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Groupoid and Topological Quotient Group

Mohammad Qasim Mann'a

Department of Mathematics, College of Science, Al-Qassim University
P.O. Box: 6644-Buraidah: 51452, Saudi Arabia.

Abstract

In this paper we induced some results which are make connection between the concept of groupoid and the quotient group. Further, we introduce some new properties to the idea of groupoid in quotient group (quotient group-groupoid). Moreover, we give definitions to topological quotient group-groupoid and product of quotient group-groupoids and discuss some possessions to these definitions.

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0. Introduction

Indeed, Mucuk [8] introduced the definition of quotient group-groupoid from the fact that; the group-groupoid is equivalent to crossed module. Actually, we need more explain to this definition to explorer and construction some new properties and notions based to this definition.

As in group theory, we discuss the relationship between the group-groupoid and normal subgroup-groupoid through the concept of quotient group to induce some results which are related to new form to the definition of groupoids in quotient group (quotient group-groupoid) and study some new properties to the idea of this definition.

Also in this article we introduce a definition to topological quotient group-groupoid and induced some properties of this definition over the notion of topological quotient group theory and construct the form of product quotient group-groupoids.

1. Preliminaries

A *groupoid* [2]G is small category consists of two sets G and O_G, called respectively the set of elements (or arrows) and the set of objects (or vertices) of the groupoid, together with, two maps α, β: $G \to O_G$, called respectively the source and target maps, the map $\varepsilon : O_G \to G$, written as $\varepsilon(x) = 1_x$, where 1_x is called the identity element at x in O_G, and ε is called the object map and the partial multiplication map,

$\gamma : G \underset{\alpha=\beta}{\times} G \to G$ written $\gamma(g,h) = g \circ h$, on the set,

$G \underset{\alpha=\beta}{\times} G = \{(g,h) \in G \times G : \alpha(g) = \beta(h)\}$.

These terms must satisfy the followings axioms:

1. $\alpha(g \circ h) = \alpha(h)$ *and* $\beta(g \circ h) = \beta(g)$,
2. $(g \circ h) \circ k = g \circ (h \circ k)$,
3. $\alpha(1_x) = \beta(1_x) = x$,
4. $g \circ 1_{\alpha(g)} = g$ *and* $1_{\beta(g)} \circ g = g$, for all $g, h, k \in G$ and $x \in O_G$.

For a groupoid G, we will denote to the inverse map by;

5. $\sigma : G \to G$, such that $g \to g^{-1}$.

A *morphism* of groupoids H and G is a functor, that is, it consists of a pair of functions $f : H \to G$ and $O_f : O_H \to O_G$ such that;

1. $f(a \circ b) = f(a) \circ f(b)$,
2. $f(a^{-1}) = f(a)^{-1}$,
3. $\beta_G . f = O_f . \beta_H$, $\alpha_G . f = O_f . \alpha_H$,
4. $f . \varepsilon_H = \varepsilon_G . O_f$. Where $a \circ b$ is defined.

A *subgroupoid* of G is a subcategory H of G which is also a groupoid. We say that, H is full (wide) subgroupoid if for all $x, y \in O_H$ then, $G_x^y = H_x^y$ ($O_H = O_G$).

Moreover, H is called *normal* subgroupoid if H wide and $a \circ H(x) \circ a^{-1} = H(y)$ or $a^{-1} \circ H(y) \circ a = H(x)$ for any $x, y \in O_G$ and $a \in G_x^y$.

2. Group-groupoid

In fact the notion of group-groupoid is equivalent to many concepts in category theory and we present these concepts as follows, the cross modules, internal category in group and the group object in category of groupoids see [9]. The following definition and properties introduced by [3].

Definition 2.1 A *group-groupoid* G is a groupoid endowed with a group structure such that the following maps which are called respectively addition, inverse and unit, are morphism of groupoids:

1. $m : G \times G \to G,\ (g,h) \mapsto gh,$ group multiplication,
2. $u : G \to G,\ (g) \mapsto \overline{g},$ group inverse map,
3. $e : \{*\} \to G,$ where $\{*\}$ is singleton.

So, by (3) if the identity for the group structure on O_G is e then, 1_e is the identity for the group structure on the arrows of G. In group-groupoid G for $g, h \in G,$ the groupoid composite is denoted by $g \circ h$ when $\alpha(g) = \beta(h)$ and the group addition by gh.

A *morphism* of group-groupoids $f : G \to H$ is a morphism of the underlying groupoids preserving the group structure. The following Proposition a pear in Brown and Spencer [3].

Proposition 2.2 *Let G be a group-groupoid. If* $b \in G(y,z)$ *and* $a \in G(x,y)$, *then*

1. $b \circ a = (b\overline{1_y})1_y \circ (1_e a) = ((b\overline{1_y}) \circ 1_e)(1_y \circ a) = b\overline{1_y}a$.
2. $a^{-1} = 1_{\alpha(a)}\overline{a}1_{\beta(a)} = 1_x\overline{a}1_y$.

Definition 2.3[7, Definition 2.1] Let G be a group-groupoid and $H \subset G$. Then H is a subgroup-groupoid of G if $(H, O_H, \cdot, \circ)$ form a group-groupoid.

Further, H is *wide* if $O_H = O_G$ and *full* if $H(x,y) = G(x,y)$ for all $x, y \in O_H$ Moreover, if H is a subgroup-groupoid and the set of arrows of H is normal subgroup then the set of objects O_H is normal subgroup [7, Proposition 2.5]. So, *H* is called *normal subgroup-groupoid* see also [8, Definition 3.10].

The following definition appears in [5].

Definition 2.4 A *topological group-groupoid*, is a group-groupoid *G* with a topologies on *G* and O_G such that all the morphisms of group-groupoid are continuous, that is: *m, u,* α, β, γ, ε and σ. A *morphism* of topological group-groupoids $f : G \to H$ is continuous.

3. Quotient Group-groupoid

In this section we study the properties of quotient group on group-groupoid to define the concept of quotient group-groupoid.

Let *G* be a group and *H* be normal subgroup of *G*. The group G/H is called the quotient group and the map $q : G \to G/H$ such that $g \mapsto gH$ is called quotient map [4].

Propostion 3.1 *Let G be a group-groupoid and H be a normal subgroup-groupoid of G then* $G \underset{\alpha=\beta}{\times} G$ *is a subgroup of* $G \times G$ *and* $H \underset{\alpha=\beta}{\times} H$ *is a normal subgroup of* $G \underset{\alpha=\beta}{\times} G$.

Proof. Let $(g,r),(h,t) \in G \underset{\alpha=\beta}{\times} G$. Since α, β and *u* are morphisms of groups, then $\alpha(\overline{h}) = \overline{\alpha(h)} = \overline{\beta(t)} = \beta(\overline{t})$ implies that $(\overline{h},\overline{t}) \in G \underset{\alpha=\beta}{\times} G$ then $(g\overline{h}, r\overline{t}) \in G \underset{\alpha=\beta}{\times} G.$

Similarly, we can show that $H \underset{\alpha=\beta}{\times} H$ is a subgroup of $G \underset{\alpha=\beta}{\times} G$.

To prove that $H \underset{\alpha=\beta}{\times} H$ is a normal subgroup of $G \underset{\alpha=\beta}{\times} G$. Let, $(g,r) \in G \underset{\alpha=\beta}{\times} G$ and $(h_1,h_2) \in H \underset{\alpha=\beta}{\times} H$, then we have that $\alpha(gh_1\overline{g}) = \beta(rh_2\overline{r})$ and this implies that, $(g,r)(h_1,h_2)(\overline{g},\overline{r}) = (gh_1\overline{g}, rh_2\overline{r}) \in H \underset{\alpha=\beta}{\times} H.$

From above result, we define the set, $G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H = \{(g,r)H \underset{\alpha=\beta}{\times} H : \alpha(g) = \beta(r)\}$ with the multiplication on it as, $(g_1, r_1)H \underset{\alpha=\beta}{\times} H(g_2, r_2)H \underset{\alpha=\beta}{\times} H = (g_1 g_2, r_1 r_2)H \underset{\alpha=\beta}{\times} H.$

So, it is clear that $G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H$ is a group (quotient group). Further, this leads to define a quotient map, $\varphi : G \underset{\alpha=\beta}{\times} G \to G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H$, $\varphi(g,r) = (g,r)H \underset{\alpha=\beta}{\times} H$. Its easy to see that φ is well-defined, onto and homomorphism.

Proposition 3.2 Let G be a group-groupoid and H be a norrmal subgroup-groupoid then, $G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H \cong (G \underset{\alpha=\beta}{\times} G)H \times H \Big/ H \times H$.

Proof. . Define $h : G \underset{\alpha=\beta}{\times} G \to (G \underset{\alpha=\beta}{\times} G)H \times H \Big/ H \times H$, $h((g,r)) = (g,r)H \times H$ its easy to see that h is onto and homomorphism of groups. Further, we have $Ker\,(h) =$ $\{(g,r) \in G \underset{\alpha=\beta}{\times} G : h(g,r) = H \times H\} = \{(g,r)G \underset{\alpha=\beta}{\times} G : (g,r) \in H \times H\} = H \underset{\alpha=\beta}{\times} H.$ So, it's obvious that $G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H \cong (G \underset{\alpha=\beta}{\times} G)H \times H \Big/ H \times H$.

Proposition 3.3 *Let G be a group-groupoid and H be a normal subgroup-groupoid of G then,* $G/H \underset{\alpha=\beta}{\times} G/H = \{(gH, rH) : \alpha(g) = \beta(r)\}$, *is a subgroup of* $G/H \times G/H$.

Proof. Clear.

Proposition 3.4 *Let G be a group-groupoid and H be a normal subgroup-groupoid of G then,* $G/H \underset{\alpha=\beta}{\times} G/H \cong G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H$ *are isomorphic groups.*

Proof. Define, $\vartheta : G \underset{\alpha=\beta}{\times} G \to G/H \underset{\alpha=\beta}{\times} G/H$, $\vartheta(g,r) = (gH, rH)$ then ϑ is well-defined, homomorphism and onto and $Ker(\vartheta) = \{(g,r)G \underset{\alpha=\beta}{\times} G : \vartheta(g,r) = (H,H)\}$ $= \{(g,r) \in G \underset{\alpha=\beta}{\times} G : (gH, rH) = (H,H)\} = H \underset{\alpha=\beta}{\times} H$. Then we have the following isomorphic, $G/H \underset{\alpha=\beta}{\times} G/H \cong G \underset{\alpha=\beta}{\times} G \Big/ H \underset{\alpha=\beta}{\times} H$.

Remark 3.5 From above result we can define the isomorphism groups as,

$\Gamma : G/H \underset{\alpha=\beta}{\times} G/H \to G \underset{\alpha=\beta}{\times} G\Big/H \underset{\alpha=\beta}{\times} H$, $\Gamma(gH, rH) = (g,r)H \underset{\alpha=\beta}{\times} H$, also we have $\Gamma^{-1} = \Psi : G \underset{\alpha=\beta}{\times} G\Big/H \underset{\alpha=\beta}{\times} H \to G/H \underset{\alpha=\beta}{\times} G/H$, $\Psi((g,r)H \underset{\alpha=\beta}{\times} H) = (gH, rH)$.

Proposition 3.6 *Let G be a group-groupoid, and H be a normal subgroup-groupoid of G then* $\rho : G \underset{\alpha=\beta}{\times} G\Big/H \underset{\alpha=\beta}{\times} H \to G/H$ *such that* $\rho((g,r)H \underset{\alpha=\beta}{\times} H) = (g \circ r)H$ *is onto morphism of groups.*

Proof. ρ Well-defined; Suppose that, $(g_1, r_1)H \underset{\alpha=\beta}{\times} H = (g_2, r_2)H \underset{\alpha=\beta}{\times} H$ then, we have, $(\overline{g_2}g_1, \overline{r_2}r_1) \in H \underset{\alpha=\beta}{\times} H$. Since H is a normal subgroup-groupoid then $\overline{g_2}g_1 \circ \overline{r_2}r_1 \in H$ and $\overline{(g_2 \circ r_2)}(g_1 \circ r_1) \in H$, implies that, $(g_1 \circ r_1)H = (g_2 \circ r_2)H$.

Homomorphism;

$\rho((g_1, r_1)H \underset{\alpha=\beta}{\times} H(g_2, r_2)H \underset{\alpha=\beta}{\times} H) = \rho((g_1g_2, r_1r_2)H \underset{\alpha=\beta}{\times} H) = (g_1g_2 \circ r_1r_2)H =$
$(g_1 \circ r_1)\,(g_2 \circ r_2)H = (g_1 \circ r_1)H\,(g_2 \circ r_2)H = \rho((g_1, r_1)H \underset{\alpha=\beta}{\times} H)\rho(g_2, r_2)H \underset{\alpha=\beta}{\times} H)$
where, $(g_1, r_1), (g_2, r_2) \in G \underset{\alpha=\beta}{\times} G$.

Onto;

Let $gH \in G/H$ since, $gH = (g \circ 1_{\alpha(g)})H$. Then there is $(g, 1_{\alpha(g)})H \underset{\alpha=\beta}{\times} H$ in $G \underset{\alpha=\beta}{\times} G\Big/H \underset{\alpha=\beta}{\times} H$ such that, $\rho((g, 1_{\alpha(g)})H \underset{\alpha=\beta}{\times} H) = (g \circ 1_{\alpha(g)})H = gH$.

In the following proposition we define the set $gH \circ rH$ as;

$gH \circ rH = \{gh_1 \circ rh_2 : h_1, h_2 \in H, \alpha(g) = \beta(r)\}$.

Proposition 3.7 *Let G be a group-groupoid and H normal subgroup-groupoid then,* $gH \circ rH = (g \circ r)H$.

Proof. Suppose that $gh_1 \circ rh_2 \in gH \circ rH$ then $\alpha(gh_1) = \beta(rh_2)$ and since the source and target maps are morphism of groups then we have, $\alpha(g)\alpha(h_1) = \beta(r)\beta(h_2)$ and $\alpha(g) = \beta(r)$ then, $\alpha(h_1) = \beta(h_2)$ implies that, $gh_1 \circ rh_2 = (g \circ r)(h_1 \circ h_2) \in (g \circ r)H$,

Implies that, $gH \circ rH \subset (g \circ r)H$.........(1)

Conversely, suppose that $(g \circ r)h \in (g \circ r)H$. Since,

$(g \circ r)h = (g \circ r)(h \circ 1_{\alpha(h)}) = gh \circ r1_{\alpha(h)} \in gH \circ rH.$

Then we have, $(g \circ r)H \subset gH \circ rH$.........(2)

From (1) and (2) we have, $gH \circ rH = (g \circ r)H$.

Proposition 3.8 *Let G be a group-groupoid, and H be a normal subgroup-groupoid of G then* $\gamma^{'} : G/H \underset{\alpha=\beta}{\times} G/H \to G/H$ *such that* $\gamma^{'}(gH, rH) = gH \circ rH$ *is onto morphism of groups.*

Proof. From Proposition 3.7, we note that $\gamma^{'}$ is well-defined.

Homomrphism;

$\gamma^{'}((g_1 H, r_1 H)(g_2 H, r_2 H))$

$= \gamma^{'}((g_1 g_2 H, r_1 r_2 H)$

$= g_1 H g_2 H \circ r_1 H r_2 H$

$= (g_2 H \circ r_2 H)$

$= \gamma^{'}((g_1 H, r_1 H)\gamma^{'}(g_2 H, r_2 H).$

Onto;

Suppose that $gH \in G/H$ then since, $gH = gH \circ 1_{\alpha(g)} H$ then there is

$(gH, 1_{\alpha(g)} H)$ in $G/H \underset{\alpha=\beta}{\times} G/H$ such that, $\gamma^{'}(gH, 1_{\alpha(g)} H)$ $gH \circ 1_{\alpha(g)} H = gH$.

In the following we introduce a definition to *quotient group-groupoid* similar to [8, Definition 3.18] at most.

Definition 3.9 Let *G* be a group-groupoid and *H* be a normal subgroup-groupoid of *G*. Then, $(G/H, O_G/O_H)$ and the following maps called *quotient group-groupoids.*

1.The source map, $\alpha^{'} : G/H \to O_G/O_H$, $\alpha^{'}(gH) = \alpha(g)O_H$,

2. The target map, $\beta^{'} : G/H \to O_G/O_H$, $\beta^{'}(gH) = \beta(g)O_H$,

3. The object map, $\varepsilon^{'} : O_G/O_H \to G/H$, $\varepsilon^{'}(xO_H) = 1_x H$,

4. The multiplication map, $\gamma^{'} : G/H \underset{\alpha=\beta}{\times} G/H \to G/H$, $\gamma^{'}((gH, rH) = gH \circ rH$,

5. The inversion map groupoid, $\sigma^{'} : G/H \to G/H$, $\sigma^{'}(gH) = g^{-1}H$,

6. The multiplication group, $m^{'} : G/H \times G/H \to G/H$, $m^{'}(gH, rH) = grH.$

7. The inverse map group, $u^{'} : G/H \to G/H$, $u^{'}(gH) = \bar{g}H.$

Remark 3.10 We note that from a bove definition,

1. All maps are well-defined,
2. The quotient map $q : G \to G/H$ are morphism of groupoids,
3. From (4). Suppose that, $(gH, rH) \in G/H \underset{\alpha=\beta}{\times} G/H$ then we have $\alpha(g) = \beta(r)$ which implies that, $\alpha(g)O_H = \beta(r)O_H$................(1)

Also, from the definition of source and target maps in quotient group-groupoid, we have respectively, $\alpha'(gH) = \alpha(g)O_H$ and $\beta'(rH) = \beta(r)O_H$.......(2)

So, from (1) and (2) we have, $\alpha'(gH) = \beta'(rH)$.

We note that from Proposition 3.4, Propositions 3.6, 3.7 and Proposition 3.8 that, $\rho \cdot \Gamma = \gamma^{'}$.

Remark 3.11 From Definition 3.9, we have the following illustration;

Since $q : G \to G/H$ and $O_q : O_G \to O_G/O_H$ are quotient maps then from the following commutative digram,

$$\begin{array}{ccc} G & \xrightarrow{q} & G/H \\ \varepsilon\uparrow \beta\downarrow \alpha\downarrow & & \downarrow\alpha^{'} \downarrow\beta^{'} \uparrow\varepsilon^{'} \\ O_G & \xrightarrow{O_q} & O_G/O_H \end{array}$$

We have, $\alpha^{'}(gH) = \alpha(g)O_H$, $\beta^{'}(gH) = \beta(g)O_H$ and $\varepsilon^{'}(xO_H) = 1_x H$. Further, the inversion map from,

$$
\begin{array}{ccc}
G & \xrightarrow{q} & G/H \\
\sigma\downarrow & & \downarrow\sigma' \\
G & \xrightarrow{q} & G/H
\end{array}
$$

Such that, $\sigma'(gH) = g^{-1}H$.

Also, the constructions of multiplication map γ' come from the following digram,

$$
\begin{array}{ccc}
G \underset{\alpha=\beta}{\times} G/H \underset{\alpha=\beta}{\times} H & \xleftarrow{\varphi} & G \underset{\alpha=\beta}{\times} G \\
\Psi \downarrow\uparrow \Gamma & \searrow^{\rho} & \downarrow q\circ\gamma \\
G/H \underset{\alpha=\beta}{\times} G/H & \xrightarrow{\gamma'} & G/H
\end{array}
$$

See Remark (3.5).

Proposition 3.12 *Let G/H be quotient group-groupoid then,*

1. $gH \circ rH = gH\overline{1_{\alpha(g)}H}rH = g\overline{1_{\alpha(g)}}rH$,

2. $(gH)^{-1} = 1_{\alpha(g)}H\overline{gH}1_{\beta(g)}H = 1_{\alpha(g)}\overline{g}1_{\beta(g)}H$.

Proof. Clear.

Remark 3.13 There is another way to prove the Proposition (3.7).

Want to prove that, $gH \circ rH = (g\circ r)H.$

Since, $gH \circ rH = gH\overline{1_{\alpha(g)}H}rH = g\overline{1_{\alpha(g)}}rH$.

On the other hand, $(g\circ r)H = g\overline{1_{\alpha(g)}}rH$.

So, we have, $gH \circ rH = (g\circ r)H.$

Proposition 3.14 *Let* G/H *be a quotient group-groupoid. Then it's satisfies the interchange law.*

Proof. Let, $a,b,g,r \in G$ where $(b \circ a)$ and $(g \circ r)$ are defined then,

$$(bH \circ aH)(gH \circ rH)$$
$$=(b \circ a)H(a \circ r)H$$
$$=(b \circ a)(g \circ r)H$$
$$=(bg \circ ar)H$$
$$=bgH \circ arH$$
$$=(bHaH) \circ (aHrH).$$

Proposition 3.15 *Let* G/H *be a quotient group-groupoid. Then the following,*

1. $m' : G/H \times G/H \to G/H$ *such that* $(gH, rH) \mapsto grH,$

2. $u' : G/H \to G/H$ *such that* $gH \mapsto \bar{g}H$ *are morphisms of groupoids.*

Proof. 1.Observe that, $m'((gH, aH) \circ (rH, bH)) = m'(gH \circ rH, aH \circ bH)$

$$=(gH \circ rH)(aH \circ bH) = (g \circ r)H(a \circ b)H$$
$$=(g \circ r)(a \circ b)H = (ga \circ rb)H$$
$$=gaH \circ rbH = (gHaH) \circ (rHbH)$$
$$= m'(gH, aH) \circ m'(rH, bH).$$

Also we have, $m'(a^{-1}H, b^{-1}H) = a^{-1}b^{-1}H = (ab)^{-1}H = m'(aH, bH)^{-1}$.

Further, $\beta'.m' = O_m.\beta' \times \beta'$, $\alpha'.m' = O_m.\alpha' \times \alpha'$ and $m'.\varepsilon' \times \varepsilon' = \varepsilon'.O_m$.

2. Similar above, $u'(aH \circ bH) = u'(aH) \circ u'(bH)$.

Also, $u'(a^{-1}H) = u'(aH)^{-1}$.

Moreover, $\beta'.u' = O_u.\beta'$, $\alpha'.u' = O_u.\alpha'$ and $u'.\varepsilon' = \varepsilon'.O_u$.

Proposition 3.16 *Let* G/H *be a quotient group-groupoid. Then* $\alpha', \beta', \varepsilon'$ *and* σ' *are morphism of groups.*

Proof. First of all we have, $\alpha^{'}(gHrH) = \alpha^{'}(grH) = \alpha(gr)O_H = \alpha(g)\alpha(r)O_H = \alpha(g)O_H\alpha(r)O_H$. On the other hand, $\alpha^{'}(gH)\alpha^{'}(rH) = \alpha(g)O_H\alpha(r)O_H$. Then, $\alpha^{'}(gHrH) = \alpha^{'}(gH)\alpha^{'}(rH)$.

Also since, $\alpha^{'}(\overline{gH}) = \alpha^{'}(\overline{g}H) = \alpha(\overline{g})O_H\,\overline{\alpha^{'}(gH)} = \overline{\alpha(g)O_H} = \alpha(\overline{g})O_H$ .Then we have, $\alpha^{'}(\overline{gH}) = \overline{\alpha^{'}(gH)}$. So, $\alpha^{'}$ is a morphism of groupoids and similar we can show that $\beta^{'}$ is morphism of groups. Moreover, it's easy to see that $\sigma^{'}(gHrH) = \sigma^{'}(gH)\sigma^{'}(rH)$, and $\sigma^{'}(\overline{gH}) = \overline{\sigma^{'}(gH)}$.

Also, $\varepsilon^{'}(xO_H\,yO_H) = 1_{xy}H = \varepsilon^{'}(xO_H)\varepsilon^{'}(yO_H)$, and $\varepsilon^{'}(\overline{xO_H}) = \varepsilon 1_{\overline{x}}H = \overline{1_x}H = \overline{\varepsilon^{'}(xO_H)}$. Therefore, the inversion and object maps are morphisms of groupoids.

Example 3.17 Let *G* be a group. Observe that $G\times G$ is a *group-groupoid* with the object set *G* and the morphims are the pairs (a,b). The source and target maps are defined by $\alpha(a,b)=b$ and $\beta(a,b)=a$, the composition $\circ$ of groupoid is defined by $(a,b)\circ(b,c)=(a,c)$ and the group multiplication is defined by $(a,b)(d,c)=(ad,bc)$. Also, the object map defined $\varepsilon(a)=(a,a)$. Therefore, $G\times G$ is a group-groupoid. Now, let *H* be a normal subgroup in *G*. Then $H\times H$ is normal subgroup-groupoid in $G\times G$. So, $G\times G/H\times H$ is a *quotient group-groupoid* with the object set G/H. and the morphims are the pairs $(a,b)H\times H$. The source, target and object maps are defined by $\alpha^{'}((a,b)H\times H) = \alpha(a,b)H = bH$, $\beta^{'}((a,b)H\times H) = aH$ and defined the object map $\varepsilon^{'}(aH) = (a,a)H\times H$ respectively. The composition $\circ$ of quotient group-groupoid defined as; $(a,b)H\times H\circ(b,c)H\times H = ((a,b)\circ(b,c))(H\times H) = (a,c)H\times H.$

Further, the group multiplication; $(a,b)H\times H(d,c)H\times H = (ad,bc)H\times H.$ Also, inverse group map; $u^{'}((a,b)H\times H) = (\overline{a},\overline{b})H\times H.$ Moreover, the inversion map groupoids defined as, $\sigma^{'}((a,b)H\times H) = (b,a)H\times H.$

Proposition 3.18 *Let* $f:G\to K$ *be onto morphism of group-groupoid, then* $Kerf$ *is normal subgroup-groupoid, where* $Ker(O_f) = O_{Kerf}$.

Proof. Since $Kerf$ and O_{Kerf} are normal subgroup. Then we need to prove that $Kerf$ is a subgroupoid. Let $a,b\in Kerf$ such that $a\circ b$ defined then,

1. $f(a\circ b)=f(a)\circ f(b)=1_e\circ 1_e=1_e$.
2. $f(a^{-1})=(f(a))^{-1}=(1_e)^{-1}=1_e$.

Proposition 3.19 *Let $f:G\to K$ be onto morphism of group-groupoids, H be a normal subgroup-groupoid such that $H\subset Kerf$. Then there exist a unique onto morphism of group-groupoids $h:G/H\to K$ such that $f=h\cdot q$, where q is a quotient map and G/H is a quotient group-groupoid.*

Proof. Define,
$$h(gH)=f(g)$$
It's easy to see that *h* a unique morphism of groups. Further, we need to prove that *h* is a morphism of groupoids.

Let, $h(gH\circ rH)=h(g\circ rH)=f(g\circ r)=f(g)\circ f(r)=h(gH)\circ h(rH)$.

Also,
$$h((gH)^{-1})=f(g^{-1})=(f(g))^{-1}=h(gH)^{-1}.$$

Further,
$$\alpha_K\cdot h(gH)=\alpha_K\cdot f(g)=O_f\cdot\alpha_G(g)=O_h(O_q\cdot\alpha_G(g))=O_h\cdot\alpha'(gH),$$

Similar we have,
$$\beta_K\cdot h(gH)=O_h\cdot\beta'(gH).$$

Moreover,
$$h\cdot\varepsilon'=\varepsilon_K\cdot O_h.$$

Remark 3.20 From a bove proposition if $H=Kerf$, then *h* will become an isomorphism of groups [4]. So, we will call to a bijection morphism of group-groupoids it an *isomorphism group-groupoids*.

Proposition 3.21 *Let G be a group-groupoid and H and K are normal subgroup-groupoid then,*

1.*HK is a normal subgroup-groupoid in G, where* $O_{HK}=O_HO_K$.

2. $H\cap K$ *is normal subgroup-groupoids, where* $O_{H\cap K}=O_H\cap O_K$.

Proof. 1. Since *HK* is a normal subgroup then it's enough to prove that *HK* is a subgroupoid. Let $a,b\in HK$ such that $a\circ b$ defined then, $a\circ b=h_1k_1\circ h_2k_2=\overline{h_1k_11_{\alpha(h_1k_1)}}h_2k_2=(h_1k_1)(1_{\alpha(\bar{h}_1)}1_{\alpha(\bar{k}_1)})(h_2k_2)$ implies that, $a\circ b\in HK$. Further, it's clear that, $a^{-1}=(h_1k_1)^{-1}=h_1^{-1}k_1^{-1}\in HK$. So, HK is normal subgroup-groupoid.

2.Clear.

Proposition 3.22 *Let H and K are subgroup-groupoid in a group-groupoid G, where H is a normal subgroup-groupoid and G/H is a quotient group-groupoid then*,

1. *H is normal subgroup-groupoid of HK,*

2. $H \cap K$ *is normal subgroup-groupoid of K,*

3. *There is isomorphism group-groupoid between* $K/H \cap K$ *and* KH/H.

Proof. 1 and 2 clear.

3.Let $k_1H, k_2H \in KH/H$ such that $k_1 \circ k_2$ is defind then, $k_1H \circ k_2H = (k_1 \circ k_2)H \in KH/H$. Also, we have $(kH)^{-1} = k^{-1}H \in KH/H$. Then, KH/H is a subgroup-groupoid and similar we can prove that, $K/H \cap K$ is a subgroup-groupoid of *G*. Now define, $f : K \to KH/H$, $f(k) = kH$. Then *f* is onto morphism of groups. Also, *f* is a morphism of groupoids from the following commutative digram:

$$\begin{array}{ccc} K & \xrightarrow{f} & KH/H \\ {\scriptstyle \varepsilon\uparrow\beta\downarrow\alpha} & & {\scriptstyle \alpha'\downarrow\beta'\uparrow\varepsilon'} \\ O_K & \xrightarrow{O_f} & O_K O_H / O_H \end{array}$$

Further, we have, $f(k_1 \circ k_2) = (k_1 \circ k_2)H = k_1H \circ k_2H = f(k_1) \circ f(k_2)$, and $f(k^{-1}) = k^{-1}H = (kH)^{-1} = f(k)^{-1}$. Also, by the second fundamental theorem of group isomorphism we have, $\theta : K/H \cap K \to HK/H$ where $Ker(f) = H \cap K$ an isomorphism of groups and it is easy to see that θ is a morphism of groupoids.

4. Toplogical Quotient Group-groupoid

In this section we introduce the definition of toplogical quotient group-groupoid and discuss some of its properties. Let *G* be a topological group [1], and *H* be a normal subgroup, since the quotient map $q : G \to G/H$ is continuous, open and onto then, G/H topological group such that:

$m' : G/H \times G/H \to G/H$, $(gH, rH) \mapsto grH$, and

$u' : G/H \to G/H$, $gH \mapsto \bar{g}H$ are continuous.

In the following we consider G is a group-groupoid, H is a normal subgroup-groupoid and G/H is a quotient group-groupoid.

Definition 4.1 A *groupoid in topological quotient group* (*topological quotient group-groupoid*) is a quotient group-groupoid G/H with a topologies on the set of arrows G/H and the set of objects O_G/O_H such that all the morphisms of a quotient group-groupoid are continuous, that is: $m,' \ u,' \ ,\alpha,' \ \beta' ,\gamma',\varepsilon'$ and $\sigma.'$

Proposition 4.2 *Let G be a topological group-groupoid and H be a normal subgroup-groupoid. Then* G/H *is topological quotient group-groupoid.*

Proof. Since G/H is topological group. Want to prove that $\alpha,' \ \beta' ,\gamma',\varepsilon'$ and σ' are continuous; Since, $\alpha' \cdot q = O_q \cdot \alpha$ then $\alpha,'$ is continuous from the fact that $\alpha' \cdot q$ is continuous and q is a quotient map. In similar way it is easy to see that β', ε' and σ' are continuous. It remains to prove that γ'. Define,

$$\begin{array}{ccc} G \underset{\alpha=\beta}{\times} G & \xrightarrow{\gamma} & G \\ \varphi \downarrow & & \downarrow q \\ G \underset{\alpha=\beta}{\times} G/H \underset{\alpha=\beta}{\times} H & \xrightarrow{\rho} & G/H \end{array}$$

Such that $q \cdot \gamma = \rho \cdot \varphi$ where q and φ are quotient maps. Then it's clear that ρ is continuous. Moreover, $\Gamma : G/H \underset{\alpha=\beta}{\times} G/H \to G \underset{\alpha=\beta}{\times} G/H \underset{\alpha=\beta}{\times} H$ is a continuous and since $\gamma' = \rho \cdot \Gamma$ then we have γ' is continuous.

Example 4.3 Let G be a topological group-groupoid then we have the set, $H = G(e) = \alpha^{-1}(e) \cap \beta^{-1}(e)$ is normal subgroup-groupoid with one object $O_H = \{e\}$. Then G/H is a topological quotient group-groupoid such that the set of arrows is G/H and the set of objects is O_G/O_H. The source, target and object maps defind, $\alpha'(aH) = \alpha(a)$, $\beta'((aH) = \beta(a)$ and $\varepsilon'(x) = 1_x H$ are continuous. Moreover, the inversion map, $\sigma'(aH) = a^{-1}H$ and the multiplication map;

$\gamma^{'}: G/H \underset{\alpha=\beta}{\times} G/H \to G/H$, $\gamma^{'}(aH, bH) = aH \circ bH = (a \circ b)H$ are continuous and the set of arrows G/H and the set of objects O_G/O_H are topological quotient groups.

Proposition 4.4 *Let G/H be a topological quotient group-groupoid.Then the set of arrows G/H and the set of objects O_G/O_H are topological groups.*

Proof. Clear.

Definiton 4.5 Let $f: G \to H$ be a continuous morphism of topological group-groupoids. Then *f* called topological isomorphism group-groupoid if and only if its bijection and open.

Proposition 4.6 *Let $f: G \to H$ be onto continuous morphism of topological group-groupoids. Then $h: G/Ker(f) \to H$ is a continuous ismorphism group-groupoids such that $f = h \cdot q$, where q is a quotient map and $G/Ker(f)$ is a quotient group-groupoid.Moreover, h is topological isomorphism group-groupoids if and only if f is open map.*

Proof. Since $f = h \cdot q$ then its clear that *h* is continuous and isomorphism of group-groupoids see Proposition (3.19) and Remark (3.20) Also, if *h* is open and since *q* is open and $f = h \cdot q$ then *f* is open. Conversley, suppose that *f* is open then since *q* is quotient map its easy to see that *h* is open.

Proposition 4.7 *Let G be a topological group-groupoid and H be a normal subgroup-groupoid such that G/H is a quotient group-groupoid.Then the arrows of G/H and the objects O_G/O_H are Hausdroff (discrete) if and only if H and O_H are closed (open) in G and O_G respectively.*

Proof. Clear. See [1, Proposition 3.18].

Proposition 4.8 *Let H and K are subgroup-groupoid in a topological group-groupoid G, where H is a normal subgroup-groupoid and let $q: G \to G/H$ the quotient map. Then, the bijection $\theta: K/H \cap K \to HK/H$ continuous.*

In topological groupoid *G*, if the target or the source maps are open then *G* is called interior (open) topological groupoid which means that the other maps in *G* are open maps.

Proposition 4.9 *Let* G/H *be a topological quotient group-groupoid. If one of* α, β *is open.Then* $\alpha^{'}, \beta^{'}$ *and* $\gamma^{'}$ *are open.*

Proof. Suppose that α is open then β and γ are open maps. Since maps *q* and O_q are quotients and $\alpha^{'} \cdot q = O_q \cdot \alpha$ then it's clear that $\alpha^{'}$ is open. Also, since $\sigma^{'}$ is open and $\beta^{'} = \alpha^{'} \cdot \sigma^{'}$ then $\beta^{'}$ is open. Moreover, since $q \cdot \gamma = \rho \cdot \varphi$ then we have ρ is open see Proposition (4.2) and since $\rho \cdot \Gamma = \gamma^{'}$ where Γ is open map then $\gamma^{'}$ is open.

5.The Product of Quotient Group-groupoids

Here we define the concept of the product of quotient group-groupoid.

Let $\{G_i : i \in I\}$ be a family of topological group-groupoids then the product $G = \prod_i G_i$ with $O_G = \prod_i O_{G_i}$ form a topological group-groupoid [6, Proposition, 4.1].

In the following paragrph we introduce the concepts of the product quotient group-groupoids. Let $\{G_i/H_i : i \in I\}$ be a family of quotient group-groupoids where H_i is a normal subgroup-groupoid for all $i \in I$. Define the set of arrows $G = \prod_{i \in I} G_i/H_i$ with the set of objects $O_G = \prod_{i \in I} O_{G_i}/O_{H_i}$ such that O_G is the set of all tuples $(x_i O_{H_i})_{i \in I}$, for each $x_i O_{H_i} \in O_{G_i}/O_{H_i}$. The set of arrows is the set of all tuples $(g_i H_i)_{i \in I}$, for each $g_i H_i \in G_i/H_i$, and the composition, multiplication and inverse defined respectively;

1. $(g_i H_i)_{i \in I} \circ (r_i H_i)_{i \in I} = ((g_i \circ r_i) H_i)_{i \in I}$, for each $(g_i H_i, r_i H_i) \in G_i/H_i \underset{\alpha_i = \beta_i}{\times} G_i/H_i$,
2. $(g_i H_i)_{i \in I} . (r_i H_i)_{i \in I} = ((g_i . r_i) H_i)_{i \in I}$, for each $(g_i H_i, r_i H_i) \in G_i/H_i \times G_i/H_i$,
3. $u^{'}((g_i H_i)_{i \in I}) = (\overline{g_i} H_i)_{i \in I}$.

Also, we define the object group multiplications;

4. $(x_i O_{H_i})_{i \in I} . (y_i O_{H_i})_{i \in I} = ((x_i . y_i) O_{H_i})_{i \in I}$,

for each $(x_iO_{H_i}, y_iO_{H_i}) \in O_{G_i}/O_{H_i} \times O_{G_i}/O_{H_i}$.

The source $\alpha^{'}$, the target $\beta^{'}$ and the object map $\varepsilon^{'}$ are defined as,

5. $\alpha^{'}((g_iH_i)_{i\in I}) = (\alpha_i(g_i)H_i)_{i\in I}$,
6. $\beta^{'}((g_iH_i)_{i\in I}) = (\beta_i(g_i)H_i)_{i\in I}$,
7. $\varepsilon^{'}((x_iO_{H_i})_{i\in I}) = (\varepsilon_i(x_i)O_{H_i})_{i\in I}$.

Also, the inversion map is,

8. $\sigma^{'}((g_iH_i)_{i\in I}) = (\sigma_i(g_i)H_i)_{i\in I}$.

Further, *G* satisfies the interchange law, i.e.

$$((g_iH_i)_{i\in I} \circ (r_iH_i)_{i\in I})\ ((k_iH_i)_{i\in I} \circ (t_iH_i)_{i\in I}) =$$

$$((g_i \circ r_i)H_i)_{i\in I}\ ((k_i \circ t_i)H_i)_{i\in I}$$

$$= ((g_i \circ r_i)\ (k_i \circ t_i)H_i)_{i\in I}$$

$$= (((g_ik_i) \circ (r_i.t_i))H_i)_{i\in I}$$

$$= ((g_i.k_i)H_i)_{i\in I} \circ ((r_i.t_i)H_i)_{i\in I}$$

$$= ((g_iH_i)_{i\in I}.(k_iH_i)_{i\in I}) \circ ((r_iH_i)_{i\in I}.(t_iH_i)_{i\in I}) .$$

Moreover, $\alpha^{'}, \beta^{'}$ and $\varepsilon^{'}$ are morphisms of groups;

$$\alpha^{'}((g_iH_i)_{i\in I}.(r_iH_i)_{i\in I}) = \alpha^{'}(((g_i.r_i)H_i)_{i\in I})$$

$$= (\alpha_i(g_i.r_i)H_i)_{i\in I}$$

$$= (\alpha_i(g_i)H_i)_{i\in I}.(\alpha_i(r_i)H_i)_{i\in I}$$

$$= \alpha^{'}((g_iH_i)_{i\in I}).\alpha^{'}((r_iH_i)_{i\in I}) .$$

Also, we can see that;

$$\beta^{'}((g_iH_i)_{i\in I}.(r_iH_i)_{i\in I}) = \beta^{'}((g_iH_i)_{i\in I}).\beta^{'}((r_iH_i)_{i\in I}),$$

$\varepsilon^{'}((x_i O_{H\,i})_{i\in I} \cdot (y_i O_{H_i})_{i\in I}) = \varepsilon^{'}(x_i O_{H\,i})_{i\in I}) \cdot \varepsilon^{'}(y_i O_{H_i})_{i\in I})$.

Propostion 5.1 *Let* $\{G_i/H_i : i \in I\}$ *be a family of topological quotient group-groupoids then the product* $G = \prod_i G_i/H_i$ *is topological quotient qroup-groupoid, where* H_i *is a normal subgroup-groupoid for all* $i \in I$.

Proof. The previous discussion indicates that *G* is a quotient group-groupoid. Also its clear that the multiplication and inverse maps of group in the following;

1. $m^{'} : G \times G \to G$, $m^{'}((g_i H_i)_{i\in I}, (r_i H_i)_{i\in I}) = ((g_i . r_i) H_i)_{i\in I}$,

2. $u^{'} : G \to G$, $u^{'}((g_i H_i)_{i\in I}) = (\overline{g_i} H_i)_{i\in I}$,

are contiuous maps, see[1, pp.237-39].

Further, we have the multiplication map of groupoids;

3. $\gamma^{'} : G \underset{(\alpha_i = \beta_i)_{i\in I}}{\times} G \to G$, $\gamma^{'}((g_i H_i)_{i\in I}, (r_i H_i)_{i\in I}) = (g_i H_i \circ r_i H_i)_{i\in I} = ((g_i \circ r_i) H_i)_{i\in I}$,

for each $(g_i H_i, r_i H_i) \in G_i / H_i \underset{\alpha_i = \beta_i}{\times} G_i / H_i$, are continuous map from the fact that,

$$P_i \cdot \gamma^{'} = \gamma^{'}{}_i \cdot (P_i \underset{\alpha_i = \beta_i}{\times} P_i).$$

Moreover, we have to prove that $\sigma^{'}, \alpha^{'}, \beta^{'}$ and $\varepsilon^{'}$ are all continuous maps. But

a) $P_i \cdot \sigma^{'} = \sigma^{'}{}_i \cdot P_i$,

b) $O(P_i) \cdot \alpha^{'} = \alpha^{'}{}_i \cdot P_i$,

c) $O(P_i) \cdot \beta^{'} = \beta^{'}{}_i \cdot P_i$,

d) $\varepsilon^{'}{}_i \cdot O(P_i) = P_i \cdot \varepsilon^{'}$.

where $P_i : G \to G_i / H_i$ and $O(P_i) : O_G \to O_{G_i} / O_{H_i}$ are projection maps. Therefore, $\sigma^{'}, \alpha^{'}, \beta^{'}$ and $\varepsilon^{'}$ are all continuous maps. This implies that *G* is a topological quotient group-groupoid.

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