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Common Fixed Point for Generalized - (ψ, α, β)-Weakly Contractive Mappings in Dislocated Metric Spaces

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Abstract

In this paper, we establish some common fixed point theorems for generalized-(ψ , α , β)-weakly contractive mappings in dislocated metric spaces. We present an example in support of our theorem.

Keywords: Generalized weakly contractive condition, weakly compatible maps, E.A. property, (CLR) property.

1. INTRODUCTION

Hitzler and Seda [4] introduced the concept of dislocated metric space (d-metric space) as follows.

Definition 1.1 Let X be a non empty set and let $d: X \times X \to [0, \infty)$ be a function and for all $x, y, z \in X$, the following conditions are satisfied:

(1) $d(x, y) = d(y, x)$;
(2) $d(x, y) = d(y, x) = 0$;
(3) $d(x, y) \leq d(x, z) + d(z, y)$;

Then d is called dislocated metric (or simply d-metric) on X and the pair (X, d) is called dislocated metric space.

Example 1.2 Let (X, d) be a metric space. The function $f: X \times X \to \mathbb{R}^+$, defined as

$d(x, y) = \max(x, y)$;   for all $x, y \in X$ is a d-metric on X.

Definition 1.3 A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be d-convergent if for every given $\in > 0$ there exist an $n \in N$ and $x \in X$ such that $d(x_n, x) < \in$ for all $n > N$ and it is denoted by $\lim_{n\to\infty} x_n = x \; or \; x_n \to x \; as \; n \to \infty$

Definition 1.4A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be d-Cauchy sequence if for every $\in > 0$ there exist $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $m, n \in n_0$

Definition 1.5 A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

We denote by Ψ the set of functions : [0, ∞) → [0, ∞) satisfying the following hypotheses:

(ψ1) ψ is continuous and monotone nondecreasing,

(ψ2) ψ(t) = 0 if and only if t = 0.

We denote by Φ the set of functions : [0, ∞) → [0, ∞) satisfying the following hypotheses:

(α1) α is continuous,

(α2) α(t) = 0 if and only if t = 0.

We denote by Γ the set of functions : [0, ∞) → [0, ∞) satisfying the following hypotheses:

(β1) β is lower semi-continuous,

(β2) β(t) = 0 if and only if t = 0.

Definition 1.6 A mapping T : X → X is said to be (ψ, α, β) weak contraction if there exists three maps ψ, α, β : [0, ∞) → [0, ∞) such that ψ(d(Tx, Ty)) ≤ α(d(x, y)) – β(d(x, y)), where

(i) ψ is continuous and monotone non-decreasing,
(ii) α is continuous,
(iii) β is lower semi-continuous,
(iv) ψ(t) = 0 = α(t) = β(t), if and only if, t = 0.

Now, we introduce the following notions:

Definition 1.7 A mapping T : X → X is said to be generalized (ψ, α, β) weak contraction if there exists three maps ψ, α, β : [0, ∞) → [0, ∞) such that

$$\psi(d(Tx, Ty)) \le \alpha(M(x, y)) - \beta(M(x, y)), \text{ where}$$

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\}$$

and

(i) ψ is continuous and monotone non-decreasing,
(ii) α is continuous,
(iii) β is lower semi-continuous,
(iv) $\psi(t) = 0 = \alpha(t) = \beta(t)$, if and only if, t = 0.

Definition 1.8 A mapping g : X → X is said to be generalized (ψ, α, β) weak contraction with respect to f : X → X if there exists three maps ψ, α, β : [0, ∞) → [0, ∞) such that

$$\psi(d(gx, gy)) \le \alpha(N(fx, fy)) - \beta(N(fx, fy)),$$

where $N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx,gx)d(fy,gy)}{1+d(fx,fy)}, \frac{d(fx,gx)d(fy,gy)}{1+d(gx,gy)}\}$ and

(i) ψ is continuous and monotone non-decreasing,
(ii) α is continuous,
(iii) β is lower semi-continuous,
(iv) $\psi(t) = 0 = \alpha(t) = \beta(t)$, if and only if, t = 0.

In 1996, Jungck et. al. [6] introduced the concept of weakly compatible maps as follows:

Definition 1.9 Two maps f and g defined on a self map X are said to be weakly compatible if they commute at their coincidence points.

In 2002, Aamri et. al. [1] introduced the notion of E.A. property as follows:

Definition 1.10 Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

In 2011, Sintunavarat et. al. [10] introduced the notion of (CLR_g) property as follows:

Definition 1.11 Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some x in X.

2. MAIN RESULTS

For proving our main results, we need the following lemma:

Lemma 2.1. Let $\{a_n\}$ be a sequence of non-negative real numbers. If

(2.1) $\psi(a_{n+1}) \leq \alpha(a_n) - \beta(a_n)$

for all $n \in \mathbb{N}$, where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma$ and

(2.2) $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$, then the following hold:

(i) $a_{n+1} \leq a_n$ if $a_n > 0$,
(ii) $a_n \to 0$ as $n \to \infty$.

Theorem 2.2. Let f and g be self mappings of a Hausdorff dislocated metric space (X, d) satisfying the followings :

(2.3) $gX \subseteq fX$,

(2.4) fX or gX is a complete subspace of X,

(2.5) $\psi(d(gx, gy)) \leq \alpha(N(fx, fy)) - \beta(N(fx, fy))$, for all x, y in X,

where $\psi \in \Psi$, $\alpha \in \Phi$ and $\beta \in \Gamma$ and satisfy condition (2.2) with

$N(fx, fy) = \max\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx,gx)d(fy,gy)}{1+d(fx,fy)}, \frac{d(fx,gx)d(fy,gy)}{1+d(gx,gy)}\}$.

Then f and g have a unique coincidence point in X.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $gX \subseteq fX$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

(2.6) $y_n = fx_{n+1} = gx_n$ for all $n \geq 0$.

Moreover, we assume that if $y_n = y_{n+1}$ for some n in $\mathbb{N}$, then there is nothing to prove. Now, we assume that $y_n \neq y_{n+1}$ for all n in $\mathbb{N}$.

We assert that

(2.7) $$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Substituting x = x_n and y = x_{n+1} in (2.5), using (2.6), we have

$\psi(d(y_n, y_{n+1})) = \psi(d(gx_n, gx_{n+1}))$

$\leq \alpha(N(fx_n, fx_{n+1})) - \beta(N(fx_n, fx_{n+1}))$

(2.8) $= \alpha(N(y_{n-1}, y_n)) - \beta(N(y_{n-1}, y_n))$, where

$N(y_{n-1}, y_n) = \max\{ d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1},y_n)d(y_n,y_{n+1})}{1+d(y_{n-1},y_n)}, \frac{d(y_{n-1},y_n)d(y_n,y_{n+1})}{1+d(y_n,y_{n+1})}\}$

$= \max\{ d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$,

since $\frac{d(y_{n-1},y_n)d(y_n,y_{n+1})}{1+d(y_{n-1},y_n)} \leq d(y_n, y_{n+1})$ and $\frac{d(y_{n-1},y_n)d(y_n,y_{n+1})}{1+d(y_n,y_{n+1})} \leq d(y_{n-1}, y_n)$.

If $d(y_{n-1}, y_n) < d(y_n, y_{n+1})$, then from (2.8), we get

$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_n, y_{n+1})) - \beta(d(y_n, y_{n+1}))$, which implies that, $d(y_n, y_{n+1}) = 0$, that is, $y_n = y_{n+1}$, which is a contradiction.

So $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$, then from (2.8), we obtain

(2.9) $\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_{n-1}, y_n)) - \beta(d(y_{n-1}, y_n))$.

Similarly, we also conclude that

(2.10) $\psi(d(y_{n+1}, y_{n+2})) \leq \alpha(d(y_n, y_{n+1})) - \beta(d(y_n, y_{n+1}))$.

Generally, we have that for each $n \in \mathbb{N}$

(2.11) $\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_{n-1}, y_n)) - \beta(d(y_{n-1}, y_n))$.

From (ii) of Lemma 2.1, we obtain that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Next, we prove that $\{y_n\}$ is a d-Cauchy sequence. Suppose that $\{y_n\}$ is not a d-Cauchy sequence. Then there exists $\varepsilon > 0$ such that for $k \in \mathbb{N}$, there are $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying

(a) m(k) and n(k) are positive integers.

(b) $d(y_{n(k)}, y_{m(k)}) \geq \varepsilon$

(c) m(k) is the smallest even number such that the condition (b) holds

Taking into account (b) and (c), we have that

$$\varepsilon \leq d(y_{n(k)}, y_{m(k)})$$

(2.12) $\leq d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$

$\leq \varepsilon + d(y_{m(k)-1}, y_{m(k)})$.

Letting k → ∞, we obtain

(2.13) $\lim_{k\to\infty} d\left(y_{n(k)}, y_{m(k)}\right) = \varepsilon$,

$\varepsilon \leq d(y_{n(k)-1}, y_{m(k)-1})$

(2.14) $\leq d(y_{n(k)-1}, y_{m(k)-2}) + d(y_{m(k)-2}, y_{m(k)-1})$

$\leq \varepsilon + d(y_{m(k)-2}, y_{m(k)-1})$.

Making k → ∞, we obtain

(2.15) $\lim_{k\to\infty} d\left(y_{n(k)-1}, y_{m(k)-1}\right) = \varepsilon$

Substituting $x = x_{n(k)}$ and $y = x_{m(k)}$ in (2.5), we have

$\psi(d(gx_{n(k)}, gx_{m(k)})) \leq \alpha(N(fx_{n(k)}, fx_{m(k)})) - \beta(N(fx_{n(k)}, fx_{m(k)}))$, that is,

(2.16) $\psi(d(y_{n(k)}, y_{m(k)})) \leq \alpha(N(y_{n(k)-1}, y_{m(k)-1})) - \beta(N(y_{n(k)-1}, y_{m(k)-1}))$,

where $d(y_{n(k)-1}, y_{m(k)-1}) \leq N(y_{n(k)-1}, y_{m(k)-1})$

$= \max\{ d(y_{n(k)-1}, y_{m(k)-1}), d(y_{n(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{m(k)}),$

$\frac{d(y_{n(k)-1}, y_{n(k)})d(y_{m(k)-1}, y_{m(k)})}{1+d(y_{n(k)-1}, y_{m(k)-1})}, \frac{d(y_{n(k)-1}, y_{n(k)})d(y_{m(k)-1}, y_{m(k)})}{1+d(y_{n(k)}, y_{m(k)})}\}.$

$= \max\{ d(y_{n(k)-1}, y_{m(k)-1}), d(y_{n(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{m(k)})\}$.

Letting k → ∞ in (2.16) and using the lower semi-continuity of β and the continuities of ψ and α, we obtain

$\psi(\varepsilon) \leq \alpha(\varepsilon) - \beta(\varepsilon)$, which implies that $\varepsilon = 0$, by (2.2), a contradiction with $\varepsilon > 0$. It follows that $\{y_n\}$ is a d-Cauchy sequence.

Since fX is complete, so there exists a point u in fX such that

(2.17) $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = u$.

Since u ∈ fX, so we can find p in X such that fp = u.

We claim that fp = gp.

From (2.5), we have

$\psi(d(fx_{n+1}, gp)) = \psi(d(gx_n, gp))$

$\leq \alpha(N(gx_n, gp)) - \beta(N(gx_n, gp))$.

Letting limit as n → ∞ and using the continuity of α and semi-continuity of β, we get

(2.18) ψ(d(fp, gp)) ≤ $\alpha(\lim_{n\to\infty} N(gx_{n,}gp))$ - $\beta(\lim_{n\to\infty} N(gx_{n,}gp))$,

where

$N(gx_{n,}gp)$ = max{d(fx_n, fp), d(fx_n, gx_n), d(fp, gp), $\frac{d(fx_{n,},gx_{n,})d(fp,gp)}{1+d(fx_{n,},fp)}, \frac{d(fx_{n,},gx_{n,})d(fp,gp)}{1+d(gx_{n,},gp)}$}.

Making limit as n → ∞, we have

$\lim_{n\to\infty} N(gx_{n,}gp)$ = max{d(fp, fp), d(fp, fp), d(fp, gp), $\frac{d(fp,fp)d(fp,gp)}{1+d(fp,fp)}, \frac{d(fp,gp)d(fp,gp)}{1+d(fp,gp)}$}

(2.19) = d(fp, gp).

So, from (2.18) and (2.19), we have

ψ(d(fp, gp)) ≤ α(d(fp, gp)) - β(d(fp, gp)), which implies that, d(fp, gp) = 0, that is,

(2.20) fp = gp = u.

Therefore, p is a point of coincidence of f and g.

The uniqueness of the point of coincidence is a consequence of condition (2.5).

Now, we show that there exists a common fixed point of f and g. Since f and g are weakly compatible, by (2.20), we have gfp = fgp, and

(2.21) gu = gfp = fgp = fu.

If p = u, then p is a common fixed point of f and g.

If p ≠ u, then by (2.5), we have

ψ(d(gp, gu)) ≤ α(N(gp, gu)) - β(N(gp, gu)),

where N(gp, gu) = max{d(fp, fu), d(fp, gp), d(fu, gu), $\frac{d(fp,gp)d(fu,gu)}{1+d(fp,fu)}, \frac{d(fp,gp)d(fu,gu)}{1+d(gp,gu)}$}

= max{d(u, gu), d(u, u), 0, 0, 0}

= d(u, gu).

Therefore, we have

ψ(d(u, gu)) ≤ α(d(u, gu)) - β(d(u, gu)), which by (2.2) implies that, d(u, gu) = 0, that is, u = gu = fu.

Consequently, u is the unique common fixed point of f and g.

Denote by Λ the set of functions : [0, ∞) → [0, ∞) satisfying the following hypotheses:

(h_1) γ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$,

(h_2) for every $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \gamma(s)ds > 0.$$

We have the following result.

Theorem 2.3. Let (X, d) be a Hausdorff dislocated metric space and f, g : X → X be self mappings satisfying (2.3), (2.4) and the following:

$$\int_0^{d(gx,gy)} \gamma_1(s)ds \leq \int_0^{N(fx,fy)} \gamma_2(s)ds - \int_0^{N(fx,fy)} \gamma_3(s)ds,$$

for all x, y in X, where $\gamma_1, \gamma_2, \gamma_3 \in \Lambda$ and satisfy condition (2.2). If f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. On taking $\psi(t) = \int_0^t \gamma_1(s)ds$, $\alpha(t) = \int_0^t \gamma_2(s)ds$ and $\beta(t) = \int_0^t \gamma_3(s)ds$ in Theorem 2.2, we get Theorem 2.3.

Taking $\gamma_3(s)$ = (1-k) $\gamma_2(s)$ for $k \in [0, 1)$ in Theorem 2.3, we obtain the following result:

Corollary 2.4. Let (X, d) be a Hausdorff dislocated metric space and f, g : X → X be self mappings satisfying (2.3), (2.4) and the following:

$$\int_0^{d(gx,gy)} \gamma_1(s)ds \leq k \int_0^{N(fx,fy)} \gamma_2(s)ds,$$

for all x, y in X, where $\gamma_1, \gamma_2 \in \Lambda$ and satisfy condition (2.2). If f and g are weakly compatible, then f and g have a unique common fixed point.

Remark 2.5. If N(fx, fy) = d(fx, fy), then (2.5) reduces to

(2.22) $\psi(d(gx, gy)) \leq \alpha(d(fx, fy)) - \beta(d(fx, fy))$,

which is condition (2.3) of Theorem 1 [3].

Remark 2.6. If f is the identity mapping, then (2.22) reduces to

(2.23) $\psi(d(gx, gy)) \leq \alpha(d(x, y)) - \beta(d(x, y))$.

Example 2.7. Let X = [0, 10] ∪ {11, 12, 13, . . .} and

(2.24) $d(x, y) = \begin{cases} |x-y|, if\ x, y \in [0,10], x \neq y \\ x+y, if\ atleast\ one\ of\ x\ or\ y \notin [0,10] and\ x \neq y \\ x, if\ x = y \end{cases}$

Then (X, d) is a Hausdorff and dislocated metric space.

Let $\psi, \alpha, \beta : [0, \infty) \rightarrow [0, \infty)$ be defined as

ψ(t) = α(t) = $\begin{cases} t, if\ 0 \le t \le 10, \\ t^2, if\ t > 10 \end{cases}$ and

β(t) = $\begin{cases} \frac{1}{5}\ t^2, if\ 0 \le t \le 10, \\ \frac{1}{5}, if\ t > 10 \end{cases}$

Let g : X → X be defined as

gx = $\begin{cases} x - \frac{1}{5}\ x^2, if\ 0 \le x \le 10, \\ x - 10, if\ x \in \{11,12,13, ...\} \end{cases}$

Without loss of generality, we assume that $x \geq y$ and discuss the following cases:

Case I: When $x > y$.

Case II: When $x = y$.

Case I: When $x > y$. It has three subcases:

Case 1. (x ∈ [0, 10]). Then

ψ(d(gx, gy)) = {x- $\frac{1}{5}x^2$} - {y- $\frac{1}{5}y^2$}

= (x - y) - $\frac{1}{5}$ (x - y) (x + y) ≤ (x - y) - $\frac{1}{5}$ $(x - y)^2$

= d(x, y) - $\frac{1}{5}$ $(d(x, y))^2$

= α(d(x, y)) - β(d(x, y)).

Case 2. (x ∈ {12, 13, . . .}). Then

d(gx, gy) = d(x-10, y- $\frac{1}{5}y^2$), if y ∈ [0, 10],

or, d(gx, gy) = x – 10 + y- $\frac{1}{5}y^2$ ≤ x + y – 10.

and

d(gx, gy) = d(x-10, y-10), if y ∈ {11, 12, 13, . . .},

or, d(gx, gy) = x – 10 + y – 10 < x + y – 10.

Consequently, we have

ψ(d(gx, gy)) = $(d(gx, gy))^2$ ≤ $(x + y - 10)^2$ < (x + y - 10) (x + y + 10)

= $(x + y)^2$ - 100 < $(x + y)^2$ - $\frac{1}{5}$

= α(d(x, y)) - β(d(x, y)).

Case 3. (x = 11). Then y ∈ [0, 10], gx = 1 and d(gx, gy) = 1 – (y- $\frac{1}{5}y^2$) ≤ 1.

So, we have ψ(d(gx, gy)) ≤ ψ(1) = 1.

Again d(x, y) = 11 + y.

So, α(d(x, y)) - β(d(x, y)) = $(11 + y)^2 - \frac{1}{5}$

$= 121 + y^2 + 22\, y - \frac{1}{5}$

$= \frac{604}{5} + 22\, y + y^2 > 1 = \psi(d(gx, gy))$.

Case II: When $x = y$.

Case 1. (x ∈ [0, 10]). So y ∈ [0, 10]). Then

ψ(d(gx, gy)) = ψ(d(gx, gx))=ψ(gx)=gx =x- $\frac{1}{5}x^2$

= α(d(x, y)) - β(d(x, y)).

Case 2. (x ∈ {12, 13, . . .}). So y ∈ {12, 13, . . .}). Then,

or, ψ(d(gx, gy)) =$(gx)^2$=$(x - 10)^2 < (x^2 - 10) < x^2 - \frac{1}{5}$

= α(d(x, y)) - β(d(x, y)).

Consequently, we have

$\psi(d(gx, gy)) = (d(gx, gy))^2 \leq (x + y - 10)^2 < (x + y - 10)(x + y + 10)$

$= (x + y)^2 - 100 < (x + y)^2 - \frac{1}{5}$

= α(d(x, y)) - β(d(x, y)).

Considering all the above cases, we conclude that the inequality (2.23) remains valid for ψ, α, and β constructed as above and consequently, g has a unique fixed point.

Clearly, it is seen that 0 is the unique fixed point of g.

3. WEAKLY COMPATIBLE AND E.A. PROPERTY

Theorem 3.1. Let f and g be self mappings of a Hausdorff dislocated metric space (X, d) satisfying (2.3), (2.5) and the following:

(3.1) f and g are weakly compatible,

(3.2) f and g satisfy the E.A. property.

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

(3.3) $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some z in X.

Since gX ⊆ fX, there exists a sequence $\{y_n\}$ in X such that $gx_n = fy_n$. Hence $\lim_{n\to\infty} fy_n = z$.

Now, we shall show that $\lim_{n\to\infty} gy_n = z$.

Let us suppose that $\lim_{n\to\infty} gy_n = t$.

From (2.5), we have

$\psi(d(gx_n, gy_n)) \le \alpha(N(fx_n, fy_n)) - \beta(N(fx_n, fy_n))$.

Letting $n \to \infty$, we have

(3.4) $\psi(d(z, t)) \le \alpha(\lim_{n\to\infty} N(fx_n, fy_n)) - \beta(\lim_{n\to\infty} N(fx_n, fy_n))$,

where, $N(fx_n, fy_n) = \max\{d(fx_n, fy_n), d(fx_n, gx_n), d(fy_n, gy_n), \frac{d(fx_n, gx_n)d(fy_n, gy_n)}{1+ d(fx_n, fy_n)},$

$\frac{d(fx_n, gx_n)d(fy_n, gy_n)}{1+ d(gx_n, gy_n)}\}$.

Letting $n \to \infty$, we have

(3.5) $\lim_{n\to\infty} N(fx_n, fy_n) = \max\{d(z, z), d(z, z), d(z, t), \frac{d(z,z)d(z,t)}{1+d(z,z)}, \frac{d(z,z)d(z,t)}{1+d(z,t)}\} = d(z, t)$.

Thus, from (3.4) and (3.5), we get

$\psi(d(z, t)) \le \alpha(d(z, t)) - \beta(d(z, t))$, which implies that $d(z, t) = 0$, that is, $z = t$.

Hence, $\lim_{n\to\infty} gy_n = z$.

Now, suppose that fX is complete subspace of X. Then, there exists u in X such that z = fu.

Subsequently, we have

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = z = fu$.

Now, we show that fu = gu.

From (2.5), we have

$\psi(d(gx_n, gu)) \le \alpha(N(fx_n, fu)) - \beta(N(fx_n, fu))$.

Letting $n \to \infty$, we have

(3.6) $\psi(d(z, gu)) \leq \alpha(\lim_{n\to\infty} N(fx_n, fu)) - \beta(\lim_{n\to\infty} N(fx_n, fu))$,

where, $N(fx_n, fu) = \max\{d(fx_n, fu), d(fx_n, gx_n), d(fu, gu), \frac{d(fx_n,gx_n)d(fu,gu)}{1+ d(fx_n,fu)},$

$$\frac{d(fx_n,gx_n)d(fu,gu)}{1+ d(gx_n,gu)}\}.$$

Letting $n \to \infty$, we have

(3.7) $\lim_{n\to\infty} N(fx_n, fu) = \max\{d(z, z), d(z, z), d(z, gu), \frac{d(z,z)d(z,gu)}{1+ d(z,fu)},$

$$\frac{d(z,z)d(z,gu)}{1+ d(z,gu)}\} = d(z, gu).$$

Thus, from (3.6) and (3.7), we get

$\psi(d(z, gu)) \leq \alpha(d(z, gu)) - \beta(d(z, gu))$, which implies that, $d(z, gu) = 0$, that is, $z = gu = fu$.

Since f and g are weakly compatible, therefore, $gfu = fgu$, implies that, $ffu = fgu = gfu = ggu$.

Now, we claim that gu is the common fixed point of f and g.

From (2.5), we have

$\psi(d(gu, ggu)) \leq \alpha(N(fu, ffu)) - \beta(N(fu, ffu))$

$= \alpha(d(fu, ffu)) - \beta(d(fu, ffu))$

$= \alpha(d(gu, ggu)) - \beta(d(gu, ggu))$, which implies that, $gu = ggu = ffu$.

Therefore, gu is the common fixed point of f and g.

For the uniqueness, let z and w be two common fixed points of f and g.

From (2.5), we have

$\psi(d(z, w)) = \psi(d(gz, gw))$

(3.8) $\leq \alpha(N(fz, fw)) - \beta(N(fz, fw))$,

where, $N(fz, fw) = \max\{d(fz, fw), d(fz, gz), d(fw, gw),$
$\frac{d(fz,gz)d(fw,gw)}{1+d(fz,fw)}, \frac{d(fz,gz)d(fw,gw)}{1+d(gz,gw)}\}$

(3.9) $= \max\{d(z, w), 0, 0, 0, 0\} = d(z, w)$.

From (3.8) and (3.9), we get

$\psi(d(z, w)) \leq \alpha(d(z, w)) - \beta(d(z, w))$, which implies that, $d(z, w) = 0$, that is, $z = w$.

Therefore, f and g have a unique common fixed point in X.

4. WEAKLY COMPATIBLE AND (CLR_F) PROPERTY

Theorem 4.1. Let f and g be self mappings of a Hausdorff dislocated metric space(X, d) satisfying (2.3), (2.5), (3.1) and the following:

(4.1) f and g satisfy (CLR_f) property.

Then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the (CLR_f) property, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ = fx, for some x in X.

From (2.5), we have

$\psi(d(gx_n, gx)) \leq \alpha(N(fx_n, fx)) - \beta(N(fx_n, fx))$.

Letting n → ∞, we have

(4.3) $\psi(d(fx, gx)) \leq \alpha(\lim_{n\to\infty} N(fx_n, fx)) - \beta(\lim_{n\to\infty} N(fx_n, fx))$,

where, $N(fx_n, fx) = \max\{d(fx_n, fx), d(fx_n, gx_n), d(fx, gx), \frac{d(fx_n,gx_n)d(fx,gx)}{1+ d(fx_n,fx)},$

$$\frac{d(fx_n,gx_n)d(fx,gx)}{1+ d(gx_n,gx)}\}.$$

Letting n → ∞, we have

(4.4) $\lim_{n\to\infty} N(fx_n, fx) = \max\{d(fx, fx), d(fx, fx), d(fx, gx), \frac{d(fx,fx)d(fx,gx)}{1+ d(fx,fx)}, \frac{d(fx,fx)d(fx,gx)}{1+ d(fx,gx)}\}$

$= d(fx, gx)$.

Thus, from (4.3) and (4.4), we get

$\psi(d(fx, gx)) \leq \alpha(d(fx, gx)) - \beta(d(fx, gx))$, which implies that d(fx, gx) = 0, that is, gx = fx.

Now, let z = fx = gx. Since f and g are weakly compatible, therefore, fgx = gfx, implies that, fz = fgx = gfx = gz.

Now, we claim that gz = z.

From (2.5), we have

$\psi(d(gz, z)) = \psi(d(gz, gx))$

(4.5) $\leq \alpha$(N(fz, fx)) – β(N(fz, fx)).

where, N(fz, fx) = max{d(fz, fx), d(fz, gz), d(fx, gx), $\frac{d(fz,gz)d(fx,gx)}{1+\,d(fz,fx)}$, $\frac{d(fz,gz)d(fx,gx)}{1+\,d(gz,gx)}$}

(4.6) = max {d(gz, z), 0, 0, 0, 0} = d(gz, z).

From (4.5) and (4.6), we get

ψ(d(gz, z)) ≤ α(d(gz, z)) – β(d(gz, z)), which implies that, d(gz, z) = 0, that is, gz = z.

Hence, gz = z = fz. So, z is the common fixed point of f and g.

For the uniqueness, let w be another common fixed point of f and g.

From (2.5), we have

ψ(d(z, w)) = ψ(d(gz, gw))

(4.7) $\leq \alpha$(N(fz, fw)) – β(N(fz, fw)),

where, N(fz, fw) = max{d(fz, fw), d(fz, gz), d(fw, gw), $\frac{d(fz,gz)d(fw,gw)}{1+\,d(fz,fw)}$, $\frac{d(fz,gz)d(fw,gw)}{1+\,d(gz,gw)}$}

(4.8) = max {d(z, w), 0, 0, 0, 0, 0} = d(z, w).

From (4.7) and (4.8), we get

ψ(d(z, w)) ≤ α(d(z, w)) – β(d(z, w)), which implies that, d(z, w) = 0, that is, z = w.

Therefore, f and g have a unique common fixed point in X.

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