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# Some Results on Block Frames

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**Abstract**

The notion of block frame is defined. A necessary and sufficient condition for the existence block frames is obtained. We also characterize orthonormal block frames and Riesz block frames. Finally, we show that block frames are compression of Riesz block frames and Parseval block frames are compression of orthonormal block frames.

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## 1. Introduction

Let $\mathcal{H}$ be a real (or complex) separable Hilbert space with inner product $\langle .,. \rangle$.

**Definition 1.1.**

(a) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ if $\{x_n\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that

$$A\sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2 \leq B\sum_{n=1}^{\infty} |a_n|^2, \text{ for all } \{a_n\}_{n=1}^{\infty} \in \ell^2$$

(b) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ is a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}. \tag{1.1}$$

The scalars $A$ and $B$ are called the *lower* and *upper frame bounds* of the frame, respectively. They are not unique. If $A = B$, then $\{x_n\}$ is called an *A-tight frame* and if $A = B = 1$, then $\{x_n\}$ is called a *Parseval frame*. The inequality in (1.1) is called the *frame inequality* of the frame. The operator $T : \ell^2 \to \mathcal{H}$ defined as

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k x_k, \ \{c_k\} \in \ell^2,$$

is called the *pre-frame operator* (or *synthesis operator*) and its adjoint operator $T^* : \mathcal{H} \to \ell^2$, is called the *analysis operator* and is given by

$$T^*(x) = \{\langle x, x_k \rangle\} \text{ for all } x \in \mathcal{H}.$$

Composing $T$ and $T^*$ we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(x) = \sum_{k=1}^{\infty} \langle x, x_k \rangle x_k \text{ for all } x \in \mathcal{H}.$$

The frame operator $S$ is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $x \in \mathcal{H}$,

$$x = SS^{-1}x = \sum_{k=1}^{\infty} \langle S^{-1}x, x_k \rangle x_k \quad \left( = \sum_{k=1}^{\infty} \langle x, S^{-1}x_k \rangle x_k \right). \tag{1.2}$$

For more details related to frames and Riesz frames, one may refer [1, 2]. In [4], Y. C. Eldar and G. D. Forney studied about the fundamental relationship between rank-one quantum measurement and tight frames and define frame matrices in analogy to the measurement matrices of quantum mechanics. Further they use Neumark's Theorem to extend tight frames to orthonormal bases. Through the motivation of Least Squares Measurement of quantum machanics, they construct optimal tight frames. Further, W. Sun [7] defined and studied $G$-frames and $g$-Riesz bases in Hilbert spaces.

## 2. Block frames

We begin this section by defining the notion of block frame in Hilbert spaces.

**Definition 2.1.** Let $\mathbb{N} = \bigcup_{n=1}^{\infty} M_n$, where $M_i$ are finite subsets of $\mathbb{N}$ with $M_i \cap M_j = \phi$ for all $i \neq j$. Let $\{\lambda_n\}$ be any sequence of scalars and a sequence $\{x_n\}$ in Hilbert space $\mathcal{H}$ is called a *block frame* with respect to $(M_n, \alpha_n)$ if there exist constants $A$ and $B$ $(0 < A \leq B < \infty)$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} \sum_{j \in M_n} |\lambda_j|^2 |\langle x, x_j \rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}. \tag{2.1}$$

If $A = B$, then $\{x_n\}$ is called a $A$*-tight block frame* with respect to $(M_n, \lambda_n)$. If $A = B = 1$, then $\{x_n\}$ is called a *Parseval block frame* with respect to $(M_n, \lambda_n)$.

**Definition 2.2.** Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{N}$ which is define in Definition 2.1. Let $\{e_n\}$ be an orthonormal basis of $\ell^2$. Define

$$\ell^2_{M_n} = \left\{ \sum_{j \in M_n} \alpha_j e_j : \alpha_j \text{ are scalars and } n \in \mathbb{N} \right\}.$$

and

$$\left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2} = \left\{ \{u_n\} : u_n \in \ell^2_{M_n} \text{ and } \sum_{n=1}^{\infty} \|u_n\|^2 < \infty \right\}.$$

Notice that $\ell^2_{M_n}$ are subspaces of $\ell^2$, for $n \in \mathbb{N}$. Define inner product in $\left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$ as

$$\langle \{u_n\}, \{v_n\} \rangle = \sum_{n=1}^{\infty} \langle u_n, v_n \rangle, \text{ for } \{u_n\}, \{v_n\} \in \left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}.$$

We can easily show that $\left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$ is a Hilbert space.

In the following result, we give a necessary and sufficient condition for the existence of block frames in Hilbert space $\mathcal{H}$.

**Theorem 2.3.** A sequence $\{x_n\}$ in a Hilbert space $\mathcal{H}$ is a block frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$ if and only if there exists a bounded linear operator $T_B$ from $\left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$

onto $\mathcal{H}$ for which

$$T_B\left(\{\sum_{i\in M_n}\alpha_i e_i\}_{n=1}^{\infty}=\sum_{n=1}^{\infty}\sum_{i\in M_n}\alpha_i\lambda_i x_i\right)$$

*Proof.* Consider

$$\begin{aligned}\left\|\left\{\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}\right\|^2 &= \sum_{n=1}^{\infty}\langle\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i,\sum_{j\in M_n}\langle x,\lambda_j x_j\rangle e_j\rangle\\ &= \sum_{n=1}^{\infty}\sum_{i\in M_n}\bar{\lambda_i}\langle x,x_i\rangle\sum_{j\in M_n}\lambda_j\overline{\langle x,x_j\rangle}\langle e_i,e_j\rangle\\ &= \sum_{n=1}^{\infty}\sum_{i\in M_n}|\lambda_i|^2|\langle x,x_i\rangle|^2.\end{aligned}$$

So, $\left\{\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}\in\left(\sum_{n\in\mathbb{N}}\bigoplus\ell^2_{M_n}\right)_{\ell^2}$. Now, define $Q:\mathcal{H}\to\left(\sum_{n\in\mathbb{N}}\bigoplus\ell^2_{M_n}\right)_{\ell^2}$ by

$$Q(x)=\left\{\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty},\ \text{for } x\in\mathcal{H}.$$

Let $\{u_n\}=\left\{\sum_{i\in M_n}\alpha_i e_i\right\}_{n=1}^{\infty}\in\left(\sum_{n\in\mathbb{N}}\bigoplus\ell^2_{M_n}\right)_{\ell^2}$ and consider

$$\begin{aligned}\langle Q(x),\{u_n\}\rangle &= \left\langle\left\{\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty},\left\{\sum_{j\in M_n}\alpha_j e_j\right\}_{n=1}^{\infty}\right\rangle\\ &= \sum_{n=1}^{\infty}\left\langle\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i,\sum_{j\in M_n}\alpha_j e_j\right\rangle\\ &= \sum_{n=1}^{\infty}\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle\sum_{j\in M_n}\bar{\alpha_j}\langle e_i,e_j\rangle\\ &= \sum_{n=1}^{\infty}\sum_{i\in M_n}\bar{\alpha_i}\langle x,\lambda_i x_i\rangle\\ &= \left\langle x,\sum_{n=1}^{\infty}\sum_{i\in M_n}\alpha_i\lambda_i x_i\right\rangle\end{aligned}$$

So, $Q^* : \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2} \to \mathcal{H}$ is given by

$$Q^*\left(\left\{\sum_{i\in M_n} \alpha_i e_i\right\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty}\sum_{i\in M_n} \alpha_i \lambda_i x_i$$

Thus, from block frame inequalities, $Q$ is one-one and $Q(\mathcal{H})$ is closed. So, by [p. 103], $Q^*$ is onto. Hence, $T_B : \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2} \to \mathcal{H}$ given by

$$T_B\left(\left\{\sum_{i\in M_n} \alpha_i e_i\right\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty}\sum_{i\in M_n} \alpha_i \lambda_i x_i$$

is a bounded linear operator from onto $\left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ on $\mathcal{H}$.

Conversely, let $x \in \mathcal{H}$, then we have

$$\begin{aligned}
\left\langle x, T_B\left(\left\{\sum_{i\in M_n} \alpha_i e_i\right\}_{n=1}^{\infty}\right)\right\rangle &= \left\langle x, \sum_{n=1}^{\infty}\sum_{i\in M_n} \alpha_i \lambda_i x_i\right\rangle \\
&= \sum_{n=1}^{\infty}\sum_{i\in M_n} \bar{\alpha_i}\langle x, \lambda_i x_i\rangle \\
&= \sum_{n=1}^{\infty}\left\langle \sum_{i\in M_n} \langle x, \lambda_i x_i\rangle e_i, \sum_{i\in M_n} \alpha_i e_i\right\rangle \\
&= \left\langle \left\{\sum_{i\in M_n} \langle x, \lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}, \left\{\sum_{i\in M_n} \alpha_i e_i\right\}_{n=1}^{\infty}\right\rangle
\end{aligned}$$

So, $T_B^*(x) = \left\{\sum_{i\in M_n} \langle x, \lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}$, for $x \in \mathcal{H}$. Also,

$$\|T_B^*(x)\|^2 = \left\|\left\{\sum_{i\in M_n} \langle x, \lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}\right\|^2 = \sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 |\langle x, x_i\rangle|^2.$$

Set $\|T_B\|^2 = B$ and clearly we have

$$\sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 |\langle x, x_i\rangle|^2 \leq B\|x\|^2, \text{ for all } x \in \mathcal{H}.$$

Since, $T_B$ is onto, so by [], $T_B^*$ is one-one and $T_B^*(\mathcal{H})$ is closed. Again, by [], there exists a constant $A > 0$ such that

$$A\|x\|^2 \le \|T_B^*(x)\|^2 = \sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 |\langle x, x_i\rangle|^2, \text{ for all } x \in \mathcal{H}.$$

■

**Remark 2.4.** The bounded operator $T_B : \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2} \to \mathcal{H}$ given by

$$T_B\left(\left\{\sum_{i\in M_n} \alpha_i e_i\right\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty}\sum_{i\in M_n} \alpha_i \lambda_i x_i$$

is called the *synthesis operator* of the block frame $\{x_n\}$ and The bounded operator $T_B^* : \mathcal{H} \to \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ given by

$$T_B^*(x) = \left\{\sum_{i\in M_n} \langle x, \lambda_i x_i\rangle e_i\right\}_{n=1}^{\infty}$$

is called the *analysis operator* of the block frame $\{x_n\}$. By composing operators $T_B$ and $T_B^*$, we obtain *frame operator* $S_B = T_B T_B^* : \mathcal{H} \to \mathcal{H}$ given by

$$S_B(x) = \sum_{n=1}^{\infty}\sum_{i\in M_n} \langle x, \lambda_i x_i\rangle \lambda_i x_i = \sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 \langle x, x_i\rangle x_i.$$

**Remark 2.5.** Let $\{x_n\}$ be block frame and $S_B$ be block frame operator. Let $x \in \mathcal{H}$ and we obtain

$$\langle S_B(x), x\rangle = \left\langle \sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 \langle x, x_i\rangle x_i, x\right\rangle = \sum_{n=1}^{\infty}\sum_{i\in M_n} |\lambda_i|^2 |\langle x, x_i\rangle|^2.$$

Thus, from the block frame inequality we have

$$A\langle x, x\rangle \le \langle S_B(x), x\rangle \le B\langle x, x\rangle, \text{ for all } x \in \mathcal{H}.$$

This clearly shows that $S_B$ is bounded linear operator which is also invertible and positive.

Next, we show the block frame coefficients $\{\langle x, S_B^{-1}\lambda_n x_n\rangle\}$ have minimal $\left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ -norm among all sequences representing $x \in \mathcal{H}$.

**Theorem 2.6.** Let $\{x_n\}$ be block frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$ and $x \in \mathcal{H}$. If $x = \sum_{n=1}^{\infty} \sum_{i \in M_n} \alpha_i \lambda_i x_i$, then

$$\sum_{n=1}^{\infty} \sum_{i \in M_n} |\alpha_i|^2 = \sum_{n=1}^{\infty} \sum_{i \in M_n} |\lambda_i|^2 |\langle x, S_B^{-1} x_i \rangle|^2 + \sum_{n=1}^{\infty} \sum_{i \in M_n} |\alpha_i - \lambda_i \langle x, S_B^{-1} x_i \rangle|^2,$$

where $S_B$ is the block frame operator.

*Proof.* We have $x = \sum_{n=1}^{\infty} \sum_{i \in M_n} |\lambda_i|^2 \langle x, S_B^{-1} x_i \rangle x_i$ and we obtain

$$\begin{aligned}
\langle x, S_B^{-1} x \rangle &= \left\langle \sum_{n=1}^{\infty} \sum_{i \in M_n} |\lambda_i|^2 \langle x, S_B^{-1} x_i \rangle x_i, S_B^{-1} x \right\rangle \\
&= \sum_{n=1}^{\infty} \sum_{i \in M_n} |\lambda_i|^2 \langle x, S_B^{-1} x_i \rangle \overline{\langle x, S_B^{-1} x_i \rangle} \\
&= \sum_{n=1}^{\infty} \left\langle \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i, \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\rangle \\
&= \left\langle \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\}, \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\rangle
\end{aligned}$$

Also,

$$\begin{aligned}
\langle x, S_B^{-1} x \rangle &= \left\langle \sum_{n=1}^{\infty} \sum_{i \in M_n} \alpha_i \lambda_i x_i, S_B^{-1} x \right\rangle \\
&= \sum_{n=1}^{\infty} \sum_{i \in M_n} \alpha_i \lambda_i \overline{\langle x, S_B^{-1} x_i \rangle} \\
&= \sum_{n=1}^{\infty} \left\langle \sum_{i \in M_n} \alpha_i e_i, \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\rangle \\
&= \left\langle \left\{ \sum_{i \in M_n} \alpha_i e_i \right\}, \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\rangle
\end{aligned}$$

From the above equations, we get

$$\left\langle \left\{ \sum_{i \in M_n} \alpha_i e_i \right\}, \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\rangle - \left\langle \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\}, \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\rangle$$

is equal to zero. From this we obtain

$$\left\langle \left\{ \sum_{i \in M_n} (\alpha_i - \langle x, S_B^{-1} \lambda_i x_i \rangle) e_i \right\}, \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\rangle = 0$$

So, $\left\{ \sum_{i \in M_n} (\alpha_i - \langle x, S_B^{-1} \lambda_i x_i \rangle) e_i \right\}$ is orthogonal to $\left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\}$. Thus,

$$\begin{aligned} \left\| \left\{ \sum_{i \in M_n} \alpha_i e_i \right\} \right\|^2 &= \left\| \left\{ \sum_{i \in M_n} (\alpha_i - \langle x, S_B^{-1} \lambda_i x_i \rangle) e_i \right\} + \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\|^2 \\ &= \left\| \left\{ \sum_{i \in M_n} (\alpha_i - \langle x, S_B^{-1} \lambda_i x_i \rangle) e_i \right\} \right\|^2 + \left\| \left\{ \sum_{i \in M_n} \langle x, S_B^{-1} \lambda_i x_i \rangle e_i \right\} \right\|^2 \end{aligned}$$

Finally, we get

$$\sum_{n=1}^{\infty} \sum_{i \in M_n} |\alpha_i|^2 = \sum_{n=1}^{\infty} \sum_{i \in M_n} |\alpha_i - \langle x, S_B^{-1} \lambda_i x_i \rangle|^2 + \sum_{n=1}^{\infty} \sum_{i \in M_n} |\lambda_i|^2 |\langle x, S_B^{-1} x_i \rangle|^2.$$

■

**Remark 2.7.** Let $\{x_n\}$ be a block frame for $\mathcal{H}$ with respect to $(M_n, \alpha_n)$. Let $T_B, T_B^*$ be synthesis, analysis operators and $S_B$ be block frame operator. We know that $S_B^{-1} S_B = I_{\mathcal{H}} = S_B^{-1} T_B T_B^*$. Therefore, $T_B^* = T_B^* S_B^{-1} T_B T_B^*$. So, $S_B^{-1} T_B$ is the pseudoinverse of $T_B^*$ and $T_B^* S_B^{-1} T_B$ is a projection from $\left( \sum_{n \in \mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$ onto $T_B^*(\mathcal{H})$.

**Remark 2.8.** If $\{x_n\}$ is a Parseval frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$, then $T_B T_B^* = I_{\mathcal{H}}$, that is $T_B^*$ is isometry.

Now, we define Riesz block frames and Parseval block in Hilbert spaces.

**Definition 2.9.**

(a) A block frame is a Riesz block frame if $T_B^*(\mathcal{H}) = \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$.

(b) A block frame is a orthonormal block frame if $T_B T_B^* = I_{\mathcal{H}}$ and $T_B^* T_B = I_{(\sum_{n\in\mathbb{N}} \oplus \ell^2_{M_n})_{\ell^2}}$

In the following results, we give a characterization of orthonormal block frames and Riesz block frames.

**Theorem 2.10.** Let $\{x_n\}$ be a block frame for $\mathcal{H}$ with respect to $(M_n, \alpha_n)$. Then, $\{x_n\}$ is an orthonormal block frame if and only if $\langle \lambda_i x_i, \lambda_j x_j \rangle = \delta_{ij}$ and $\{x_n\}$ is a Parseval block frame.

*Proof.* Since, $\{x_n\}$ is an orthonormal block frame, so $T_B^* T_B = I_{(\sum_{n\in\mathbb{N}} \oplus \ell^2_{M_n})_{\ell^2}}$ and $T_B T_B^* = I_{\mathcal{H}}$. Also, we have

$$\langle \lambda_i x_i, \lambda_j x_j \rangle = \langle T_B(e_i), T_B(e_j) \rangle = \langle T_B^* T_B e_i, e_j \rangle = \delta_{ij}.$$

Conversely, since $\{x_n\}$ is a Parseval frame, so $T_B T_B^* = I_{\mathcal{H}}$.
Let $\left\{\sum_{i\in M_n} a_i e_i\right\} \in \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ and we have

$$\begin{aligned} T_B^* T_B \left(\left\{\sum_{i\in M_n} a_i e_i\right\}\right) &= T_B^*\left(\sum_{n=1}^{\infty} \sum_{i\in M_n} a_i \lambda_i x_i\right) \\ &= \left\{\sum_{j\in M_k} \lambda_j \langle \sum_{n=1}^{\infty} \sum_{i\in M_n} a_i \lambda_i x_i, x_j \rangle e_j\right\} \\ &= \left\{\sum_{j\in M_k} \sum_{n=1}^{\infty} \sum_{i\in M_n} a_i \langle \lambda_i x_i, \lambda_j x_j \rangle e_j\right\} \\ &= \left\{\sum_{j\in M_n} a_j e_j\right\} \end{aligned}$$

Which shows $T_B^* T_B = I_{(\sum_{n\in\mathbb{N}} \oplus \ell^2_{M_n})_{\ell^2}}$. ■

**Theorem 2.11.** Let $\{x_n\}$ be a block frame for $\mathcal{H}$ with respect to $(M_n, \alpha_n)$. Then, $\{x_n\}$ is a Riesz block frame if and only if $\langle \lambda_i x_i, \lambda_j S_B^{-1} x_j \rangle = \delta_{ij}$.

*Proof.* If, $\{x_n\}$ is a Riesz block frame, then $T_B^* S_B^{-1} T_B = I_{(\sum_{n\in\mathbb{N}} \oplus \ell^2_{M_n})_{\ell^2}}$. Also, we have

$$\langle \lambda_i x_i, S_B^{-1} \lambda_j x_j \rangle = \langle T_B(e_i), S_B^{-1} T_B(e_j) \rangle = \langle T_B^* S_B^{-1} T_B e_i, e_j \rangle = \delta_{ij}.$$

Conversely, let $\left\{ \sum_{i\in M_n} a_i e_i \right\} \in \left( \sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$ and we have

$$\begin{aligned} T_B^* S_B^{-1} T_B \left( \left\{ \sum_{i\in M_n} a_i e_i \right\} \right) &= T_B^* S_B^{-1} \left( \sum_{n=1}^{\infty} \sum_{i\in M_n} a_i \lambda_i x_i \right) \\ &= \left\{ \sum_{j\in M_k} \langle \sum_{n=1}^{\infty} \sum_{i\in M_n} a_i S_B^{-1} \lambda_i x_i, \lambda_j x_j \rangle e_j \right\} \\ &= \left\{ \sum_{j\in M_k} \sum_{n=1}^{\infty} \sum_{i\in M_n} a_i \langle \lambda_i x_i, S_B^{-1} \lambda_j x_j \rangle e_j \right\} \\ &= \left\{ \sum_{j\in M_n} a_j e_j \right\} \end{aligned}$$

Which shows $T_B^* S_B^{-1} T_B = I_{(\sum_{n\in\mathbb{N}} \oplus \ell^2_{M_n})_{\ell^2}}$. ■

**Lemma 2.12.** Let $\{x_n\}$ be a block frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$. Then,

$$\left( \sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2} = ker T_B \oplus T_B^*(\mathcal{H}).$$

*Proof.* From Remark 2.7, we know that $T_B^* S_B^{-1} T_B$ is an orthonormal projection from $\left( \sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2}$ onto $T_B^*(\mathcal{H})$. So, we have

$$\left( \sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2} = ker T_B^* S_B^{-1} T_B \oplus T_B^*(\mathcal{H}).$$

Also, $T_B = T_B T_B^* S_B^{-1} T_B$ and let $\{u_n\} \in Ker T_B^* S_B^{-1} T_B$. So, $T_B^* S^{-1} T_B(\{u_n\}) = 0$. From here we have $T_B(\{u_n\}) = 0$. Thus, $\{u_n\} \in ker T_B$. Also, if $T_B(\{u_n\}) = 0$, then $T_B^* S_B^{-1} T_B(\{u_n\}) = 0$ and $\{u_n\} \in ker T_B^* S_B^{-1} T_B$. Hence,

$$\left( \sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n} \right)_{\ell^2} = ker T_B \oplus T_B^*(\mathcal{H}).$$

■

Next, we show that block frames are compressions of of Riesz block frames.

**Theorem 2.13.** Let $\{x_n\}$ be a block frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$. Then, there exists a Hilbert space $\mathcal{K}$ with $\mathcal{H}$ as a subspace of $\mathcal{K}$ and Riesz block frame $\{y_n\}$ for $\mathcal{K}$ with respect to $(M_n, \lambda_n)$ such that $P_{\mathcal{H}}(y_n) = x_n$, for all $n \in \mathbb{N}$, where $P_{\mathcal{H}}$ is a projection from $\mathcal{K}$ onto $\mathcal{H}$.

*Proof.* Let $\mathcal{K} = \mathcal{H} \oplus kerT_B$ and $P_{kT_B}$ be orthogonal projection from $\left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ onto $kerT_B$. Define $y_i = x_i + \frac{1}{\lambda_i} P_{kT_B}(e_i) \in \mathcal{K} = \mathcal{H} \oplus kerT_B$, $x \in \mathcal{H}$ and $v = \left\{\sum_{i\in M_n} v_i e_i\right\} \in kerT_B$. Define $L_B : \mathcal{K} \to \left(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n}\right)_{\ell^2}$ as

$$L_B(x \oplus v) = \left\{\sum_{i\in M_n} \langle x \oplus v, \lambda_i y_i\rangle e_i\right\}.$$

Notice that

$$\begin{aligned}
\left\{\sum_{i\in M_n} \langle x \oplus v, \lambda_i y_i\rangle\right\} &= \left\{\sum_{i\in M_n} \langle x \oplus v, \lambda_i x_i + P_{kT_B}(e_i)\rangle e_i\right\} \\
&= \left\{\sum_{i\in M_n} (\langle x, \lambda_i x_i\rangle + \langle \left\{\sum_{i\in M_n} v_i e_i\right\}, P_{kT_B}(e_i)\rangle) e_i\right\} \\
&= \left\{\sum_{i\in M_n} (\langle x, \lambda_i x_i\rangle + v_i) e_i\right\}.
\end{aligned}$$

First we will show that $L_B$ is well defined, consider

$$\begin{aligned}
\|L_B(x\oplus v)\|^2 &= \langle\{\sum_{i\in M_n}(\langle x,\lambda_i x_i\rangle+v_i)e_i\},\{\sum_{i\in M_n}(\langle x,\lambda_i x_i\rangle+v_i)e_i\}\rangle\\
&= \sum_{n=1}^{\infty}\langle\sum_{i\in M_n}(\lambda_i\langle x,x_i\rangle+v_i)e_i,\sum_{j\in M_n}(\lambda_j\langle x,x_j\rangle+v_i)e_j\rangle\\
&= \sum_{n=1}^{\infty}\langle\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i+\sum_{i\in M_n}v_ie_i,\sum_{j\in M_n}\langle x,\lambda_j x_j\rangle e_j+\sum_{j\in M_n}v_je_j\rangle\\
&= \sum_{n=1}^{\infty}[\langle\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i,\sum_{j\in M_n}\langle x,\lambda_j x_j\rangle e_j\rangle+\langle\sum_{i\in M_n}v_ie_i,\sum_{j\in M_n}v_je_j\rangle\\
&+ \langle\sum_{i\in M_n}\langle x,\lambda_i x_i\rangle e_i,\sum_{j\in M_n}v_je_j\rangle+\langle\sum_{i\in M_n}v_ie_i,\sum_{j\in M_n}\langle x,\lambda_j x_j\rangle e_j\rangle]\\
&= \sum_{n=1}^{\infty}\sum_{i\in M_n}|\lambda_i|^2|\langle x,x_i\rangle|^2+\sum_{n=1}^{\infty}\sum_{i\in M_n}|v_i|^2<\infty.
\end{aligned}$$

Let $v=P_{kT_B}(b)$, for some $b\in\left(\sum_{n\in\mathbb{N}}\bigoplus\ell^2_{M_n}\right)_{\ell^2}$. Also, let $a=\{\sum_{i\in M_n}a_ie_i\}$ and $x\in\mathcal{H}$. Then, we have

$$\begin{aligned}
\langle L_B(x\oplus v),\{\sum_{i\in M_n}a_ie_i\}\rangle &= \langle\{\sum_{i\in M_n}(\langle x,\lambda_i x_i\rangle+v_i)e_i\},\{\sum_{i\in M_n}a_ie_i\}\rangle\\
&= \sum_{n=1}^{\infty}\langle\sum_{i\in M_n}(\langle x,\lambda_i x_i\rangle+v_i)e_i,\sum_{j\in M_n}a_je_j\rangle\\
&= \sum_{n=1}^{\infty}\sum_{i\in M_n}(\langle x,\lambda_i x_i\rangle+v_i)\sum_{j\in M_n}\bar{a}_i\langle e_i,e_j\rangle\\
&= \sum_{n=1}^{\infty}\sum_{i\in M_n}\bar{a}_i(\langle x,\lambda_i x_i\rangle+v_i)\\
&= \langle x,\sum_{n=1}^{\infty}\sum_{i\in M_n}a_i\lambda_i x_i\rangle+\sum_{n=1}^{\infty}\sum_{i\in M_n}v_i\bar{a}_i\\
&= \langle x,T_B(a)\rangle+\langle P_{kT_B}(b),a\rangle\\
&= \langle x\oplus v,T_B(a)\oplus P_{kT_B}(a)\rangle.
\end{aligned}$$

Thus, we obtain $L_B^*(a)=T_B(a)\oplus P_{kT_B}(a)$. Now, we proceed to show that $L_B^*$ is invertible. Let us first show $L_B^*$ is onto. Let $x\oplus v\in\mathcal{K}$. Since, $T_B$ is onto, so there exists $\xi \in (\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n})_{\ell^2}$ such that $x = T_B(\xi)$. Now, choose $a = \{\sum_{i\in M_n} a_i e_i\} \in (\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n})_{\ell^2}$ in such a way that $a = T_B^* S_B^{-1} T_B(\xi) \oplus v$. Since, $T_B^* S_B^{-1} T_B$ is a projection from $T_B^*(\mathcal{H}) \oplus kerT_B$ onto $T_B^*(\mathcal{H})$, so we obtain

$$T_B^* S_B^{-1} T_B(a) = T_B^* S_B^{-1} T_B(T_B^* S_B^{-1} T_B(\xi) \oplus v) = T_B^* S_B^{-1} T_B(\xi).$$

Let $Q_B = I - T_B^* S_B^{-1} T_B$ be a projection from $T_B^*(\mathcal{H}) \oplus kerT_B$ onto $kerT_B$. Also

$$v = I(a) - T_B^* S_B^{-1} T_B(\xi) = I(a) - T_B^* S_B^{-1} T_B(a) = Q_B(a).$$

Also, $T_B = T_B T_B^* S_B^{-1} T_B$, so we have

$$x = T_B(\xi) = T_B T_B^* S_B^{-1} T_B(\xi) = T_B T_B^* S_B^{-1} T_B(a) = T_B(a).$$

Therefore, $x \oplus v = T_B(a) \oplus Q_B(a) = L_B^*(a)$. Hence, $L_B^*$ is onto. Next, we will show $L_B^*$ is one-one. Let $a = \{\sum_{i\in M_n} a_i e_i\} \in kerL_B^*$, then $L_B^*(a) = 0$. $T_B(a) = 0$ and $Q_B(a) = 0$ that is $a = T_B^* S_B^{-1} T_B(a)$. From here, one can notice that $a = 0$. Hence, $L_B^*$ is one-one. Also, we can easily see that

$$P_{\mathcal{H}}(y_n) = P_{\mathcal{H}}(x_n \oplus \frac{1}{\lambda_n} P_{kT_B}(e_n)) = x_n, \text{ for all } n \in \mathbb{N}.$$

■

In a similar manner, we can also show that Parseval block frames are compressions of orthonormal block frames.

**Theorem 2.14.** Let $\{x_n\}$ be a Parseval block frame for $\mathcal{H}$ with respect to $(M_n, \lambda_n)$. Then, there exists a Hilbert space $\mathcal{K}$ with $\mathcal{H}$ as a subspace of $\mathcal{K}$ and orthonormal block frame $\{y_n\}$ for $\mathcal{K}$ with respect to $(M_n, \lambda_n)$ such that $P_{\mathcal{H}}(y_n) = x_n$, for all $n \in \mathbb{N}$, where $P_{\mathcal{H}}$ is a projection from $\mathcal{K}$ onto $\mathcal{H}$.

*Proof.* Let $\mathcal{K} = \mathcal{H} \oplus kerT_B$ and $P_{kT_B}$ be orthogonal projection from $(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n})_{\ell^2}$ onto $kerT_B$. Define $y_i = x_i + \frac{1}{\lambda_i} P_{kT_B}(e_i) \in \mathcal{K} = \mathcal{H} \oplus kerT_B$, $x \in \mathcal{H}$ and $v = \{\sum_{i\in M_n} v_i e_i\} \in kerT_B$. Define $L_B : \mathcal{K} \to (\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n})_{\ell^2}$ as

$$L_B(x \oplus v) = \{\sum_{i\in M_n} \langle x \oplus v, \lambda_i y_i \rangle\}.$$

As proved in Theorem 2.13, we have

$$\begin{aligned}\|L_B(x \oplus v)\|^2 &= \sum_{n=1}^{\infty}\sum_{i\in M_n}|\lambda_i|^2|\langle x, x_i\rangle|^2 + \sum_{n=1}^{\infty}\sum_{i\in M_n}|v_i|^2 \\ &= \|T_B^*(x)\|^2 + \|v\|^2 = \|x\|^2 + \|v\|^2 \\ &= \|x \oplus v\|^2\end{aligned}$$

Thus, $L_B$ is an isometry that is $L_B^* L_B = I_{\mathcal{K}}$. Also, we know that $Q_B = I - T_B^* T_B$ is a projection from $(\sum_{n\in\mathbb{N}} \bigoplus \ell^2_{M_n})_{\ell^2}$ onto $ker T_B$. And we have

$$\begin{aligned}\langle \lambda_i y_i, \lambda_j y_j\rangle &= \langle \lambda_i (x_i + \frac{1}{\lambda_i} Q_B(e_i)), \lambda_j (x_j + \frac{1}{\lambda_i} Q_B(e_j))\rangle \\ &= \langle \lambda_i x_i, \lambda_j x_j\rangle + \langle Q_B(e_i), Q_B(e_j)\rangle \\ &= \langle \lambda_i x_i, \lambda_j x_j\rangle + \langle e_i, Q_B(e_j)\rangle \\ &= \langle \lambda_i x_i, \lambda_j x_j\rangle + \langle e_i, (I - T_B^* T_B)(e_j)\rangle \\ &= \langle \lambda_i x_i, \lambda_j x_j\rangle + \langle e_i, e_j\rangle - \langle e_i, T_B^* T_B e_j\rangle \\ &= \langle \lambda_i x_i, \lambda_j x_j\rangle + \langle e_i, e_j\rangle - \langle T_B e_i, T_B e_j\rangle \\ &= \langle e_i, e_j\rangle = \delta_{ij}.\end{aligned}$$

■

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