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Approximate Solution of Real Definite Integrals Over a Circle in Adaptive Environment

Kumudini Meher, Saumya Ranjan Jena and Arjun Kumar Paul

Dept of Mathematics, School of Applied Sciences,
KIIT University, Bhubaneswar-751024 , Odisha, India.

Abstract

In this note, a mixed quadrature rule with an adaptive scheme is implemented over the circular surface in the Cartesian two dimensional space. The double mathematical transformations which transform the circular surface to a standard square space. The mixed quadrature rule has been tested in adaptive scheme taking 5 numerical tests and it is found to be more effective than that of Boole's rule .

Keywords: Mixed quadrature rule, Degree of precision, Error bound, Adaptive quadrature scheme, Circular region.

MSC 2010: 65D30, 65D32

1. INTRODUCTION

The applications of mixed quadrature rule for the approximation of real integrals $I(f)=\int_c^d f(x)dx$ and $I(f)=\int_c^d\int_e^f f(x,y)dy\,dx$ have been used by several authors [1,2].The symmetric Gaussian quadrature formula for integrating arbitrary functions of two variables over the surface of a triangle was proposed by [3,4]. The symmetric integration formula with higher order precision up to degree ten was given by [5,6]. An alternative integration formula for triangular finite elements was proposed by [7]. Lastly, the mixed quadrature on real definite integrals with finite element methods has been suggested by [8].P. Dash and S.R. Jena [9] had implemented the mixed quadrature on real definite integrals over triangles and spheres with finite element methods respectively. The generalised Gaussian quadrature formula for integrating arbitrary functions of two variables over the surface of a circle was proposed by Shivaram [10].

In this paper we adopt an application of mixed quadrature with adaptive scheme over circular surface $\left\{(x,y): -a \le x \le a, -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}\right\}$ in the Cartesian two dimensional (x,y) space. The mathematical transformation from (x,y) space to (u,v) space maps the standard circle in (x,y)space to a standard 2-square space $\{(u,v): 0 \le u,v \le 1\}$. Then the another transformation transforms the 2-square (u,v) space to the space $\{(\xi,\eta): -1 \le \xi,\eta \le 1\}$.Here taking the advantage of the fact that Boole's rule and Lobatto four-point rule are of same precision (i.e. precision 5), a mixed quadrature rule of higher precision (i.e. precision 7) has been obtained by taking the linear combination of these rules. The mixed quadrature rule so formed has been tested on different definite integrals giving better results than Boole's rule in adaptive scheme.

For a real integrable function g, an interval $[l,m]$ and a prescribed tolerance ε, it is desired to compute an approximation B to the integral $I = \int_l^m g(x)dx$ so that $|B - I| \le \varepsilon$ .The basic principle for adaptive quadrature is the additive property of a definite integral of the form $Q = R + S$ with the adaptive integration schemes [11,12,13]. $Q = \int_l^m g(x)dx$ $R = \int_l^r g(x)dx$ $S = \int_r^m g(x)dx$

Where r is any point between l and m.

In adaptive integration, the points at which the integrand is evaluated are so chosen in such a way that depends on the nature of the integrand. The idea is that if we can approximate each of the two integrals R and S within a specified tolerance, then the sum Q gives us the desired result. If not, we can recursively apply the additive property to each of the intervals $[l,r]$ and $[r,m]$.Adaptive subdivision of course has geometrical appeal. It seems intuitive that points should be concentrated in regions where the integrand is badly behaved. The whole interval rules can take no direct account of this.

This paper is designed as follows. Sec-1 contains Introduction. In sec-2, we discuss about the integrals of an arbitrary function over a quarter circular region. In sec-3, we will construct a mixed quadrature rule of degree of precision seven by taking two constituent rules Boole's rule and Lobatto four-point rule each of degree of precision five. The error analysis and error bound are introduced in sec-4. The numerical verification of our proposed rule is experimented on some suitable real integrals in sec-5 and finally the conclusion follows in sec-6.

2. CONSTRUCTION OF INTEGRALS OVER A QUARTER CIRCULAR REGION

The quarter circular region is transformed to a unit square.

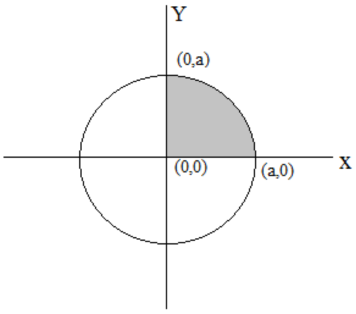


Fig-1 (circular region)

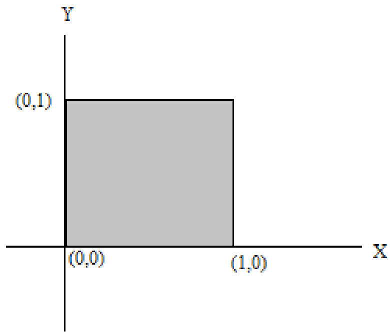


Fig-2 (square region)

The numerical integration of an arbitrary function f over a quarter circular region is,

$$I = \iint_C f(x,y)dxdy = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx \qquad (2.1)$$

The integral I of eqn (2.1) can be transformed into an integral over the surface of the square $\{(u,v), 0 \le u, v \le 1\}$ by substitution,

$$x = au, y = av\sqrt{1-u^2}$$

The determinant of the Jacobian and differential area are,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = a^2\sqrt{1-u^2}$$

Area $= dxdy = \dfrac{\partial(x,y)}{\partial(u,v)} dudv = a^2\sqrt{1-u^2}\,du\,dv$

Now eqn (2.1) becomes,

$$I = \int_0^1\int_0^1 f\left(au, av\sqrt{1-u^2}\right)a^2\sqrt{1-u^2}\,du\,dv \qquad (2.2)$$

The integral I of eqn (2.2) can be further transformed into an integral over the standard 2-square: $\{(\xi,\eta), -1 \le \xi, \eta \le 1\}$ by monomial transformation,

$$u = \frac{1+\xi}{2}, v = \frac{1+\eta}{2} \qquad (2.3)$$

Then the determinant of the Jacobian and the differential area are

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} = \frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} = \frac{1}{2}\times\frac{1}{2} - 0\times 0 = \frac{1}{4}$$

$$dudv = \frac{\partial(u,v)}{\partial(\xi,\eta)} d\xi d\eta = \frac{1}{4} d\xi d\eta \qquad (2.4)$$

Now using eqn (2.3) and eqn (2.4) in eqn (2.2),

$$I = \int_{-1}^{1}\int_{-1}^{1} f\left(a\left(\frac{1+\xi}{2}\right), a\left(\frac{1+\eta}{2}\right)\sqrt{1-\left(\frac{1+\xi}{2}\right)^2} \right) \frac{a^2}{4}\sqrt{1-\left(\frac{1+\xi}{2}\right)^2}\, d\xi\, d\eta$$

Where a is the radius of the circle.

3. CONSTRUCTION OF MIXED QUADRATURE RULE OF DEGREE OF PRECISION SEVEN

Here the mixed quadrature rule of degree of precision seven is formed by taking the linear convex combination of two constituent rules i.e Boole's rule and Lobatto four point rule each of degree of precision five.

For approximate evaluation of real definite integral $I(f) = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy \qquad (3.1)$

We choose Boole's rule,

$$I(f) \cong R_B(f) = \frac{1}{2025}\left[\begin{array}{l} 7\left\{7f(1,1)+7f(1,-1)+32f\left(1,\frac{1}{2}\right)+32f\left(1,-\frac{1}{2}\right)+12f(1,0)\right\}+ \\ 7\left\{7f(-1,1)+7f(-1,-1)+32f\left(-1,\frac{1}{2}\right)+32f\left(-1,-\frac{1}{2}\right)+12f(-1,0)\right\}+ \\ 32\left\{7f\left(\frac{1}{2},1\right)+7f\left(\frac{1}{2},-1\right)+32f\left(\frac{1}{2},\frac{1}{2}\right)+32f\left(\frac{1}{2},-\frac{1}{2}\right)+12f\left(\frac{1}{2},0\right)\right\}+ \\ 32\left\{7f\left(-\frac{1}{2},1\right)+7f\left(-\frac{1}{2},-1\right)+32f\left(-\frac{1}{2},\frac{1}{2}\right)+32f\left(-\frac{1}{2},-\frac{1}{2}\right)+12f\left(-\frac{1}{2},0\right)\right\}+ \\ 12\left\{7f(0,1)+7f(0,-1)+32f\left(0,\frac{1}{2}\right)+32f\left(0,-\frac{1}{2}\right)+12f(0,0)\right\} \end{array}\right] \qquad (3.2)$$

Lobatto- four point rule,

$$I(f)\cong R_{L4}(f)=\frac{1}{36}\left[\begin{array}{l}\left\{f(1,1)+5f\left(1,\frac{1}{\sqrt{5}}\right)+5f\left(1,-\frac{1}{\sqrt{5}}\right)+f(1,-1)\right\}+\\ 5\left\{f\left(\frac{1}{\sqrt{5}},1\right)+5f\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)+5f\left(\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)+f\left(\frac{1}{\sqrt{5}},-1\right)\right\}+\\ 5\left\{f\left(-\frac{1}{\sqrt{5}},1\right)+5f\left(-\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)+5f\left(-\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)+f\left(-\frac{1}{\sqrt{5}},-1\right)\right\}+\\ \left\{f(-1,1)+5f\left(-1,\frac{1}{\sqrt{5}}\right)+5f\left(-1,-\frac{1}{\sqrt{5}}\right)+f(-1,-1)\right\}\end{array}\right] \quad (3.3)$$

where each rule of eqn(3.2) and eqn(3.3) is of degree of precision five.

Expanding each term of eqn(3.2) and eqn(3.3) using Maclaurin's series,

$$\begin{aligned}I(f)&=R_B(f)+E_B(f)\\ I(f)&=R_{L4}(f)+E_{L4}(f)\end{aligned} \quad (3.4)$$

where

$$\begin{aligned}R_B(f)&=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]\\ &+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{2}{3\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\\ &+\frac{1}{3600}f_{4,4}(0,0)+\frac{56}{9\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{19}{30\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots\end{aligned} \quad (3.5)$$

$$\begin{aligned}R_{L4}(f)&=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]\\ &+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{52}{75\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\\ &+\frac{1}{3600}f_{4,4}(0,0)+\frac{1456}{225\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{84}{125\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots\end{aligned} \quad (3.6)$$

We can write eqn(3.1) using Maclaurin's expansion,

$$\begin{aligned}I(f)&=4f_{0,0}(0,0)+\frac{2}{3}\left[f_{2,0}(0,0)+f_{0,2}(0,0)\right]+\frac{1}{30}\left[f_{4,0}(0,0)+f_{0,4}(0,0)\right]\\ &+\frac{1}{9}f_{2,2}(0,0)+\frac{1}{180}\left[f_{4,2}(0,0)+f_{2,4}(0,0)\right]+\frac{4}{7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right]+\\ &+\frac{1}{3600}f_{4,4}(0,0)+\frac{1}{7560}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right]+\frac{4}{9!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right]+\ldots\end{aligned} \quad (3.7)$$

Error associated with Boole's rule and Lobatto four-point rule respectively are,

$$E_B(f) = I(f) - R_B(f)$$
$$= -\frac{2}{21\times 6!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] - \frac{8}{9\times 8!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right] - \frac{17}{90\times 8!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right] \quad (3.8)$$

$$E_{L4}(f) = I(f) - R_{L4}(f)$$
$$= -\frac{64}{75\times 7!}\left[f_{6,0}(0,0)+f_{0,6}(0,0)\right] - \frac{256}{25\times 9!}\left[f_{6,2}(0,0)+f_{2,6}(0,0)\right] - \frac{256}{125\times 9!}\left[f_{8,0}(0,0)+f_{0,8}(0,0)\right] \quad (3.9)$$

Now multiplying $\left(\frac{32}{25}\right)$ in eqn (2.4) and subtracting eqn (2.5) from eqn (2.4) we get,

$$I(f) = R_{BL4}(f) + E_{BL4}(f) \quad (3.10)$$

where $R_{BL4}(f) = \frac{1}{7}\left[32R_B(f) - 25R_{L4}(f)\right]$ (3.11)

This is the required mixed quadrature rule.

The error associated with the rule $R_{BL4}(f)$ is,

$$E_{BL4}(f) = \frac{1}{7}\left[32E_B(f) - 25E_{L4}(f)\right]$$

4. ERROR ANALYSIS

The error analysis of the said mixed quadrature rule can be represented by the following Theorems.

Theorem-4.1

Let $f(x,y)$ be sufficiently differentiable function in the closed interval $[-1,1]$. The bound of truncated error $E_{BL4}(f)$ associated with the rule $R_{BL4}(f)$ is given by $\left|E_{BL4}(f)\right| \cong \frac{16}{315\times 8!}\left|f_{8,0}(0,0)+f_{0,8}(0,0)\right|$.

Proof- From eqn (3.10),

$$I(f) = R_{BL4}(f) + E_{BL4}(f)$$

where $R_{BL4}(f) = \frac{1}{7}\left[32R_B(f) - 25R_{L4}(f)\right]$

$$E_{BL4}(f) = \frac{1}{7}\left[32E_B(f) - 25E_{L4}(f)\right]$$

Hence $\left|E_{BL4}(f)\right| \cong \frac{16}{315\times 8!}\left|f_{8,0}(0,0)+f_{0,8}(0,0)\right|$

Theorem-4.2

The bounds for the truncated error $E_{BL4}(f)=I(f)-R_{BL4}(f)$ is $\left|E_{BL4}(f)\right| \le \frac{64M}{147\times 6!}\left|\eta_2-\eta_1\right|$ for $\eta_1,\eta_2 \in [-1,1]$ and $M=\max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|f_{7,0}(x,*)+f_{0,7}(*,y)\right|$

Proof: We have $E_B(f)=-\frac{2}{21\times 6!}\left[f_{6,0}(\eta_1,0)+f_{0,6}(0,\eta_1)\right]$

$E_{L4}(f)=-\frac{64}{525\times 6!}\left[f_{6,0}(\eta_2,0)+f_{0,6}(0,\eta_2)\right]$ where $\eta_1,\eta_2\in[-1,1]$

$$E_{BL4}(f)=\frac{1}{7}\left[32E_B(f)-25E_{L4}(f)\right]$$

$$=\frac{64}{147\times 6!}\left[f_{6,0}(\eta_2,0)+f_{0,6}(0,\eta_2)-f_{6,0}(\eta_1,0)-f_{0,6}(0,\eta_1)\right]$$

$$=\frac{64}{147\times 6!}\left[\int_{\eta_1}^{\eta_2} f_{7,0}(x,0)\,dx+\int_{\eta_1}^{\eta_2} f_{0,7}(0,y)\,dy\right]$$

$$=\frac{64}{147\times 6!}\int_{\eta_1}^{\eta_2}\int_{\eta_1}^{\eta_2}\left[f_{7,0}(x,*)+f_{0,7}(*,y)\right]dx\,dy$$

$$\left|E_{BL4}(f)\right| \le \frac{64M}{147\times 6!}\left|\eta_2-\eta_1\right|$$

Where $M=\max\limits_{\substack{-1\le x\le 1\\ -1\le y\le 1}}\left|f_{7,0}(x,*)+f_{0,7}(*,y)\right|$

Which gives only the truncational error bound as η_1,η_2 are unknown points in $[-1,1]$ and it is obvious that the error in approximation will be less if the points η_1,η_2 are closed to each other.

Corollary-4.1

The error bound for the truncational error $E_{BL4}(f)$ is given by $\left|E_{BL4}(f)\right| \le \frac{128M}{147\times 6!}$ when $\left|\eta_1-\eta_2\right|\le 2$ [14].

5. NUMERICAL VERIFICATION

Here there is a comparison of mixed quadrature rule with Boole's rule for approximation of some real double definite integrals in adaptive routine. The integrals under considerations are,

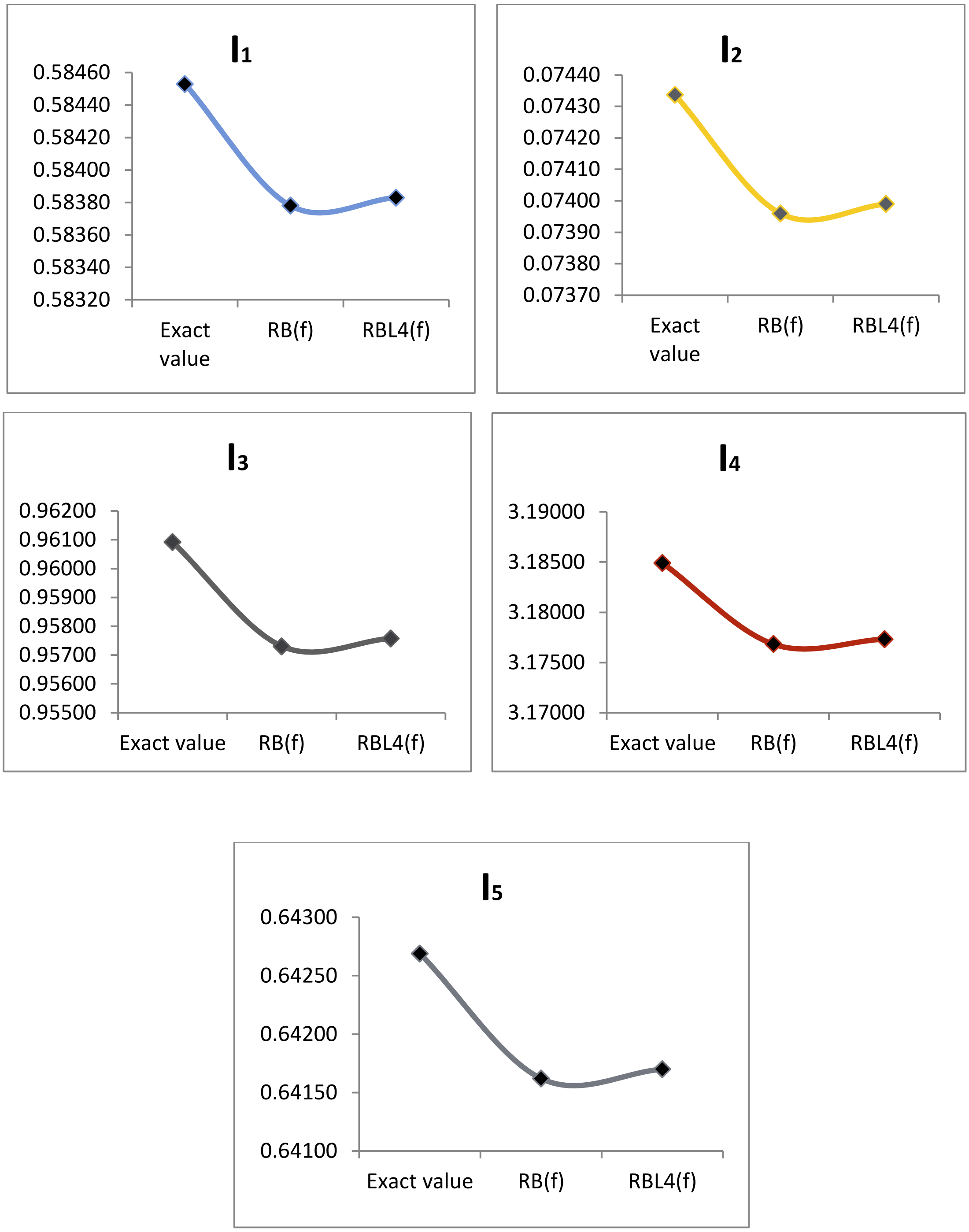
$$I_1 = \int_0^1 \int_0^{\sqrt{1-x^2}} (1+x+y)^{-\frac{1}{2}} dy\, dx \,, I_2 = \int_0^{0.5} \int_0^{\sqrt{(0.5)^2-x^2}} e^{-y^2} \sin(x+y) dy\, dx \,,$$

$$I_3 = \int_0^2 \int_0^{\sqrt{4-x^2}} \frac{\log(4+x+y)}{(4+x+y)} dy\, dx \,, I_4 = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{(x+2y)} dy\, dx \,, I_5 = \int_0^1 \int_0^{\sqrt{1-x^2}} (x+y)^2\, dy\, dx \,.$$

Table-1

(Comparison of mixed quadrature rule with Boole's rule in approximation of some real definite integrals over the quarter circle in adaptive quadrature method with stopping criterion ε)

Exact value of the integrals	*Boole's rule* $R_B(f)$ *by adaptive quadrature method*	*No of Intervals for* $R_B(f)$	*Mixed quadrature rule* $R_{BL4}(f)$ *by adaptive quadrature method*	*No of Intervals for* $R_{BL4}(f)$	*Maximum admissible absolute error* (ε)
I_1=0.584537736438196	0.583784028750921	5	0.583839740762067	3	$\varepsilon_1 = 0.00069799$
I_2=0.074337571387362	0.073967462858595	2	0.073997357836509	1	$\varepsilon_2 = 0.00034021$
I_3=0.960923721236226	0.957317245340701	2	0.957584271666643	1	$\varepsilon_3 = 0.00333944$
I_4=3.184910663940983	3.176830034273946	2	3.177330715007788	1	$\varepsilon_4 = 0.00757994$
I_5=0.642699001698724	0.641622176853936	5	0.641702636785224	3	$\varepsilon_5 = 0.00099636$



Comparison of Exact value with $R_B(f)$ **and** $R_{BL4}(f)$ **of** I_1, I_2, I_3, I_4 **and** I_5.

6. CONCLUSION

The effectiveness of mixed quadrature rule is obvious from above examples in Table-1 and from the graphical representations in adaptive scheme. The mixed quadrature rule $R_{BL4}(f)$ reduces the number of steps required to approximate an integral in adaptive quadrature method in comparison to its constituent Boole's rule. This work may be extended to any circular region instead of quarter circle for any arbitrary radius for the real definite integrals with the application of mixed quadrature rule with adaptive quadrature scheme.

REFERENCES

[1]. A.K. Tripathy, R.B. Dash etal (2015), "A mixed quadrature rule blending Lobatto and Gauss-Legendre three-point rule for approximate evaluation of real definite integrals ", Int-J-Computing science and Mathematics,6(1),366-377.

[2]. S.R. Jena , S.C. Mishra (2016), "Mixed quadrature rule for double integrals of Simpson's $\frac{1}{3}rd$ and Gauss-Legendre two-point rule in two variables ", Global journal of Science frontier research,vol-16,issue-1,Online ISSN: 2249-4626 and Print ISSN: 0975-5896,31-37.

[3]. P.C. Hammer and A.H. Stroud (1958), " Numerical evaluation of multiple integrals" , Math. Tables other Aids Computation, 12, 272-280.

[4] G.R. Cowper (1973),"Gaussian quadrature formulae for triangles", Int. J.Num.Math.Engg,7,405-408.

[5]. J.N. Lyness and D. Jespersen (1975), "Moderate Degree Symmetric Quadrature Rules for the Triangle", J.Inst.Math.Applic,15,19-32.

[6]. F.G. Lannoy (1977), "Triangular finite elements and numerical integration", Computer Struct, 7,613.

[7]. C.T. Reddy and D.J. Shippy (1981), "Alternative integration formulae for triangular finite elements", Int.J.Num.Math.Engg, 17, 133-139.

[8]. S.R. Jena and R.B. Dash (2009) , "Mixed quadrature of real definite integral over triangles", Pacific-Asian Journal of Mathematics,vol-3,1-2.

[9]. P. Dash and S.R. Jena(2015), "Mixed quadrature over sphere", Global Journal of Pure and Applied Mathematics,vol-11,415-425.

[10]. K.T. Shivaram (2013), "Generalised Gaussian quadrature over a sphere", Int. Journal of Scientific and Engg. Research, 4, 1530-1534.

[11]. Walter G, Walter G, "Adaptive quadrature – Revisited ", BIT Numerical Mathematics, 2000,40(1), pp .84-109.

[12]. Oliver J. “A doubly adaptive Clenshaw-Curtis quadrature method ”, Computing centre, University of Essex, Wivenhoe park, Colchester, Essex. 1971,15(2), pp.141-147.

[13]. P. Patra , D. Das etal, “Approximation of Singular Integrals by a Mixed Quadrature of Anti-Gauss and Steffensen’s Quadrature Rules in the Adaptive Environment ”, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 11, Number 1 (2016), pp. 79-92

[14]. S. Conte and C. De. Boor (1980),“Elementary Numerical Analysis”, Tata Mac-Graw Hill.