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Common Fixed Point Theorems under Rational Contraction in Complete G-Metric Space

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Abstract

In this paper, we prove some common fixed point theorems governed by rational contraction on complete *G*-metric spaces for self-mappings satisfying various contractive condition, we also discuss that these mappings are *G*-continuous on such a fixed point.

Keywords: G-metric space, fixed point, rational inequality
A M S *Mathematics subject Classification*: 47H10, 54H25

1. INTRODUCTION

Fixed point is so important that the study finds applications in many important areas as diverse as differential equations, operation research, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians such as G¨ahler [5, 6] (called 2-metric spaces) and Dhage [3,4] (called D-metric spaces). Moreover, it was shown that Dhage's notion of D-metric space is flawed by errors and most of the results established by him and others are invalid. Recently, Mustafa et.al.studied many fixed point theorems for mappings satisfying various contractive conditions on complete G-metric spaces. Subsequently, some authors like Renu Chugh et.al.[2] have generalized some results of Mustafa et.al. [10,11] and studied some fixed point results for self-mapping in a complete G-metric space under some contractive conditions related to a non-decreasing map φ :

$[0, +\infty) \to [0, +\infty)$ with $\lim_{n\to\infty}\varphi_n(t) = 0$ for all $t \in (0, +\infty)$. Our results generalize some recent results in the setting of G-metric space.

PRELIMINARIES

We begin with

Definition 1. ([11]). *Let X be a non-empty set and $G: X^3: [0, \infty)$ be a function satisfying the following axioms:*

(G1) $G(x,y,z) = 0$ if $x=y=z$,

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$

(G3) $G(x,x,y) \leq G(x,y,z)$, for all $x, y,z\ X \in$ with $z \neq y$

(G4) $G(x,y,z)=G(x,z,y)= G(y,z,x)= \cdots$(symmetry in all three variables),

(G5) $G(x,y,z)=G(x,a,a)+G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality)

Then the function *G* is called a generalized metric, or specifically a *G*-metric on X and the pair *(X, G)* is called a *G*-metric Space.

Definition 2. ([11]). *Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $G(x,x_n,x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ or $x_n = x$ as $n \to \infty$, in a G-metric space (X, G), then for each $\varepsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \varepsilon$ for all $m, n \in N$.*

Now, we state some results from the papers ([2]-[6]) which are helpful for proving our main results.

Proposition 1. ([11]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

i. $\{x_n\}$ *is G-convergent to x,*
ii. $G(x_n,x_n, x) \to 0$ as $n \to \infty$,
iii. $G(x_n, x, x) \to 0$ as $n \to \infty$,
iv. $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 3.([10]). *Let (X, G) be a G-metric space. A sequence* $\{x_n\}$ *is called G-Cauchy if, for each* $\varepsilon > 0$*, there exists a positive integer N such that* $G(x_n, x_m, x_l) < \varepsilon$*, for all* $n, m, l \in N$*, i.e., if* $G(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to \infty$.

Definition 4. ([10]). *If (X, G) and* (X', G') *be two G-metric space and let* $f : (X, G) \to (X', G')$ *be a function, then f is said to be G-continuous at a point* $x_0 \in X$ *if given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for* $x, y \in X$ *and* $G(x_0, x, y) < \delta$ *implies* $G'(f(x_0), f(x), f(y)) < \varepsilon$. *A function f is G-continuous at X if and only if it is G-continuous at all* $x_0 \in X$ *or function f is said to be G-continuous at a point* $x_0 \in X$ *if and only if it is G-sequentially continuous at* x_0*, that is, whenever* $\{x_n\}$ *is G-convergent to* x_0*,* $\{f(x_n)\}$ *is G-convergent to* $f(x_0)$.

Proposition 2. ([9]). *Let (X, G) be a G-metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables*.

Definition 5. ([11]). *A G-metric space (X, G) is called a symmetric G-metric space if* $G(x, y, y) = G(y, x, x)$ *for all* $x, y \in X$.

Proposition 3. ([11]). *Every G-metric space (X, G) will define a metric space* (X, d_G) *by*

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x)$ *for all* $x, y \in X$.

If (X, G) is a symmetric G-metric space, then

(ii) $d_G(x, y) = 2G(x, y, y)$ *for all* $x, y \in X$.

However, if *(X, G)* is not symmetric, then it follows from the *G*-metric properties that

(iii) $\frac{3}{2}G(x, y, y) \le d_G(x, y) \le 3G(x, y, y)$ for all $x, y \in X$.

Definition 6. ([10]). *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.*

Proposition 4. ([11]). *A G-metric space (X, G) is said to be G-complete if and only if* (X, d_G) *is a complete metric space.*

Proposition 5. ([9]). *Let (X, G) be a G-metric space. Then, for any x, y, z, a in X, it follows that:*

(i) *if* $G(x, y, z) = 0$*, then* $x = y = z$,

(ii) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \le 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

2. MAIN RESULT

We need the following Lemma to prove our main results:

Lemma 1.[10]. *Let (X, G) be a G-metric space and T be a self-map on X satisfying*

(1) $G(Tx,Ty,Tz) \leq qG(x,y,z)$

for all x, y, z $\in X$*, where* $0 \leq q < 1$*, and* $x_n = T x_{n-1} = T(T x_{n-2}) = = T^n(x_0)$*, for some* $x_0 \in X$*, then* $\{x_n\}$ *is a G-Cauchy sequence in X.*

Theorem 1. Let (X, G) be a complete G-metric space and T : X → X be the mapping satisfying the following :

$$G(Tx,Ty,Tz) \leq$$

$$k\max\left(\begin{array}{c}\frac{G(Tx,Tz,Tz)}{G(Tx,Ty,Tz)+G(y,y,z)}\,.\,G(x,Tx,Tx),\frac{G(Tx,Tx,Tz)}{G(Tz,Ty,Ty)+G(Tx,z,Ty)}\,.\,G(Tz,Ty,Ty),\\ \frac{G(Tx,Tx,Ty)}{G(x,Tz,Tz)+G(Tx,z,y)}\,.\,G(Ty,Ty,Tz),\\ G(y,Tx,Tx),G(z,Tx,Tx),G(x,Ty,Ty),\\ G(z,Tz,Tz),G(y,Tz,Tz),G(z,Tz,Tz)\end{array}\right)$$

for all $x, y, z \in X$, where $0 \leq k < \frac{1}{2}$, then T has a unique fixed point and T is G-continuous at the fixed point.

Proof. Suppose T satisfy condition (2) and $x_0 \in X$ be an arbitrary point

Step 1. We inductively construct the sequence $\{x_n\}$ of point in X as:

$$x_1 = T(x_0)$$

$$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$$

$$x_3 = T(x_2) = T(T^2(x_0)) = T^3(x_0)$$

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$$x_n = T(x^{n-1}) = T(T^{n-1}(x_0)) = T^n(x_0)$$

Clearly $\{x_n\}$ is a sequence of images of x_0, under repeated application of T.

Step 2. *$\{x_n\}$ is a G-Cauchy sequence in X. Assume $x_n \neq x_{n+1}$ for all n. Since if there exist an n such that $x_n = x_{n+1}$ then, $T^n(x_0) = T(T^n(x_0))$, yields $T^n(x_0)$ is a fixed point.*

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq k\, max \begin{pmatrix} \frac{G(Tx_{n-1},Tx_n,Tx_n)}{G(Tx_{n-1},Tx_n,Tx_n)+G(x_n,x_n,x_n)} \cdot G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), \\ \frac{G(Tx_{n-1},Tx_{n-1},Tx_n)}{G(Tx_n,Tx_n,Tx_n)+G(Tx_{n-1},x_n,Tx_n)} \cdot G(Tx_n, Tx_n, Tx_n), \\ \frac{G(Tx_{n-1},Tx_n,Tx_n)}{G(x_{n-1},Tx_n,Tx_n)+G(Tx_{n-1},Tx_{n-1},x_n)} \cdot G(Tx_n, Tx_n, Tx_n), \\ G(x_n, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_{n-1}, Tx_{n-1}), \\ G(x_{n-1}, Tx_n, Tx_n), \\ G(x_n, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n), \\ G(x_n, Tx_n, Tx_n) \end{pmatrix}$$

$$\leq k\, max \begin{pmatrix} \frac{G(x_n,x_{n+1},x_{n+1})}{G(x_n,x_{n+1},x_{n+1})+G(x_n,x_n,x_n)} \cdot G(x_{n-1}, x_n, x_n), \\ \frac{G(x_n,x_n,x_{n+1})}{G(x_{n+1},x_{n+1},x_{n+1})+G(x_n,x_n,x_{n+1})} \cdot G(x_{n+1}, x_{n+1}, x_{n+1}), \\ \frac{G(x_n,x_{n+1},x_{n+1})}{G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)} \cdot G(x_{n+1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_n, x_n), G(x_n, x_n, x_n), \\ G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) \end{pmatrix}$$

$$\leq k\, max\left(G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1})\right)$$

Case 1. If

$$max\begin{pmatrix} G(x_{n-1}, x_n, x_n), \\ G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) \end{pmatrix} = G(x_{n-1}, x_n, x_n),$$

then, using (3), we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq k\, G(x_{n-1}, x_n, x_n)$$

Thus by Lemma 1, we have $\{x_n\}$ is a G-cauchy sequence in *X.*

Case 2. If

$$max\begin{pmatrix} G(x_{n-1}, x_n, x_n), \\ G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) \end{pmatrix} = G(x_{n-1}, x_{n+1}, x_{n+1}),$$

then, from (3) and using G_5 of Definition 1.1, ones obtain

$G(x_n, x_{n+1}, x_{n+1}) \leq$k $G(x_{n-1}, x_{n+1}, x_{n+1})$

$\leq$k $\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})\}$

this implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{k}{1-k}\, G(x_{n-1}, x_n, x_n)$$

$$G(x_n, x_{n+1}, x_{n+1}) \leq q\, G(x_{n-1}, x_n, x_n)$$

Where $q=\frac{k}{1-k}$, $0\leq k < \frac{1}{2}$

Thus again by lemma 1. We have $\{x_n\}$ is a *G*-Cauchy sequence in *X.*

Case 3. Finally, if

$$\max\begin{pmatrix} G(x_{n-1}, x_n, x_n), \\ G(x_{n-1}, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) \end{pmatrix} = G(x_n, x_{n+1}, x_{n+1}),$$

$G(x_n, x_{n+1}, x_{n+1}) \leq$k $G(x_n, x_{n+1}, x_{n+1})$

which is a contradiction, as $k<\frac{1}{2}$.

Hence in all cases the sequence $\{x_n\}$ is a *G*-Cauchy sequence.

Step 3. Since *(X, G)* is a complete *G*-metric space, by definition, there exists $u \in X$ such that $x_n \to u$.

Step 4. *u* is a fixed point of *T* .

Suppose, if possible, that $Tu \neq u$, using (3), we

$$(4)\quad G(x_n, Tu, Tu) = \leq k\, max \begin{pmatrix} \frac{G(x_n,Tu,Tu)}{G(x_n,Tu,Tu)+G(u,u,u)} \,.\, G(u, Tu, Tu), \\ \frac{G(x_n,x_n,Tu)}{G(Tu,Tu,Tu)+G(x_n,u,Tu)} \,.\, G(Tu, Tu, Tu), \\ \frac{G(x_n,x_n,Tu)}{G(x_{n-1},Tu,Tu)+G(x_n,u,u)} \,.\, G(u, Tu, Tu), \\ G(u, x_n, x_n), \\ G(u, x_n, x_n), G(x_{n-1}, Tu, Tu), G(u, Tu, Tu), \\ G(u, Tu, Tu), G(u, Tu, Tu) \end{pmatrix}$$

$$G(x_n, Tu, Tu) = k\, max\big(G(u, Tu, Tu), G(u, x_n, x_n), G(x_{n-1}, Tu, Tu)\big)$$

Taking the limit as $n \to \infty$, and using the fact that function *G* is continuous on its variable, we obtain

$$G(x_n, Tu, Tu) \leq k\, G(u, Tu, Tu)$$

which arises a contradiction, since, $0 \leq k < \frac{1}{2}$.

Hence, $Tu = u$, i.e., u is a fixed point of T.

Step 5. Uniqueness of fixed point u of T.

Suppose that, v (≠u) is another fixed point of T, such that $Tv = v$,

$$G(u,v,v) \leq k\ max\begin{pmatrix} \frac{G(u,v,v)}{G(u,v,v)+G(v,v,v)} \cdot G(u,u,u), \frac{G(u,u,v)}{G(v,v,v)+G(u,v,v)} \cdot G(v,v,v), \\ \frac{G(u,u,v)}{G(v,v,v)+G(u,u,v)} \cdot G(v,v,v), \\ G(v,u,u), G(v,u,u), G(u,v,v), G(v,v,v), G(v,v,v), G(v,v,v) \end{pmatrix}$$

$$= k\ max\big(G(u,v,v), G(v,u,u)\big)$$

Which reduces to,

$$(5) \qquad G(u,v,v) \leq k\, G(v,u,u)$$

Again by same argument we will find

$$(6)\ G(v,u,u) \leq k\, G(u,v,v)$$

Which by repeated use of (5) and (6), implies

$$G(v,v,u) \leq k^2 G(v,u,u) \leq k^3 G(v,u,u) \leq \cdots \leq k^n G(v,u,u)$$

Proceeding limit as $n \to \infty$, we have $u = v$, i.e., u is a unique fixed point of T .

Step 6. T is G-continuous at the fixed point u.

Let $\{y_n\}$ be any sequence in X, such that $\lim_{n\to\infty} y_n = u$, then, by (2), we obtain

$$G(Ty_n, Tu, Ty_n) \leq k\ max\begin{pmatrix} \frac{G(Ty_n,Ty_n,Ty_n)}{G(Ty_n,Tu,Ty_n)+G(u,u,y_n)} \cdot G(y_n, Ty_n, Ty_n), \\ \frac{G(Ty_n,Ty_n,Ty_n)}{G(Ty_n,Tu,Tu)+G(Ty_n,y_n,Tu)} \cdot G(Ty_n, Tu, Tu), \\ \frac{G(Ty_n,Ty_n,Tu)}{G(y_n,Ty_n,Ty_n)+G(Ty_n,y_n,u)} \cdot G(Tu, Tu, Ty_n)\ , \\ G(u, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n), G(y_n, Tu, Tu), \\ G(y_n, Ty_n, Ty_n), G(u, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n) \end{pmatrix}$$

This deduces to

$$(7) G(Ty_n, u, Ty_n) \leq \text{k max}\begin{pmatrix} \frac{G(Ty_n,Ty_n,u)}{G(y_n,Ty_n,Ty_n)+G(Ty_n,y_n,u)} \cdot G(u,u,u)\ , \\ G(u, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n), \\ G(y_n, u, u) \end{pmatrix}$$

$$= \text{k max}\begin{pmatrix} G(u, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n), \\ G(y_n, u, u) \end{pmatrix}$$

$$= \text{k max}\big(G(u, Ty_n, Ty_n), (y_n, u, u)\big)$$

Proceeding the limit as $n \to \infty$, we have, $G(u, Ty_n, Ty_n)) \to 0$, and so by definition of G-continuity of G-metric space *(X, G)* we have $T(y_n) \to u = T(u)$, this implies that T is G-continuous at u.

Hence completes the theorem.

Remark 1. If the G-metric space is bounded, i.e., for some m > 0, we have $G(x, y, z) \le m$, for all x, y, z ϵ X, then an argument similar to that used above establishes the result for $0 \le k < 1$.

Corollary 1. *Let (X, G) be a complete G-metric space and let $T : X \to X$ be the mapping which satisfy the following condition for $m \in N$ and for all $x, y, z \in X$:*

$$G(T^m x, T^m y, T^m z) \le k$$

$$max\begin{pmatrix} \frac{G(T^m x, T^m z, T^m z)}{G(T^m x, T^m y, T^m z) + G(y, y, z)} \cdot G(x, T^m x, T^m x), \\ \frac{G(T^m x, T^m x, T^m z)}{G(T^m z, T^m y, T^m y) + G(T^m x, z, T^m y)} \cdot G(T^m z, T^m y, T^m y), \\ \frac{G(T^m x, T^m x, T^m y)}{G(x, T^m z, T^m z) + G(T^m x, z, y)} \cdot G(y, T^m y, T^m y), \\ G(y, T^m x, T^m x), G(z, T^m x, T^m x), G(z, T^m y, T^m y), \\ G(x, T^m z, T^m z), G(y, T^m z, T^m z), G(z, T^m z, T^m z) \end{pmatrix}$$

Where $0 \le k < 1$, then T has unique fixed point (say)*u* and T^m is *G*-continuous at u.

Proof. Using Theorem 1, ones obtain, T^m has a unique fixed point (say) u, that is, $T^m u = u$ and T^m is *G*-continuous at u. But $Tu = T(T^m u) = T^{m+1} u = T^m T u$, so Tu is another fixed point of T^m by uniqueness $Tu = u$, i.e., u is a unique fixed point of T .

Theorem 2. Let (X, G) be complete *G*-metric space and $T : X \to X$ be the mapping satisfying the following condition:

(9) $G(Tx, Ty, Tz)$

$$\le k \, max\begin{pmatrix} \frac{G(x, Tx, Tx)}{G(x, Tx, Tx) + G(y, y, z)} \cdot G(x, y, z) + G(z, Tz, Tz) + G(y, Tz, Tz), \\ \frac{G(Tx, y, Tz)}{G(Tz, Ty, Ty) + G(Tx, z, Ty)} \cdot G(x, Ty, Ty) + G(z, Ty, Tz) + G(z, Tx, Tx) \\ \frac{G(x, Ty, Ty)}{G(x, Tz, Tz) + G(z, z, y)} \cdot G(y, Ty, Ty) + G(x, Tz, Tz) + G(y, Tx, Tx) \end{pmatrix}$$

for all x, y, z ϵ X, where $0 \leq k < \frac{1}{4}$ *, then T has a unique fixed point (u) and T is G-continuous at the fixed point.*

Proof. Suppose that T satisfy condition (9) and let x_0 be any arbitrary point of X.

Step 1. We inductively construct the sequence $\{x_n\}$ of point in X as:

$$x_1 = Tx_0$$

$$x_2 = Tx_1 = T\,T\,x_0 = T^2x_0$$

$$x_3 = Tx_2 = TT^2x_0 = T^3x_0$$

$$\vdots$$

$$x_n = T(x_{n-1}) = T\,(T^{n-1}(x_0)) = T^n(x_0),$$

Clearly $\{x_n\}$ is a sequence of images of x_0, under repeated application of T .

Step 2. $\{x_n\}$ is a Cauchy sequence in X. Assume $x_n \neq x_{n+1}$ for all n. Since if there exist an n such that

$x_n = x_{n+1}$ then, $T^nx_0 = TT^nx_0$, yields T^nx_0 is a fixed point.

By (9), we have

(10) $\quad G(x_n, x_{n+1}, x_{n+1})$

$$\leq k\,max\begin{pmatrix} \dfrac{G(x_{n-1},x_n,x_n)}{G(x_{n-1},x_n,x_n)+G(x_n,x_n,x_n)} \\ .\,(G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})+G(x_n,x_{n+1},x_{n+1})), \\ \dfrac{G(x_n,x_n,x_{n+1})}{G(x_{n+1},x_{n+1},x_{n+1})+G(x_n,x_n,x_{n+1})}\,. \\ (G(x_{n-1},x_{n+1},x_{n+1})+G(x_{n+1},x_{n+1},x_{n+1})+G(x_{n+1},x_{n+1},x_{n+1})) \\ \dfrac{G(x_{n-1},x_{n+1},x_{n+1})}{G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)}\,. \\ (G(x_n,x_{n+1},x_{n+1})+G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_n,x_n)) \end{pmatrix}$$

(11)

$$= k\,max\begin{pmatrix} G(x_{n-1},x_n,x_n)+2G(x_n,x_{n+1},x_{n+1}), \\ G(x_{n-1},x_{n+1},x_{n+1})+G(x_n,x_{n+1},x_{n+1}), \\ G(x_n,x_{n+1},x_{n+1})+G(x_{n-1},x_{n+1},x_{n+1}) \end{pmatrix}$$

$$= k\, max \begin{pmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1}) \end{pmatrix}$$

Case 1. If

$$\max \begin{pmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1}) \end{pmatrix} = G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})$$

Then (10) becomes,

$$G(x_n, x_{n+1}, x_{n+1}) \leq k \left(G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}) \right)$$

Which can be written as

$$G(x_n, x_{n+1}, x_{n+1}) \leq qG(x_{n-1}, x_n, x_n),$$

Where q = $\left(\frac{k}{1-2k}\right)$, and q<1, as $0 \leq k < \frac{1}{4}$

Then by lemma 1, we have $\{x_n\}$ is a *G*-Cauchy sequence in *X*.

Case 2. If

$$max \begin{pmatrix} G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1}), \\ G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1}) \end{pmatrix}$$
$$= G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1})$$

Then (10) reduces to

(11) $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1})$

Using G_5 of *Definition 1*. We have

(12) $G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n-1}, x_{n+1}, x_{n+1})$

Now (11) becomes,

$$G(x_n, x_{n+1}, x_{n+1}) \leq q\, G(x_{n-1}, x_n, x_n),$$

Where $q = \left(\frac{k}{1-2k}\right)$, and q<1, as $0 \leq k < \frac{1}{4}$.

Then again by using Lemma *1*, we obtain $\{x_n\}$ is a *G*-Cauchy sequence in *X*.

Hence in both cases $\{x_n\}$ is a *G*-Cauchy sequence in *X*.

Step 3. Since (*X, G*) is a complete *G*-metric space, by definition, there exists a point (say) $u \in X$ such that $x_n \to u$.

Step 4. *u* is fixed point of *T* , suppose, if possible, that $Tu \neq u$, using (9), ones obtain

$\mathrm{G}(x_n, \mathrm{Tu}, \mathrm{Tu})$

$$\leq k\, max \begin{pmatrix} \frac{G(x_{n-1}, x_n, \mathrm{u})}{G(x_{n-1}, x_n, \mathrm{u}) + G(\mathrm{u}, u, u)} . G(x_{n-1}, u, u) + G(u, Tu, Tu) + G(u, Tu, Tu), \\ \frac{G(x_n, u, Tu)}{G(Tu, Tu, Tu) + G(x_n, u, Tu)} . G(x_{n-1}, Tu, Tu) + G(u, Tu, Tu) + G(u, x_n, x_n) \\ \frac{G(x_{n-1}, Tu, Tu)}{G(x_{n-1}, Tu, Tu) + G(\mathrm{u}, u, u)} . G(\mathrm{u}, Tu, Tu) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n) \end{pmatrix}$$

$$= k\, max \begin{pmatrix} G(x_{n-1}, u, u) + G(u, Tu, Tu) + G(u, Tu, Tu) \\ \mathrm{G}(x_{n-1}, Tu, Tu) + G(u, Tu, Tu) + G(u, x_n, x_n) \\ G(\mathrm{u}, Tu, Tu) + G(x_{n-1}, Tu, Tu) + G(u, x_n, x_n) \end{pmatrix}$$

$$= k\, max \begin{pmatrix} G(x_{n-1}, u, u) + 2G(u, Tu, Tu), \\ \mathrm{G}(x_{n-1}, Tu, Tu) + G(u, Tu, Tu) + G(u, x_n, x_n) \end{pmatrix}$$

Taking the limit as n → ∞, and using the fact that function *G* is continuous in its variable, we get

$\mathrm{G}(x_n, \mathrm{Tu}, \mathrm{Tu}) \leq k\, max \begin{pmatrix} 2G(u, Tu, Tu), \\ 2G(u, Tu, Tu) \end{pmatrix} \leq 2\mathrm{k}\, G(u, Tu, Tu)$

Which is a contradiction, since $0 \leq k < \frac{1}{4}$. Hence *u*= *Tu*, i.e., u is a fixed point of *T*.

Step 5. Uniqueness of fixed point u of T .

G(u,v,v)

$$\leq k\, max \begin{pmatrix} \frac{G(u, u, v)}{G(u, u, v) + G(v, v, v)} . G(u, v, v) + G(v, v, v) + G(v, v, v), \\ \frac{G(u, v, v)}{G(v, v, v) + G(u, v, v)} . G(u, v, v) + G(v, v, v) + G(v, u, u) \\ \frac{G(u, v, v)}{G(u, v, v) + G(v, v, v)} . G(v, v, v) + G(u, v, v) + G(v, u, u) \end{pmatrix}$$

$$= k\, max\big(G(u, v, v) + G(v, u, u)\big)$$

G(u,v,v)$\leq \mathrm{k}\, \big(G(u, v, v) + G(v, u, u)\big)$

This implies that

$$G(u,v,v) \leq \frac{k}{1-k} G(v, u, u).$$

Now, by the same argument, we have

$$G(v,u,u) \leq \frac{k}{1-k} G(u,v,v).$$

Therefore, we get

$$G(u,v,v) \leq \left(\frac{k}{1-k}\right)^2 G(v,u,u).$$

But $0 \leq \frac{k}{1-k} < 1$.

Hence, we reach at the same contradiction, so $u=v$, that is, the fixed point is unique.

Step 6. Finally, to prove T is G-continuous at fixed point u. For this, let us suppose that $\{y_n\}$ be a sequence in X such that $y_n \to u$ in (X, G), we obtain

(13) $G(Ty_n, Tu, Tu)$

$$\leq k \max \begin{pmatrix} \frac{G(y_n, Ty_n, u)}{G(y_n, Ty_n, u) + G(u,u,u)} . G(y_n, u, u) + G(u, Tu, Tu) + G(u, Tu, Tu), \\ \frac{G(Ty_n, u, Tu)}{G(Tu, Tu, Tu) + G(Ty_n, u, Tu)} . G(y_n, Tu, Tu) + G(u, Tu, Tu) + G(u, Ty_n, Ty_n), \\ \frac{G(y_n, Tu, Tu)}{G(y_n, Tu, Tu) + G(u,u,u)} . G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(u, Ty_n, Ty_n) \end{pmatrix}$$

$$= k \max \begin{pmatrix} G(y_n, u, u) + 2G(u, Tu, Tu), \\ G(y_n, Tu, Tu) + G(u, Tu, Tu) + G(u, Ty_n, Ty_n), \\ G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(u, Ty_n, Ty_n) \end{pmatrix}$$

$$= k \max \begin{pmatrix} G(y_n, u, u) + 2G(u, Tu, Tu), \\ G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(u, Ty_n, Ty_n) \end{pmatrix}$$

Case 1. if

$$\max \begin{pmatrix} G(y_n, u, u) + 2G(u, Tu, Tu), \\ G(u, Tu, Tu) + G(y_n, Tu, Tu) + G(u, Ty_n, Ty_n) \end{pmatrix}$$

$$= G(y_n, u, u) + 2G(u, Tu, Tu)$$

Then (13) becomes,

$$G(Ty_n, Tu, Tu) \leq k\,(G(y_n, u, u) + 2G(u, Tu, Tu)),$$

Letting limit $n \to \infty$, and using $Tu = u$, and $y_n \to u$, we get

(14) $G(Ty_n, u, u) \leq k\,(G(u, u, u) + 2G(u, u, u))$,

This implies (14) reduce to, $G(Ty_n, u, u) = 0$. But $G(Ty_n, u, u) \geq 0$, Hence $G(Ty_n, u, u) = 0$. So,

$Ty_n \to u = Tu$, which shows that T is G-continuous at a fixed point u.

Case 2.if

$$max\begin{pmatrix} G(y_n,u,u)+2G(u,Tu,Tu), \\ G(u,Tu,Tu)+G(y_n,Tu,Tu)+G(u,Ty_n,Ty_n) \end{pmatrix}$$

$$= G(u,Tu,Tu)+G(y_n,Tu,Tu)+G(u,Ty_n,Ty_n)$$

Then (13) becomes,

$$G(Ty_n,Tu,Tu) \leq k\,(G(u,Tu,Tu)+G(y_n,Tu,Tu)+G(u,Ty_n,Ty_n))$$

Letting limit $n \to \infty$, and using $Tu = u$, we get

(15)$G(Ty_n,u,u) \leq k\,(G(u,u,u)+G(u,u,u)+G(u,Ty_n,Ty_n))$

$$\leq k\,G(u,Ty_n,Ty_n)$$

By (iii) of Proposition 5, $G(u,Ty_n,Ty_n)$ $2G(Ty_n,u,u)$, with this (15), reduces $G(Ty_n,u,u)$ 0, but $G(Ty_n,u,u)=0$, hence, $G(Ty_n,u,u) = 0$.

So, $Ty_n \to u = Tu$, which shows that T is G-continuous at the fixed point u.

Therefore in both cases T is G-continuous at point u. Hence completes the theorem.

Corollary 2.*Let (X, G) be a complete G-metric space and let $T : X \to X$ be the mapping which satisfy the following condition for $m \in N$ and for all $x, y, z \in X$:*

$$\leq\ k\max\begin{pmatrix} \dfrac{G(x,T^mx,z)}{G(x,T^mx,z)+G(y,y,z)}\,. \\ (G(x,y,z)+G(z,T^mz,T^mz)+G(y,T^mz,T^mz)), \\ \dfrac{G(T^mx,y,T^mz)}{G(T^mz,T^myT^my)+G(T^mx,z,T^my)}\,. \\ \big(G(x,T^my,T^my)+G(z,T^my,T^mz)+G(z,T^mx,T^mx)\big), \\ \dfrac{G(x,T^my,T^my)}{G(x,T^mz,T^mz)+G(z,z,y)}\,. \\ (G(y,T^my,T^my)+G(x,T^mz,T^mz)+G(y,T^mx,T^mx)) \end{pmatrix}$$

$$G(T^mx,T^my,T^mz)$$

Where $0\leq k <1$, then T has unique fixed point (say)u and T^m is G-continuous at u.

Proof. Using Theorem 2, ones obtain, T^m has a unique fixed point (say) u, that is, $T^m(u) = u$ and T^m is G-continuous. But $Tu = TT^mu = T^{m+1}u = T^mTu$, so $T(u)$ is another fixed point of T^m by uniqueness $Tu = u$, i.e., u is a fixed point of T .

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