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# On A Natural Transformation Between Generalized Topologies and Strongly Generalized Interior Operators

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**Abstract**

We prove that generalized topologies, monotonic maps on power sets and strong generalized interior operators extend to functors such that the isomorphism between generalized topologies and strong generalized interior operators is natural.

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## 1. Introduction

Since the introduction of generalized open sets and generalized topologies by Cśasźar ([2], [3], [4]), various authors have investigated their properties and their relationship with generalized systems. In particular Min [6] has obtained an isomorphism between the generalized topologies and the strongly generalized interior operators (Theorem 1.3 below). In this paper it is our purpose to show that the concepts of generalized topologies and monotonic maps extend to functors on the category of SETS (Theorem 2.1 and 2.2 below) such that there is a natural transformation between them (Theorem 2.3 below). This natural transformation motivates the setting up of strongly generalized interior operator as a functor (Theorem 2.4 below). It is shown thereby that the isomorphism obtained by Min in [6] is in fact a natural isomorphism (Theorem 2.5 below) in the sense of category theory.

We refer to the following concepts and result:

**Definition 1.1. [3]** A collection **g** of subsets of a non-empty set X is called a generalized topology on X if **g** contains the empty set and is closed under arbitrary unions. The members of **g** are called generalized open sets. The collection of all generalized topologies on X is denoted by GT(X).

For more detailed information, we refer to [2], [3] and [4].

**Definition 1.2. [6]** A function $I : \mathbf{P}(X) \to \mathbf{P}(X)$between the power set of a set X is called a strongly generalized interior operator if it satisfies the following three conditions;

(C1) I(A) ⊂ A, for all A ⊂ X

(C2) I is monotonic; I(A) ⊂ I(B), whenever A ⊂ B for subsets A and B of X

(C3) I is idempotent; I(I(A)) = I(A), for all subsets A of X. SGI(X) denotes the collection of all strongly generalized interior operators on X.

We shall also refer to the following Theorem of Min

**Theorem 1.3. (Theorem 3.7 [6])** Let X be a nonempty set. Then there exists a bijection between SGI(X) and GT(X) namely $\eta : GT(X) \to SGI(X)$ defined by $\eta(g) = i_g$, where $i_g : P(X) \to P(X)$ is given by $i_g(A) = \bigcup\{G : G \in g, G \subset A\}$ that is the largest generalized open set in the generalized topology g on X.

We shall use the terminology of categories and functors. For the standard notions we refer to [1] and [5]. In particular we refer to the following definitions of functor and natural transformation between functors:

**Definition 1.4. [1]**

(1) A functor $F : \mathbf{C} \to \mathbf{D}$ between categories **C** and **D** is a rule that maps objects and arrows of **C** to objects and arrows of **D** such that if $g : A \to B$ in **C** then $F(g) : F(A) \to F(B)$ in **D** and when $g \circ h$ is defined in **C**, $F(g \circ h) = F(g) \circ F(h)$ in **D**. Further F must preserve identities, i.e. if $1_A : A \to A$ is identity of A, then $F(1_A) = 1_{FA}$

(2) For categories **C**, **D** and functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation $\theta : F \to G$ is a family of arrows in **D**, $\{\theta_X : FX \to GX\}_{X \in \mathbf{C}}$ such that, for any $f : X \to Y$ in **C**, $\theta_Y \circ F(f) = G(f) \circ \theta_X$, that is the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\theta_Y} & G(Y) \end{array}$$

We shall work exclusively in the category of SETS.

## 2. Results

We begin by observing that since any function $h : X \to Y$ is well behaved under arbitrary unions, therefore generalized topologies on X, should be mapped to generalized topologies on Y by the image map $h : \mathbf{P}(X) \to \mathbf{P}(Y)$. In fact we have the following theorem.

**Theorem 2.1.** The generalized topologies extend to a covariant functor on SETS.

*Proof.* We construct the functor $GT : SETS \to SETS$ by defining GT(X) as the collection of all the generalized topologies on X. Further for any $h : X \to Y$, since $h(\bigcup V_\alpha) = \bigcup h(V_\alpha)$ for all $V_\alpha \subset X$, therefore for a generalized topology g on X, $\{h(V) : V \in g\}$ is a generalized topology on Y. Therefore we define $GT(h)(g) = \{h(V) : V \in g\}$. Then $GT : SETS \to SETS$ maps X to GT(X) and h to GT(h). Further if $h_1 : X \to Y$ and $h_2 : Y \to Z$, then $GT(h_1) : GT(X) \to GT(Y)$ and $GT(h_2) : GT(Y) \to GT(Z)$, $(GT(h_2)oGT(h_1))(g) = GT(h_2)(\{h_1(V) : V \in g\}) = \{h_2(h_1(V)) : V \in g\} = \{(h_2oh_1)(V) : V \in g\} = (GT(h_2oh_1))(g)$. Therefore $GT(h_2)oGT(h_1) = GT(h_2oh_1)$. Finally if $1_X : X \to X$ is identity map then $GT(1_X) : GT(X) \to GT(X)$ is identity map. Therefore GT is a functor. ■

In our theorem 2.2 below we show that the collection $\Gamma(X)$ of all monotonic maps defined in [2] also extends to a covariant functor $\Gamma$.

**Theorem 2.2.** There is a functor $\Gamma : SETS \to SETS$ such that $\Gamma(X) = \{\gamma : \mathbf{P}(X) \to \mathbf{P}(X) : \gamma \text{ is monotonic}\}$.

*Proof.* We need only define $\Gamma$ on functions. For this if $h : X \to Y$ is any function, define $\Gamma(h) : \Gamma(X) \to \Gamma(Y)$ by $\Gamma(h)(\gamma) = h \circ \gamma \circ h^{-1}$, where $h : \mathbf{P}(X) \to \mathbf{P}(Y)$ and $h^{-1} : \mathbf{P}(Y) \to \mathbf{P}(X)$ are image and inverse image maps respectively. If $\gamma$ is monotonic, then $h \circ \gamma \circ h^{-1}$ is monotonic, therefore $\Gamma(h)$ is well defined. For composition consider any functions $h_1 : X \to Y$ and $h_2 : Y \to Z$, then $\Gamma(h_1) : \Gamma(X) \to \Gamma(Y)$ and $\Gamma(h_2) : \Gamma(Y) \to \Gamma(Z)$, $\Gamma(h_2)o\Gamma(h_1))(\gamma) = (\Gamma(h_2))(h_1o\gamma oh_1^{-1}) = h_2o(h_1o\gamma oh_1^{-1})oh_2^{-1} = (h_2oh_1)o\gamma o(h_2oh_1)^{-1} = (\Gamma(h_2oh_1))(\gamma)$. Therefore $\Gamma(h_2)o\Gamma(h_1) = \Gamma(h_2oh_1)$. Also if $1_X : X \to X$ is identity map then $\Gamma(1_X) : \Gamma(X) \to \Gamma(X)$ is identity map. Therefore $\Gamma$ is a functor. ■

In view of the fact that the operator $i_g$ in theorem 1.3 is monotonic and therefore for every generalized topology on X, $i_g$ gives us an element of $\Gamma(X)$, we are therefore able to set up a natural transformation below between the functors GT and $\Gamma$.

**Theorem 2.3.** There is a natural transformation $\phi$ from GT to $\Gamma$.

*Proof.* To prove the existence of a natural transformation $\phi : GT \to \Gamma$, we need to find a collection $\{\phi_X : X \text{ a set}\}$ such that whenever $h : X \to Y$ is any function, the following diagram is commutative.

$$\begin{array}{ccc} GT(X) & \xrightarrow{\phi_X} & \Gamma(X) \\ \downarrow{\scriptstyle GT(h)} & & \downarrow{\scriptstyle \Gamma(h)} \\ GT(Y) & \xrightarrow{\phi_Y} & \Gamma(Y) \end{array}$$

For this define for any set X, $\phi_X : GT(X) \to \Gamma(X)$ by $\phi_X(g) = i_g$, since $i_g$ is monotonic, therefore $\phi_X$ is well-defined. To check the commutativity of fig 1,as for naturality of $\phi = \{\phi_X\}$, if $h : X \to Y$ is any function, then $(\phi_Y \circ GT(h))(g) = \phi_Y(\{h(V) : V \in g\}) = \phi_Y(g')$ where $g' = \{h(V) : V \in g\}, \phi_Y(g') : \mathbf{P}(Y) \to \mathbf{P}(Y)$ given by $\phi_Y(g')(B) = \bigcup\{G' : G' \in g', G' \subset B\} = \bigcup\{h(V) : V \in g, h(V) \subset B\} = \bigcup\{h(V) : V \in g, V \subset h^{-1}B\}$, also $(\Gamma(h) \circ \phi_X)(g) = \Gamma(h)(\phi_X(g)) = h \circ (\phi_X(g)) \circ h^{-1}$ where $h \circ (\phi_X(g)) \circ h^{-1} : \mathbf{P}(Y) \to \mathbf{P}(Y)$ given by $(h \circ (\phi_X(g)) \circ h^{-1})(B) = h(\phi_X(g)(h^{-1}B)) = h(\bigcup\{V : V \in g, V \subset h^{-1}B\}) = \bigcup\{h(V) : V \in g, V \subset h^{-1}B\}$, therefore $(\phi_Y \circ GT(h))(g) = (\Gamma(h) \circ \phi_X)(g)$, therefore $(\phi_Y \circ GT(h)) = (\Gamma(h) \circ \phi_X)$. Hence the diagram is commutative and therefore $\phi = \{\phi_X\}$ is natural transformation. ∎

For our next result it is important to notice that $\phi_X(g) = i_g$ defined in theorem 2.3 above, is not only a monotonic map but infact a strong generalized interior operator in the sense of [6]. It was proved by Min[6] that there is an isomorphism between generalized topologies on a set X and the collection of all strong generalized interior operators. Since $\phi_X(g)$ is a strong generalized interior operator and $\phi$ is a natural transformation, therefore it is imperative to check if the isomorphism of Min is natural or not. But for this we must first establish strong generalized interior operator as a functor. This is done in our theorem 2.4 below.

**Theorem 2.4.** The strong generalized interior operators extend to a covariant functor on SETS.

*Proof.* We construct the functor $SGI : SETS \to SETS$ by defining SGI(X) as the collection of all the strong generalized interior operators on X. If $h : X \to Y$ is any function, we define $SGI(h) : SGI(X) \to SGI(Y)$ by $SGI(h)(p) = h \circ p \circ h^{-1}$, where $h : \mathbf{P}(X) \to \mathbf{P}(Y)$ and $h^{-1} : \mathbf{P}(Y) \to \mathbf{P}(X)$ are image and inverse image map respectively. p satisfies (C1),(C2) and (C3). For any subset B of Y, $(h \circ p \circ h^{-1})(B) = h(p(h^{-1}B)) \subset h(h^{-1}B) \subset B$, $SGI(h)(p)$ satisfies (C1). If $B_1 \subset B_2$,then $(h \circ p \circ h^{-1})(B_1) = h(p(h^{-1}B_1)) \subset h(p(h^{-1}B_2)) = (h \circ p \circ h^{-1})(B_2)$, $SGI(h)(p)$ satisfies (C2). $(h \circ p \circ h^{-1})((h \circ p \circ h^{-1})(B)) = (h \circ p \circ h^{-1} \circ h \circ p \circ h^{-1})(B) \supset (h \circ p) \circ (p \circ h^{-1})(B) = (h \circ p \circ p \circ h^{-1})(B) = (h \circ p \circ h^{-1})(B), also (h \circ p \circ h^{-1})(B) \subset B \Rightarrow (h \circ p \circ h^{-1})(h \circ p \circ h^{-1})(B) \subset (h \circ p \circ h^{-1})(B)$, so $(h \circ p \circ h^{-1})(h \circ p \circ h^{-1})(B) = (h \circ p \circ h^{-1})(B)$, $SGI(h)(p)$ satisfies (C3). Therefore $SGI(h)(p) \in SGI(Y)$ and so

$SGI(h)$ is well defined. If $h_1 : X \to Y$ and $h_2 : Y \to Z$ are any functions then $SGI(h_1) : SGI(X) \to SGI(Y)$ and $SGI(h_2) : SGI(Y) \to SGI(Z), SGI(h_2) \circ SGI(h_1))(p) = (SGI(h_2))(h_1 \circ p \circ h_1^{-1}) = h_2 \circ h_1 \circ p \circ h_1^{-1} \circ h_2^{-1} = (h_2 \circ h_1) \circ p \circ (h_2 \circ h_1)^{-1} = (SGI(h_2 \circ h_1))(p)$.

Therefore $SGI(h_2) \circ SGI(h_1) = SGI(h_2 \circ h_1)$. Also if $1_X : X \to X$ is identity map then $SGI(1_X) : SGI(X) \to SGI(X)$ is identity map. SGI is a functor. ■

**Theorem 2.5.** The functors GT and SGI are naturally isomorphic.

*Proof.* Define $\eta_X : GT(X) \to SGI(X)$ by $\eta_X(g) = \phi_X(g)$. Since GT and SGI are functors and $\eta_X$ is an isomorphism [6].Therefore it follows from the commutative diagram in theorem 2.3 that $\eta = \{\eta_X\}$ is a natural isomorphism. In particular if $h : X \to Y$ is any function then the following diagram is commutative.

$$\begin{array}{ccc} GT(X) & \underset{\beta_X}{\overset{\eta_X}{\rightleftarrows}} & SGI(X) \\ \downarrow{\scriptstyle GT(h)} & & \downarrow{\scriptstyle SGI(h)} \\ GT(Y) & \underset{\beta_Y}{\overset{\eta_Y}{\rightleftarrows}} & SGI(Y) \end{array}$$

where $\beta_X : SGI(X) \to GT(X)$ is defined by $\beta_X(p) = \{A \subset X : A = pA\}$. ■

**Remarks 2.6.** $\beta : SGI \to GT$ and $\eta : GT \to SGI$ are natural transformations which are inverses of each other.

## References

[1] Steve Awodey,*Category Theory*, Clarendon Press, Oxford, 2006.

[2] Á. Cśasźar, *Generalized open sets*, Acta Math. Hungar., **75**(1-2)(1997), 65–87.

[3] Á. Cśasźar, *Generalized topology, Generalized continuity*, Acta Math. Hungar., **96**(2002), 351–357.

[4] Á. Cśasźar, *Generalized open set in generalized topologies*, Acta Math. Hungar., **106**(2005), 53–66.

[5] S. Mac Lane, *Categories for the working Mathematicians*, Springer, New York, 1998 (Second Edition).

[6] W.K. Min, *Some Results on generalized topological spaces and generalized systems*, Acta Math. Hungar., **108**(1-2) (2005), 171–181.