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Even Square Group Rings

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Abstract

In this note we introduce the notion of even square rings to group rings. We characterize when the group ring RG is an even square ring. It is noted that the class of even square group rings is quite large and includes all group rings FG as well where F is a field of characteristic $\neq 2$ and G be a commutative multiplicative group. In addition we note that if each element of a ring R is nil and G be a commutative multiplicative group then even square group ring RG is a Wedderburn radical and PI ring.

Keywords: even square ring, zero square ring, finite ring, group ring, Wedderburn radical, PI ring.

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1. INTRODUCTION

We have studied some properties of even square rings and nil elements in [1-2]. Here we introduce the notion of even square rings to group rings. Recall that a ring R is called an even square ring if $a^2 \in 2R, \forall a \in R$ and R is called an even ring if $a \in 2R, \forall a \in R$. Recall that an element $a \in R$ is called a nil element if $a^2 = 2a = 0$ and an element $a \in R$ is called an even square element if $a^2 \in 2R$ [1]. Clearly each nil element is nilpotent but each nilpotent element is not necessarily a nil element.

It is worth to note that a particular class of even square rings has also been studied by other authors [3-4].

Here we characterize when the group ring RG is an even square ring. It may be noted that the class of even square group rings is quite large and includes all group algebras FG as well. Here F is any field of characteristic $\neq 2$ and G be any commutative multiplicative group.

It may be noted that in this article we assume that G is a commutative multiplicative group because it is necessary for the group ring RG to be a commutative ring and in [1] it has been noted that the sum of two even square elements of a ring R is an even square element if R is a commutative ring and in the case of noncommutative rings the sum of two even square elements is not necessarily an even square element.

RESULTS

Proposition 1. Let R be an even square ring. Let G be a commutative multiplicative group then the group ring RG is an even square group ring provided R is a commutative ring.

Proof. Let R be a commutative even square ring and G be a usual multiplicative commutative group. Let $x_i \in R, y_i \in G$ then $(x_1y_1 + x_2y_2 + ... + x_ny_n) \in RG$ .We have to prove that $(x_1y_1 + x_2y_2 + ... + x_ny_n)^2 \in 2RG$. We have $(x_1y_1 + x_2y_2 + ... + x_ny_n)^2$ $= (x_1^2y_1^2 + ...x_n^2y_n^2) + 2\sum_{i,j=1}^{n}(x_ix_j)y_iy_j$ (since R and G are commutative). R is an even square ring implies $x_i^2 \in 2R$ for each $i = 1,2,...,n$ and therefore $(x_1^2y_1^2 + ... + x_n^2y_n^2)$ $\in 2RG$. Clearly $2\sum_{i,j=1}^{n}(x_ix_j)y_iy_j \in 2RG$. Thus $(x_1y_1 + x_2y_2 + ... + x_ny_n)^2 \in 2RG$. Hence RG is an even square ring.

Proposition 2. Let R be an even square ring and G be any multiplicative commutative group. If each element of R is nil then each element of the even square group ring RG is nil.

Proof. Let R be an even square ring and G be any multiplicative commutative group. Let each element of R is nil. Let $x_i \in R, y_i \in G$. Let $x = (x_1y_1 + x_2y_2 + ... + x_ny_n) \in RG$. Since each element of R is nil therefore $x_i^2 = 2x_i = 0$ for each $x_i \in R$. Clearly $2x = \sum_{i=1}^{n}(2x_i)y_i = 0$ and $x^2 = (x_1y_1 + x_2y_2 + ... + x_ny_n)^2 = (x_1^2y_1^2 + x_2^2y_2^2 + ... + x_n^2y_n^2) + \sum_{i<j;i,j=1}^{n}(x_ix_j + x_jx_i)y_iy_j$.
Since each element of R is nil therefore R is commutative [1] and

so $\sum_{i<j;i,j=1}^{n}(x_i x_j + x_j x_i)y_i y_j = \sum_{i,j=1}^{n}(2x_i x_j)y_i y_j = 0$. Thus $x^2 = 2x = 0$. Hence each element of RG is nil.

Proposition 3. Let R be an even ring and G be a commutative multiplicative group then the group ring RG is an even square group ring.

Proof. Let R be an even (commutative or noncommutative) ring and G be a commutative multiplicative group. Let $x_i \in R, y_i \in G$. Now using the definition of even ring we have $x_i \in R \Rightarrow x_i \in 2R$. This gives $x_i^2 \in 2R$ and $x_i^2 y_i^2 \in 2R$. Therefore $(x_1^2 y_1^2 + ... + x_n^2 y_n^2) \in 2RG$. Also, $x_i x_j \in 2R$, $x_j x_i \in 2R$ since $x_i, x_j \in 2R$. Therefore it follows that $\sum_{i<j;i,j=1}^{n}(x_i x_j + x_j x_i)y_i y_j \in 2RG$. Hence $(x_1 y_1 + x_2 y_2 + ... + x_n y_n)^2 \in 2RG$. Thus RG is an even square group ring.

Proposition 4. Let F is any field of characteristic $\neq 2$ and G be a commutative multiplicative group then the group algebra FG is an even square group ring.

Remark . Since each field of characteristic $\neq 2$ is an even ring [5] therefore by above proposition 3, FG is an even square group ring.

Proposition 5. Let R be an even square ring and G be any multiplicative commutative group. If each element of R is nil then the even square group ring RG is a Wedderburn radical and a PI ring.

Proof. Let R be an even square ring and G be any multiplicative commutative group. If each element of R is nil then by proposition 2 each element of even square ring RG is also nil. Therefore RG is a commutative ring. Clearly RG is a commutative nil ring and hence a Wedderburn radical. In addition RG is a nil ring of bounded index therefore it is a PI ring.

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