Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 7 (2017), pp. 2997-3004

http://www.ripublication.com

Fixed Point Theorem in Cone B-Metric Spaces Using Contractive Mappings

Neetu Sharma*
**Maulana Azad National Institute of Technology, Bhopal (M.P.) India.*

Abstract

The concept of a cone b-metric space has been introduced recently as a generalization of a b-metric space and a cone metric space in 2011. The aim of this paper is to prove fixed point theorem of contractive mapping in cone b-metric spaces without using the normality condition. This result improves and generalizes some fixed point results in metric spaces and b-metric spaces and also expand the result in cone metric spaces.

Keyword: Fixed point, Cone metric, Cone b-metric, Contractive mapping, partial ordering.
2000 AMS Subject Classification No.: 54H25, 47H10.

INTRODUCTION

Fixed point theory plays a basic role in applications of many branches of mathematics. Finding a fixed point of contractive mappings becomes the centre of strong research activity. The concept of b-metric spaces introduced by Bakhtin[3]. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. In recent investigations, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects (see [7-10]). The author defines an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. In [7], Huang and Zhang introduced cone metric spaces as a generalization of metric spaces. Moreover, they proved some fixed point theorems for contractive mappings that expanded certain results of fixed points in metric spaces. In [10], Hussain and Shah introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Throughout this paper, I proved fixed point theorem of contractive mapping using rational expression without the assumption of normality in cone b-metric spaces.

PRELIMINARIES:

The following definitions and Lemmas will be needed in the sequel.

Let E be a real Banach space and P be a subset of E .We denote the zero element of E by θ and the interior P by int P. The subset P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(iii) $P \cap (-P) = \{\theta\}$.

We define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. we shall write x < y to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int\ P$. Write $\| . \|$ as the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E,\ \theta \leq x \leq y$ implies $\| x \| \preccurlyeq K \| y \|$. The least positive number satisfying the above is called the normal constant of P. It is well known that $K \geq 1$.

In the following, we always suppose that E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\preccurlyeq$ is a partial ordering with respect to P.

Definition 1.1. Let X be a nonempty set. Suppose that the mappings $d: X \times X \to E$ satisfies:

(d1) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.1. Let $\mathrm{E} = \mathrm{R}^2,\ \mathrm{P} = \{ (\mathrm{u}, \mathrm{v}) \in \mathrm{E} / \mathrm{u},\ \mathrm{v} \geq 0\} \subset \mathrm{R}^2,\ \mathrm{X} = \mathrm{R}$ and $\mathrm{d}: \mathrm{X} \times \mathrm{X} \to \mathrm{E}$

such that $\mathrm{d}\ (\mathrm{u}, \mathrm{v}) = \{|\mathrm{u} - \mathrm{v}|, \alpha|\mathrm{u} - \mathrm{v}|\}$, where $\alpha \leq 0$ is a constant.

Then (X, d) is a cone metric space.

Definition 1.2. Let X be a nonempty set and $\mathrm{s} \geq 1$ be a given real number. A mapping $\mathrm{d}: \mathrm{X} \times \mathrm{X} \to \mathrm{E}$ is said to be cone b-metric if and only if, for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{X}$, the following conditions are satisfied:

(d1) $\theta < d(x,y)$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Then pair (X, d) is called a cone b-metric spaces.

Example 1.2.

Let $X = l^p \; with \; 0 < p < 1 \; where \; l^p = \left\{\{u_n\} \subset R: \sum_{n=1}^{\infty} |u_n|^p < \infty\right\}$. Let $d: X \times X \to R$ be defined as $d(u,v) = (\sum_{n=1}^{\infty} |u_n - v_n|^p)^{1/p}$, where $u = \{u_n\}$ and $v = \{v_n\} \in l^p$.

Then (X, d) is a b-metric space.

Put $E = l^1, P = \{\{u_n\} \in E: u_n \geq 0, for \; all \; n \geq 1\}$

Letting the map $d^*: X \times X \to E$ be defined by $d^* (u.v) = \left\{\frac{d(u,v)}{2^n}\right\}$ for n ≥ 1.

We conclude that (X, d^*) is a cone b-metric space with the coefficient $r = 2^{1/p} > 1$,but it is not a cone metric space.

It is clear that the class of cone b-metric space is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. **Therefore, it is obvious that cone b-metric spaces generalized b-metric spaces and cone metric spaces.**

Definition 1.3. Let (X, d) be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d\,(x_n, x) \ll c$ for all $n \geq N$. we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x \; (n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d\,(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

In particular when we dealing with Cone b-metric spaces in which Cone need not be normal the following lemmas are generally used.

Lemma 1.1: Let P be a cone and $\{a_n\}$ be a sequence in E. If $c \in \text{int}\, P$ and $\theta \leq a_n \to \theta$ (as $n \to \infty$), then there exists N such that for all n > N, we have $a_n \ll c$.

Lemma 1.2: Let $x, y, z \in E$, if $x \leq y$ and $y \ll z$, then $x \ll z$.

Lemma 1.3: Let P be a cone and $\theta \leq u \ll c$ for each $c \in$ int P, then $u = \theta$.

Lemma 1.4: Let P be a cone, if $u \in p$ and $u \leq ku$ for some $0 \leq k < 1$, then $u = \theta$.

Lemma 1.5: Let P be a cone and $a \leq b + c$ for each $c \in$ int P, then $a \leq b$.

MAIN RESULT

In this section, I will present some fixed point theorems for contractive mappings in the setting of cone b-metric spaces.

Theorem 1.1: Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T: X \rightarrow X$ satisfies the following condition

$$d(Tx, Ty) \leq \alpha \left[\frac{d(x,Tx)\, d(y,Ty)}{d(x,y)}\right] + \beta\, d(x,y) \quad (1)$$

for $x, y \in X$ and $\alpha, \beta \in [0,1]$ with $\alpha+\beta<1$. Then T has a unique fixed point in X.

Proof: Choose $x_0 \in X$. We construct a sequence $\{x_n\}$ by using iterative method, where $x_n = Tx_{n-1}$, $n \geq 1$,

i. e. $x_{n+1} = Tx_n = T^{n+1}x_0$.

Now from equation (1)

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \alpha \left[\frac{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)}{d(x_{n-1}, x_n)}\right] + \beta\, d(x_{n-1}, x_n)$$

$$\leq \alpha\, d(x_n, Tx_n) + \beta\, d(x_{n-1}, x_n)$$

$$\leq \alpha\, d(x_n, x_{n+1}) + \beta\, d(x_{n-1}, x_n)$$

$$(1-\alpha) d(x_n, x_{n+1}) \leq \beta\, d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{\beta}{(1-\alpha)}\, d(x_{n-1}, x_n)$$

Let $k = \frac{\beta}{(1-\alpha)}$, $\alpha + \beta < 1$, $0 < k < 1$

then $d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n)$

Now by induction

$d(x_n, x_{n+1}) \leq \ k\ d(x_{n-1}, x_n)$

.

.

.

.

$d(x_n, x_{n+1}) \leq \ k^n\ d(x_0, x_1)$

Now for any $m \geq 1, p \geq 1$ it follows that

$$d\big(x_m, x_{m+p}\big) \leq s\big[d(x_m\ , x_{m+1}) +\ d(x_{m+1}, x_{m+p})\big]$$

$$\leq sd(x_m\ , x_{m+1}) +\ sd(x_{m+1}, x_{m+p})$$

$$\leq sd(x_m\ , x_{m+1}) +\ s^2\big[d(x_{m+1}, x_{m+2}) + d\big(x_{m+2}, x_{m+p}\big)\big]$$

..

..

..

$$\leq s^1k^m d(x_0\ , x_1) +\ s^2k^{m+1}d(x_0, x_1) +\ s^{p-1}k^{m+p-2}d(x_0, x_1) +\ s^{p-1}k^{m+p-1}d(x_0, x_1)$$

$$\leq \frac{s^pk^{m+1}}{s-k}d(x_0\ , x_1) +\ s^{p-1}k^m d(x_0, x_1) \quad \to \theta\ \ as\ m \to \infty \text{ for any p.}$$

Now by lemma **1.1** we obtained $m_0 \in N$ such that

$$\frac{s^pk^{m+1}}{s-k}d(x_0\ , x_1) +\ s^{p-1}k^m d(x_0, x_1) \quad \ll c,\ \ for\ each\ m > m_0$$

Thus

$$d\big(x_m, x_{m+p}\big) \leq \quad \frac{s^pk^{m+1}}{s-k}d(x_0\ , x_1) +\ s^{p-1}k^m d(x_0, x_1) \quad \ll c$$

for all $\boldsymbol{m > m_0}$, for any p.

Using lemma **1.2** $\{x_n\}$ is a Cauchy sequence in (X, d).

Since (X, d) is a complete cone b-metric space, there exists u$\in X$ such that $x_n \to u$. Take $n_0 \in N$ such that

$$d\,(x_n, u) \ll \frac{c}{s(k+1)} \qquad \textbf{for all } n > n_0.$$

Hence

$$d(Tu, u) \;\leq s[d(Tu, Tx_n) + \; d(Tx_n, u)]$$

Using inequality (1) we have

$$d(Tu, u) \ll c \text{ for each } n \; > \; n_0,$$

Now by lemma **1.3**, we obtained that

$$d(Tu, u) = \theta \quad i.e. \;\; Tu = u$$

That is u is a fixed Point of T.

Uniqueness

Now we show that the fixed point is unique.

Suppose $v \in X$ be another fixed point then by condition (1) and lemma (1.4) we have

$$u \; = \; v$$

Hence T has a unique fixed Point in X.

The proof is completed.

Definition1.4: Suppose P is a normal cone b-metric space in E then mapping $\phi\colon [a, b] \to P$ is said to be an integrable on [a, b] with respect to cone integrable function if and only if for all partition Q of [a, b].

$$\lim_{n\to\infty} l_n^{cone}\;(\phi, Q) = \; S^{cone} = \; \lim_{n\to\infty} u_n^{cone}\,(\phi, Q)$$

Where S^{cone} must be unique and $l_n^{cone} = \sum_{i=0}^{n-1} \phi(x_i)\;\; \| x_i \; - \; x_{i+1} \|$ and

$$u_n^{cone} = \; \sum_{i=0}^{n-1} \phi(x_{i+1})\;\; \| x_i \; - \; x_{i+1} \|$$

$$S^{cone} \text{ be denoted by } \; S^{cone} = \int_a^b \phi(x)\; dP(x) = \int_a^b \phi\; dP$$

where $L'([a, b],\; P)$ is the set of all cone integrable function $\phi\colon [a, b] \to P$

Theorem 1.2: Let (X, d) be a cone b-metric space and p be a normal cone. Let $\phi: P \rightarrow P$ be a non zero mappings. Let $T: X \rightarrow X$ be a mapping such that for all $x, y \in X$ be a mapping such that for all $x, y \in X$

$$\int_0^{d(Tx,\ Ty)} \phi(t)\, dpt \leq \alpha \int_0^{M(x,\ y)} \phi(t)\, dpt$$

Where $M\,(x,y) = \alpha\, \frac{d(x,Tx)\, d(y,Ty)}{d\,(x,y)} + \beta\, d(x,y)$ with $\alpha, \beta \in [0,1]$ and $\alpha + \beta < 1.$ Also suppose that the function of $\int_0^y \phi\, dP$ is invertible and that the inverse is continuous in 0.

Then T has a unique fixed point.

REFERENCES

[1]. Deimling, K; Nonlinear Functional Analysis. Springer, Berlin (1985).
[2]. Banach, S; sur les operations dans les ensembles abstrait et leur application aux equations, integrals. Fundam. Math. 3, 133-181 (1922).
[3]. Bakhtin, IA; the contraction mapping principle in atmost metric spaces. Funct. Anal., Gos. Ped. Inst. Unianowsk 30, 26-37(1989).
[4]. Czerwik, S: Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin.Mat Fis. Univ.Modena 46, 263-276 (1998).
[5]. Boriceanu, M., Bota M., Petrusel A.; Multivalued fractals in b-metric spaces. Cent. Eur. J. Math. 8(2), 367-377 (2010).
[6]. Bota M., Molnar A., Csaba V.; On Ekeland's variational principle in b-metric spaces. Fixed Point Theory 12, 21-28 (2011).
[7]. Huang L-G, Zhang X.; Cone metric spaces and fixed point theorem of contractive mappings. J. Math. Anal. Appl. 332(2), 1468-1476 (2007).
[8]. Rezapour S., Hamlbarani R.; Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. J. Math. Anal. Appl. 345, 719-724 (2008).
[9]. Jankovic S., Kadelburg Z., Radenovic S.; on cone metric spaces: a survey. Nonlinear Anal. 4(7), 2591-2601(2011).
[10]. Hussain N., Shah MH,; KKM mappings in cone b-metric spaces, a survey Nonlinear Anal. 4(7), 2591-2601 (2011).

[11]. Cho, S-H, Bae, J-S; Common fixed point theorems for mappings satisfying property (E.A.) on cone metric space. Math. Comput. Model. 53, 945-951 (2011).