2}c_i^2 y_n^{''} + \frac{h^3}{6}c_i^3 y_n^{'''} + h^4\sum_{j=1}^{i-1} a_{ij}k_j) \tag{6.8}$$

for $i = 2, 3, \ldots, s$.

The parameters of RKM method are $c_i, a_{ij}, b_i, b_i^{'}, b_i^{''}, b_i^{'''}$ for $i, j = 1, 2, \ldots, s$ are assumed to be real. If $a_{ij} = 0$ for $i \le j$, it is an explicit method and otherwise implicit method. The RKM method can be expressed in Butcher notation using the table of coefficients as follows:

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & b^{'T} \\ & b^{''T} \\ & b^{'''T}. \end{array}$$

[28], [17] & [18] have derived the order conditions of new methods up to order seven for solving fourth-order ODEs. To obtain the order conditions, they used the Taylor series expansion and Rooted trees approaches. Using the same technique, they have derived the order conditions up to order five for solving fourth-order ODEs.

The Butcher Tableau for the RKM4, RKM5 & RKM6 methods as follow:

Table 13: The Butcher Tableau RKM4 Method

$$\begin{array}{c|ccc}
0 & 0 & & \\
\frac{5}{6} & \frac{1}{2} & 0 & \\
\frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 0 \\
\hline
 & \frac{1}{60} & \frac{1}{3240} & \frac{2}{81} \\
 & \frac{1}{20} & \frac{1}{180} & \frac{1}{9} \\
 & \frac{1}{10} & \frac{1}{15} & \frac{1}{3} \\
 & \frac{1}{10} & \frac{2}{5} & \frac{1}{2}
\end{array}$$

7. General RKM Integrators for Solving Fifth-Order Ordinary Differential Equations

The initial value of fifth-order problem ODEs of the form,

$$y^{(5)}(x) = f(x, y(x)); \qquad x \geq x_0, \tag{7.1}$$

with initial conditions,

$$y(x_0) = \alpha^0,$$
$$y^{'}(x_0) = \alpha^1.$$
$$y^{''}(x_0) = \alpha^2.$$
$$y^{'''}(x_0) = \alpha^3.$$

and,

$$y^{''''}(x_0) = \alpha^4.$$

where,

$$f \;:\; \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N,$$

Table 14: The Butcher Tableau RKM5 Method

0	0		
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$-\frac{643}{625}+\frac{317}{2500}\sqrt{6}$	0	
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	$\frac{1}{54}$	$\frac{5}{432}+\frac{\sqrt{6}}{216}$	$\frac{5}{432}-\frac{\sqrt{6}}{216}$
	$\frac{1}{18}$	$\frac{1}{18}+\frac{\sqrt{6}}{48}$	$\frac{1}{18}-\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{10915}{33023}$	$\frac{3397}{58206}$
	$\frac{1}{9}$	$\frac{4}{9}+\frac{\sqrt{6}}{36}$	$\frac{4}{9}-\frac{\sqrt{6}}{36}$

with,

$$y(x) = [y_1(x), y_2(x), \ldots, y_N(x)],$$

$$f(x, y) = [f_1(x, y), f_2(x, y), \ldots, f_N(x, y)],$$

$$\begin{aligned}
\alpha^0 &= [\alpha_1^0, \alpha_2^0, \ldots, \alpha_N^0],\\
\alpha^1 &= [\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1],\\
\alpha^2 &= [\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2],\\
\alpha^3 &= [\alpha_1^3, \alpha_2^3, \ldots, \alpha_N^3],\\
\alpha^4 &= [\alpha_1^4, \alpha_2^4, \ldots, \alpha_N^4],
\end{aligned}$$

when the ODE (7.1) in n dimension space, then we can simplified to

$$z^{(5)}(x) = g(z(x)), \tag{7.2}$$

Table 15: The Butcher Tableau RKM6 Method

0	0			
$\frac{1}{2}$	$\frac{3}{160}-\frac{\sqrt{15}}{240}$			
$\frac{1}{2}-\frac{\sqrt{15}}{10}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	$-\frac{51}{100}-\frac{22}{75}\sqrt{15}$	$\frac{1}{100}+\frac{\sqrt{15}}{5}$	$\frac{1}{2}$	0
	0	$\frac{1}{108}$	$\frac{7}{432}+\frac{\sqrt{15}}{240}$	$\frac{7}{432}-\frac{\sqrt{15}}{240}$
	0	$\frac{1}{18}$	$\frac{1}{18}+\frac{\sqrt{15}}{72}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$
	0	$\frac{2}{9}$	$\frac{5}{36}+\frac{\sqrt{15}}{36}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$

using the following assumption,

$$z(x)=\begin{pmatrix} y_1(x)\\ y_2(x)\\ y_3(x)\\ \vdots\\ y_N(x)\\ x \end{pmatrix},\ g(z)=\begin{pmatrix} f_1(z_1,z_2,\dots,z_N,z_{N+1})\\ f_2(z_1,z_2,\dots,z_N,z_{N+1})\\ f_3(z_1,z_2,\dots,z_N,z_{N+1})\\ \vdots\\ f_N(z_1,z_2,\dots,z_N,z_{N+1})\\ 0 \end{pmatrix},$$

with the initial conditions,

$$\begin{aligned} z(x_0) &= \overline{\alpha}^0, \\ z^{'}(x_0) &= \overline{\alpha}^1, \\ z^{''}(x_0) &= \overline{\alpha}^2, \\ z^{'''}(x_0) &= \overline{\alpha}^3, \\ z^{''''}(x_0) &= \overline{\alpha}^4, \end{aligned}$$

where,

$$\begin{aligned} \overline{\alpha}^0 &= [\alpha_1^0, \alpha_2^0, \ldots, \alpha_N^0, x_0], \\ \overline{\alpha}^1 &= [\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1, 1], \\ \alpha^2 &= [\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2, 0]. \\ \alpha^3 &= [\alpha_1^3, \alpha_2^3, \ldots, \alpha_N^3, 0]. \\ \alpha^4 &= [\alpha_1^4, \alpha_2^4, \ldots, \alpha_N^4, 0]. \end{aligned}$$

Proposed RKM methods

In this section, [30] concerned with fifth-order ordinary differential equation with no appearance for the first, second, third and fourth derivatives $w^{(i)}(x)$, for $i = 1, 2, 3, 4$. It can be written in the following form:

$$w^{(5)}(x) \;=\; g(x, w(x)), \qquad x \geq x_0, \tag{7.3}$$

subject to initial condition,

$$w^{(i)}(x_0) \;=\; \gamma^i,$$

for$i = 0, 1, \ldots, 4$.
where,

$$g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

and

$$\begin{aligned} w(x) &= [w_1(x), w_2(x), \ldots, w_N(x)] \\ g(x, y) &= [g_1(x, y), g_2(x, y), \ldots, g_N(x, y)] \\ \gamma^i &= [\gamma_1^i, \gamma_2^i, \ldots, \gamma_N^i] \end{aligned}$$

for $i = 0, 1, \ldots, 4$.
knowing that, N is the components number of the vector of independent variables of the system of ordinary differential equations (7.3). To convert the function $g(x, w(x))$ which depends on two variables, to a function which depends only on one variable $w(x)$, using

high dimension we can work in $N+1$ dimension using the assumption $w_{N+1}(x) = x$, then (7.3) can be simplified to following equation:

$$v^{(5)}(x) = h(v(x)) \tag{7.4}$$

using the following consideration

$$v(x) = \begin{pmatrix} w_1(x) \\ w_2(x) \\ w_3(x) \\ \cdots \\ \cdots \\ \cdots \\ w_N(x) \\ x \end{pmatrix}, h(v) = \begin{pmatrix} g_1(v_1, v_2, \ldots, v_N, v_{N+1}) \\ g_2(v_1, v_2, \ldots, v_N, v_{N+1}) \\ g_3(v_1, v_2, \ldots, v_N, v_{N+1}) \\ \cdots \\ \cdots \\ \cdots \\ g_N(v_1, v_2, \ldots, v_N, v_{N+1}) \\ 0 \end{pmatrix}.$$

subject to the initial condition,

$$w^{(i)}(x_0) = \overline{\gamma^i},$$

for$i = 0, 1, \ldots, 4$.
where,

$$\overline{\gamma}^i = [\gamma_1^i, \gamma_2^i, \ldots, \gamma_N^i, x_0]$$

for $i = 0, 1, \ldots, 4$.
These class of ODEs are found in many engineering and physical problems. Some of scientists and engineers can solve the equation (7.3) or (7.4) using one of multistep methods. Almost, they used to solve higher-order ODE by converting it to equivalent system of first-order ODEs and can solve using a classical RK method [13]. However, it would be more efficient if ODEs of order five can be solved using the proposed direct RKM method. The proposed method solve equation (7.3) or (7.4) directly be more efficient since it has less function evaluations and computational time in implementation.

In this section, we are concerned with the one-step RKM integrators for directly solving fifth-order ODEs. To obtain the order conditions he used the Taylor series expansion approach. Consequently, he have derived two of direct RKM integrators based on the algebraic equations of order conditions of RKM integrators.

Table 16: The Butcher Tableau RKM5 Method

0	0		
$\frac{3}{5}-\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	0	
$\frac{3}{5}+\frac{\sqrt{6}}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	1	0	$-\frac{119}{120}$
	$-\frac{1}{40}-\frac{\sqrt{6}}{360}$	$\frac{1}{60}+\frac{\sqrt{6}}{360}$	0
	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{6}}{48}$	$\frac{1}{18}+\frac{\sqrt{6}}{48}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{18}-\frac{\sqrt{6}}{18}$
	$\frac{1}{9}$	$\frac{7}{36}-\frac{\sqrt{6}}{48}$	$\frac{7}{18}-\frac{\sqrt{6}}{18}$

Proposed RKM Methods

[30] proposed formula of explicit RKM integrator with s-stage for solving fifth-order ODEs (7.3) as follow:

$$w_{n+1} = w_n + w_n^{'} + \frac{h^2}{2!}w_n^{''} + \frac{h^3}{3!}w_n^{'''} + \frac{h^4}{4!}w_n^{''''} + h^5 \sum_{i=1}^{s} b_i k_i \tag{7.5}$$

$$w_{n+1}^{'} = w_n^{'} + hw_n^{''} + \frac{h^2}{2!}w_n^{'''} + \frac{h^3}{3!}w_n^{''''} + h^4 \sum_{i=1}^{s} b_i^{'} k_i \tag{7.6}$$

$$w_{n+1}^{''} = w_n^{''} + hw_n^{'''} + \frac{h^2}{2!}w_n^{''''} + h^3 \sum_{i=1}^{s} b_i^{''} k_i \tag{7.7}$$

$$w_{n+1}^{'''} = w_n^{'''} + hw_n^{''''} + h^2 \sum_{i=1}^{s} b_i^{'''} k_i \tag{7.8}$$

$$w_{n+1}^{''''} = w_n^{''''} + h \sum_{i=1}^{s} b_i^{''''} k_i \tag{7.9}$$

Table 17: The Butcher Tableau RKM6 Method

c				
0	0			
$\frac{1}{2}$	$\frac{1}{2}$	0		
$\frac{1}{2}+\frac{1}{10}\sqrt{15}$	0	$\frac{1}{2}$	0	
$\frac{1}{2}-\frac{1}{10}\sqrt{15}$	0	$\frac{1}{2}$	$-\frac{359}{200}$	0
	$\frac{1}{180}$	$-\frac{359}{360}$	$\frac{1}{2}$	$\frac{1}{2}$
	$-\frac{67}{4}-\frac{109\sqrt{15}}{360}$	$-\frac{59}{60}-\frac{109\sqrt{540}}{360}$	$\frac{1}{2}$	$\frac{145}{72}+\frac{109\sqrt{15}}{216}$
	0	$\frac{1}{18}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$	$\frac{1}{18}+\frac{\sqrt{15}}{72}$
	0	$\frac{2}{9}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$	$\frac{5}{36}-\frac{\sqrt{15}}{36}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$

where

$$k_1 = g(x_n, w_n) \tag{7.10}$$

$$k_i = g\left(x_n + c_i h,\ w_n + h c_i w_n^{'} + \frac{h^2}{2!} c_i^2 w_n^{''} + \frac{h^3}{3!} c_i^3 w_n^{'''} + \frac{h^4}{4!} c_i^4 w_n^{''''} + h^5 \sum_{j=1}^{i-1} a_{ij} k_j\right) \tag{7.11}$$

for $i = 2, 3, \ldots, s$. and h is the step-size. The parameters of RKM integrator are $a_{ij}, c_i, b_i^{''''}, b_i^{'''}, b_i^{''}, b_i^{'}$, & b_i for $i = 1, 2, \ldots, s$ & $j = 1, 2, \ldots, s$ are real. It is an explicit integrator if $a_{ij} = 0$ for $i \le j$ and otherwise its implicit integrator. We have expressed the coefficients RKM method in Butcher Table as follows:

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & b^{'T} \\ & b^{''T} \\ & b^{'''T} \\ & b^{''''T}. \end{array}$$

Using Taylor enpension technique, he has derived the algebraic equations of order conditions of RKM methods for solving special fifth-order ODEs.

The Butcher Tableau for the RKM method as follow:

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8. Discussion and Conclusion

In this paper, we have studied the generalized Runge-Kutta integrators for Solving first-, second-, third, fourth-, & fifth-order ordinary differential equations. we have served several recent papers which are generalized Runge-Kutta integrators for solving ordinary differential equations (ODEs). Among them are some methods specially tuned to integrate ordinary differential problems of first-, second, third-, fourth- & fifth-order. The main contribution of these papers is to derive of direct explicit integrators of Runge-Kutta type for solving ODEs. For this purpose, we have introduced the generalized integrators of Runge-Kutta type for solving special first-, second-, third- fourth- and fifth-order ODEs (RK, RKN, RKD, RKT, RKFD & RKM). Using Taylor expansion or rooted trees approaches, they have derived order conditions for the proposed integrators. Based on these conditions, direct numerical methods with different stages are derived.

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