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Metric Dimension of Some Graphs under Join Operation

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Abstract

In this paper, we find the metric dimension for join of two paths and generalized the Caser's result for join of a path and a complete graph. We have also improved the results given by Shahida, A. T. and M. S. Sunitha.

Throughout the paper, finite and simple graphs have been considered.

Keywords: Resolving set, Basis, Metric Dimension.

1. INTRODUCTION

Navigation can be studied in a graph structure framework in which the navigation agent moves from node to node of a graph space. The robot can locate itself by the presence of distinctly labeled landmark nodes in a graph space. If the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph, what are the fewest number of landmarks needed, and where they should be located, so that the distance to the landmarks uniquely determines the robot's position on the graph? A minimum set of landmarks which uniquely determines the robot's position is called basis, and the minimum number of landmarks is called the metric dimension of graph.

Motivated by this problem, the concept of metric dimension was introduced by Slater [7] and independently by Harary and Melter [1].

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Let $W = \{w_1, w_2, w_3, ..., w_k\}$ be an ordered subset of $V(G)$; then the metric representation of $v \in V(G)$ with respect to W is defined as the $k-$tuple

$r(v/W) = \{d(v,w_1), d(v,w_2), d(v,w_3), ..., d(v,w_k)\}.$

The set W is called a resolving set of G if for all $u \neq v$ and $u, v \in V(G)$ satisfy

$r(v/W) \neq r(u/W).$

A resolving set W of G with the minimum cardinality is the basis of G. The number of elements in basis is called metric dimension of G and is denoted by $\beta(G)$, thus $\beta(G) = \min\{|W| : W \textit{ is a resolving set of } G\}$. Khuller et al. [9] studied the metric dimension motivated by the robot navigation while Chartrand et al. [2] characterizes all the graphs of order n having metric dimension $n-2$. Caceres et al. [4] have determined the metric dimension for Cartesian product of graphs. Saputro et al. [10] have shown metric dimension of comb product of graphs. Buczkowski et al. [8] determined the metric dimension of wheel $W_n = K_1 + C_n$, for $n \geq 3$. Caceres et al. [5] the metric dimension of fan $f_n = K_1 + P_n$, for $n \geq 1$ and Tomescu and Javaid [3] the metric dimension of Jahangir graph J_{2n}, for $n \geq 2$.

Motivated by [3], [4], [5], [6], [8], [10] and [11]; we have obtained some results under the join operation of two graphs.

2. JOIN OF TWO GRAPHS

The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$; is a graph with vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. In the graph $G_1 + G_2$ each vertex of G_1 is adjacent to the vertices of G_2 and vise versa i.e. $d(u_i, v_j) = 1; \ \forall\ u_i \in V(G_1), \ \forall\ v_j \in V(G_2).$

Khuller et al. [9] have derived the following results:

2.1: Metric dimension of a graph G is 1 if and only if G is a path.

2.2: If $G = (V, E)$ is a graph with metric dimension 2 and $\{a, b\} \subset V(G)$ is a basis for G, then

(i) There is a unique shortest path between a and b.

(ii) The degrees of vertices a and b are at most three.

(iii) Other vertices lying on unique shortest path between a and b have degree at most five.

3. MAIN RESULT

On the basis of above results, we have obtained necessary condition for the basis of join of two graphs:

***Lemma* 3.1:** If $V(G_1)$ and $V(G_2)$ are vertex sets of two non null graphs G_1 and G_2 respectively and W is a basis of graph G_1+G_2, then $W \cap V(G_i) \neq \phi, \ \forall \ i=1,2$.

***Proof*:** Let $V(G_1)=\{u_1,u_2,u_3...,u_m\}$; $V(G_2)=\{v_1,v_2,v_3...,v_n\}$; be the vertex sets of graphs G_1 and G_2 respectively and let W be the basis for the graph G_1+G_2.

If $W \subseteq V(G_1)$, then we have $r(v_j/W)=r(v_k/W)=(1,1,1,...,1); \quad \forall\ j \neq k;\ v_j,v_k \in V(G_2)$ and it gives a contradiction.

Similarly, if $W \subseteq V(G_2)$, then $r(u_j/W)=r(u_k/W)=(1,1,1,...,1); \quad \forall\ j \neq k;\ u_j,u_k \in V(G_1)$ and a contradiction again. So W can't be a basis of G_1+G_2. Thus W must contain at least one vertex of each graph.

In the next lemma we have obtained the lower bound for metric dimension of join of two paths.

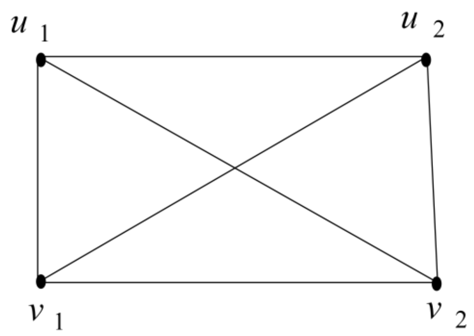
***Lemma* 3.2:** If P_m and P_n; $m \geq 2$, $n \geq 2$ are two paths, then $\beta(P_m+P_n) \geq 3$.

Proof: Let P_m and P_n be two paths and W be a basis of P_m+P_n. Then there are following possible cases arise:

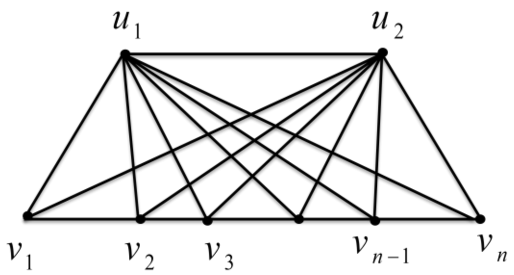
Case I: If $m=n=2$, obviously $P_2+P_2 \cong K_4$ and therefore $\beta(P_2+P_2)=3$.

Case II: If $m=2$ and $n>2$, clearly $\beta(P_2+P_n)>1$. Now we assume that $\beta(P_2+P_n)=2$, then $W=\{u_i,v_j\}; u_i \in V(P_2)$ and $v_j \in V(P_n)$; forms a basis of P_2+P_n. Degree of each u_i in this case is at least 4 in P_2+P_n; which is contradiction of result 2.2(ii), therefore W containing only two elements cannot be a basis for P_2+P_n.

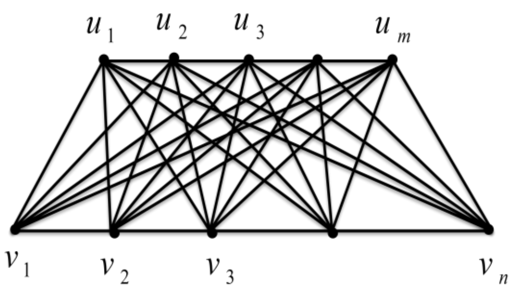
Case III: If $m>2$ and $n>2$. Then clearly degree of each vertex of P_m+P_n is at least 4, which is again a contradiction of result 2.2 (ii). Thus we conclude that no two vertices of P_m+P_n form a basis for P_m+P_n i.e. $\beta(P_m+P_n) \geq 3$. In the following Fig. 1, Fig. 2 and Fig. 3, we have shown all the above three cases.



$Fig.1$: $P_2 + P_2 \cong K_4$



$Fig.2: P_2 + P_n$



$Fig.3: P_m + P_n$

In the following theorem we have obtained sufficient condition for basis of join of two paths.

***Theorem* 3.3:** Let $W_1 \subset V(P_m)$ and $W_2 \subset V(P_n)$ be two ordered sets of graph $P_m + P_n$ such that

(i) $r(u_i / W_1) \neq r(u_j / W_1);\ \forall\ i \neq j\ and\ u_i, u_j \in V(P_m)$ in context of graph $P_m + P_n$.

(ii) $r(v_s / W_2) \neq r(v_t / W_2);\ \forall\ s \neq t\ and\ v_s, v_t \in V(P_n)$ in context of graph $P_m + P_n$.

(iii) For each $1 \leq i \leq m\ and\ 1 \leq s \leq n$; both $r(u_i / W_1)$ and $r(v_s / W_2)$ should not as $(1,1,1,...,1)$ at the same time i.e. if $r(u_i / W_1) = (1,1,1,...,1)$, then $r(v_s / W_2) \neq (1,1,1,...,1)$ and vice-versa.

Then the ordered set $W = W_1 \cup W_2$ is a resolving set for the graph $P_m + P_n$ and if both W_1 and W_2 are smallest sets also; then W is a basis for the graph $P_m + P_n$.

***Proof*:** Let us take $W = W_1 \cup W_2 = \{u_1, u_2, u_3..., u_l, v_1, v_2, v_3..., v_k\}$ such that

$r(u_i / W_1) \neq r(u_j / W_1)$, in $P_m + P_n\ \forall i \neq j$.

Then

$$r(u_i / W) \neq r(u_j / W),\ \forall i \neq j \quad (1)$$

Similarly

$$r(v_s / W) \neq r(v_t / W),\ \forall s \neq t \quad (2)$$

Now $r(u_i / W) = r(v_s / W)$ is possible only when,

$r(u_i / W) = r(v_s / W) = (1,1,1,...,1)$

But from condition (iii) of the theorem both $r(u_i / W_1)$ and $r(v_s / W_2)$ should not as $(1,1,1,...,1)$ at the same time. Therefore

$$r(u_i / W) \neq r(v_s / W);\ \forall\ 1 \leq i \leq m\ and\ 1 \leq s \leq n. \quad (3)$$

Then by (1), (2) & (3), we conclude that

$r(w_i / W) \neq r(w_j / W),\ \forall\ w_i,\ w_j \in V(P_m + P_n)\ \forall\ i \neq j.$

Thus W *i.e.* $W_1 \cup W_2$ is a resolving set for $P_m + P_n$. If W_1 and W_2 are smallest then obviously $W = W_1 \cup W_2$ is also a smallest set. Hence $W = W_1 \cup W_2$ is a smallest resolving set i.e. a basis for the graph $P_m + P_n$.

***Example* 3.4:** Let P_4 and P_6 be two paths, then consider two ordered subsets $W_1 = \{u_1, u_2\}$ and $W_2 = \{v_2, v_4\}$ of P_4 and P_6 respectively.

Now $r(u_3 / W_1) = (2,1)$ in context of graph $P_4 + P_6$.

$r(u_4 / W_1) = (2,2)$ in context of graph $P_4 + P_6$.

$r(v_1 / W_2) = (1,2)$ in context of graph $P_4 + P_6$.

$r(v_3/W_2)=(1,1)$ in context of graph P_4+P_6.

$r(v_5/W_2)=(2,1)$ in context of graph P_4+P_6.

$r(v_6/W_2)=(2,2)$ in context of graph P_4+P_6.

Here we find that

(i) $r(u_i/W_1)\neq r(u_j/W_1)$; $\forall\ i\neq j$ and $u_i,u_j\in V(P_4)$ in context of graph P_4+P_6.

(ii) $r(v_s/W_2)\neq r(v_t/W_2)$; $\forall\ s\neq t$ and $v_s,v_t\in V(P_6)$ in context of graph P_4+P_6.

(iii) $r(u_i/W_1)\neq(1,1,)$; $\forall\ 1\leq i\leq 4$ in context of graph P_4+P_6.

Then by theorem 3.3; $W_1\cup W_2=\{u_1,u_2,v_2,v_4\}$ is a resolving set for P_4+P_6. Since both W_1 and W_2 are smallest sets; therefore $W_1\cup W_2=\{u_1,u_2,v_2,v_4\}$ is a basis for the graph P_4+P_6 and $\beta(P_4+P_6)=4$.

***Theorem* 3.5:** Let P_m and P_n be two paths, then metric dimension of P_m+P_n is

$$\beta(P_m+P_n)=\begin{cases}3; & 2\leq m\leq 5 \text{ and } 2\leq n\leq 3,\\ 4; & 2\leq m\leq 5 \text{ and } n=6 \text{ or } 4\leq m\leq 6 \text{ and } 4\leq n\leq 5,\\ 5; & m=n=6.\\ \left\lceil \frac{n}{2}\right\rceil; & 2\leq m\leq 3 \text{ and } n\geq 7,\\ \left\lceil \frac{n}{2}\right\rceil+1; & 4\leq m\leq 6 \text{ and } n\geq 7,\\ \left\lceil \frac{n}{2}\right\rceil+\left\lceil \frac{m}{2}\right\rceil-2; & m\geq 7 \text{ and } n\geq 7,\end{cases}$$

***Proof*:** Let $\{u_i\mid u_i\in V(P_m)\}$ and $\{v_i\mid v_i\in V(P_n)\}$ be the vertex sets of paths P_m and P_n respectively. Then

Case I: If $2\leq m\leq 5$ and $2\leq n\leq 3$.

(a) If $2<m\leq 5$ and $2\leq n\leq 3$. Suppose $W=\{v_1,u_{m-2},u_{m-1}\}\subset V(P_m+P_n)$, we show that W is a resolving set for P_m+P_n. For this we take the representation of any vertex of $V(P_m+P_n)\setminus W$ with respect to W:

$r(v_2 / W) = (1,1,1)$
$r(v_3 / W) = (2,1,1)$
.
.
.
$r(u_m / W) = (1,2,1)$

Since these representations are pair wise distinct it follows that $\beta(P_m + P_n) \le 3$. But $\beta(P_m + P_n) \ge 3$. Therefore $\beta(P_m + P_n) = 3$; when $2 < m \le 5$ and $2 \le n \le 3$.

(b) If $m = 2$ and $2 \le n \le 3$, we consider the set 2

Now the representations of $V(P_m + P_n) \setminus W$ with respect to W:

$r(v_2 / W) = (1,1,1)$
$r(v_3 / W) = (2,1,1)$

Proceeding in same way as above, we observe that there are no two vertices having the same metric representations with respect to W, implying that $\beta(P_m + P_n) \le 3$ and $\beta(P_m + P_n) \ge 3$. So $\beta(P_m + P_n) = 3$, for $m = 2$ and $2 \le n \le 3$.

Case II: If $2 \le m \le 5$ and $n = 6$ or $4 \le m \le 6$ and $4 \le n \le 5$, then

(a) If $2 \le m \le 5$ and $n = 6$, Consider the set $W = \{v_2, v_4, u_1, u_m\} \subset V(P_m + P_n)$. Now the representations of $V(P_m + P_n) \setminus W$ with respect to W:

$r(v_1 / W) = (1,2,1,1)$
$r(v_3 / W) = (1,1,1,1)$
$r(v_5 / W) = (2,1,1,1)$
$r(v_6 / W) = (2,2,1,1)$
$r(u_{m-3} / W) = (1,1,1,2)$
$r(u_{m-2} / W) = (1,1,2,2)$
$r(u_{m-1} / W) = (1,1,2,1)$

We find that there are no two vertices having same representations implying that $\beta(P_m + P_n) \le 4$. Now we show that $\beta(P_m + P_n) \ge 4$, by proving that there is no resolving set W with cardinality 3.

Now following sub cases arise:

(i) If W contains one vertex of P_m and two vertices of P_n i.e. $W = \{u_i, v_j, v_k\}$, we observe that

(a) For $j = 2$ and $k = 4$ *or* $j = 3$ and $k = 5$ different vertices of P_n have different representations with respect to W but in this case

$$r(v_{j+1} / W) = (1,1,1);\ \ j < k \tag{1}$$

and

$$r(u_l / W) = (1,1,1);\ u_l \in N(u_i) \tag{2}$$

Therefore $r(v_{j+1} / W) = r(u_l / W)$; thus a contradiction.

(b) If $j \neq 2$ and $k \neq 4$ *or* $j \neq 3$ and $k \neq 5$; then there exist at least two vertices of $V(P_m + P_n) \setminus W$ having the same metric representation with respect to W, a contradiction again. Therefore cardinality of W can't be three in this case i.e. $\beta(P_m + P_n) \geq 4$. So $\beta(P_m + P_n) = 4$.

(ii) Similarly if W contains two vertices of P_m and one vertex of P_n; we again get a contradiction.

Case II (b): If $4 \leq m \leq 6$ and $4 \leq n \leq 5$. Suppose $W = \{v_2, v_4, u_3, u_4\} \subset V(P_m + P_n)$, then

$r(v_1 / W) = (1,2,1,1)$
$r(v_3 / W) = (1,1,1,1)$
$r(v_5 / W) = (2,1,1,1)$
$r(v_6 / W) = (2,2,1,1)$
$r(u_1 / W) = (1,1,2,2)$
$r(u_2 / W) = (1,1,1,2)$
$r(u_5 / W) = (1,1,2,1)$

There are no two vertices having the same metric representations with respect to W, implying that $\beta(P_m + P_n) \leq 4$. Now proceeding on the same manner as above $\beta(P_m + P_n) \geq 4$, therefore $\beta(P_m + P_n) = 4$; when $4 \leq m \leq 6$ and $4 \leq n \leq 5$.

Case III: For $m = n = 6$; consider the set $W = \{v_3, v_4, v_6, u_3, u_5\}$, then the metric representations of the vertices of $V(P_m + P_n) \setminus W$ with respect to W :

$r(v_1/W) = (2,2,2,1,1)$
$r(v_2/W) = (1,2,2,1,1)$
$r(v_5/W) = (2,1,1,1,1)$
$r(u_1/W) = (1,1,1,2,2)$
$r(u_2/W) = (1,1,1,1,2)$
$r(u_4/W) = (1,1,1,1,1)$
$r(u_6/W) = (1,1,1,2,1)$

$\Rightarrow$Different vertices of $V(P_m + P_n)\setminus W$ have different metric representations with respect to W, therefore

$\beta(P_m + P_n) \le 5.$ (3) Now

in order to prove that $\beta(P_m + P_n) \ge 5$; we show that there is no resolving set W such that $|W| \le 4$. For this following possible cases arises:

(i) When W contains one vertex of P_m and three vertices of P_n i.e. $W = \{u_i, v_j, v_k, v_l\}$ then $\exists$ at least two vertices $u_\alpha, u_\beta \in V(P_m)$ such that $r(u_\alpha/W) = (2,1,1,1) = r(u_\beta/W)$; a contradiction.

(ii) Similarly when W contains one vertex of P_n and remaining three vertices of P_m, then $\exists$ at least two vertices $v_\alpha, v_\beta \in V(P_n)$ such that $r(v_\alpha/W) = r(v_\beta/W)$; again a contradiction.

(iii) When W contains equal number of vertices from both P_m and P_n i.e. $W = \{u_i, u_j, v_k, v_l\}$; then we observe that

(a) For $i = 2$ and $j = 4$ *or* $i = 3$ and $j = 5$ different vertices of P_m have different representations with respect to W and

$r(u_{i+1}/W) = (1,1,1,1);\ i < j$ (4)

Similarly for $k = 2$ and $l = 4$ *or* $k = 3$ and $l = 5$ different vertices of P_m have different representations with respect to W and in this case

$r(v_{k+1}/W) = (1,1,1,1);\ k < l$ (5)

By (4) and (5) we conclude that $r(v_{i+1}/W) = r(u_{k+1}/W)$; and this is a contradiction.

(b) For $i \ne 2$ and $j \ne 4$ or $i \ne 3$ and $j \ne 5$; $k \ne 2$ and $l \ne 4$ or $k \ne 3$ and $l \ne 5$; then there exist at least two vertices of $V(P_m + P_n)\setminus W$ having the same metric representation with respect to W, which is a contradiction. Hence W can't be a resolving set with $|W| = 4$ *i.e.* $\beta(P_m + P_n) \ne 4$ and also

$\beta(P_m + P_n) = 3;\ 2 \le m \le 5$ and $2 \le n \le 3$ so $\beta(P_m + P_n) \ne 3$ in this case. Therefore
$\beta(P_m + P_n) \ge 5$ (6)
By (3) and (6) $\Rightarrow \beta(P_m + P_n) = 5;$ when $m = n = 6.$

Case IV: For $2 \le m \le 3$ and $n \ge 7$

(a) If n is even and $n \ge 7,$ in this we take W_1 consisting of a single vertex of P_m and an ordered set W_2 of vertices of P_n such that $W_1 = \{u_1\}$ and $W_2 = \{v_3, v_5, ..., v_{n-1}\}$, we get

$r(u_2 / W_1) = (1)$ in the context of graph $P_m + P_n$.

$r(u_3 / W_1) = (2)$ in the context of graph $P_m + P_n$.

Obviously, $r(u_i / W_1) \ne r(u_j / W_1);\ \forall i \ne j;$ in the context of graph $P_m + P_n$ and $2 \le i, j \le 3.$

And

$r(v_1 / W_2) = (2, 2, ..., 2)$ in the context of graph $P_m + P_n$.

$r(v_2 / W_2) = (1, 2, ..., 2)$ in the context of graph $P_m + P_n$.

$r(v_4 / W_2) = (1, 1, ..., 2)$ in the context of graph $P_m + P_n$.

So on, $r(v_n / W_2) = (2, 2, ..., 1)$ in the context of graph $P_m + P_n$.

Here we find that

(i) $r(v_i / W_2) \ne r(v_j / W_2)$ in the context of graph $P_m + P_n;\ \forall i \ne j.$

(ii) $r(v_i / W_2) \ne (1, 1, 1, ..., 1)\ \ \forall i$ in the context of graph $P_m + P_n$.

Then by theorem 3.3, $W_1 \cup W_2$ is a resolving set for $P_m + P_n$. To make $|W_1 \cup W_2|$ minimum; both $|W_1|$ and $|W_2|$ should be minimum separately. Here we have to make $|W_2|$ minimum; since $|W_1|$ containing one element is minimum.

For making $|W_2|$ minimum; we follow the steps given below:

(i) We omit end vertices of P_n.

(ii) We select alternating vertices as the elements of W_2.

Thus; for P_n; $|W_2| = \dfrac{n-2}{2}$, $n \ge 7$ is minimum and $\beta(P_m + P_n) = \dfrac{n}{2}$ (7)

(b) If n is odd, then similarly $W_1 = \{u_1\}$ is a smallest ordered set in P_m such that $r(u_i / W_1) \neq r(u_j / W_1)$ *in* the context of graph $P_m + P_n$; $\forall i \neq j$ and $2 \leq i, j \leq 3$.

Now let $W_2 = \{v_3, v_5, v_7, ...v_n\}$; then

$r(v_1 / W_2) = (2, 2, ..., 2)$ in the context of graph $P_m + P_n$

$r(v_2 / W_2) = (1, 2, ..., 2)$ in the context of graph $P_m + P_n$

$r(v_4 / W_2) = (1, 1, ..., 2)$ in the context of graph $P_m + P_n$

So on finally $r(v_{n-1} / W_2) = (2, ..., 1, 1)$ in the context of graph $P_m + P_n$.

Here we see that

(i) $r(v_i / W_2) \neq r(v_j / W_2)$ in the context of graph $P_m + P_n$ $\forall i \neq j$.

(ii) $r(v_i / W_2) \neq (1, 1, 1, ..., 1)$ $\forall i$ in the context of graph $P_m + P_n$.

Therefore $W = W_1 \cup W_2$ is a resolving set for $P_m + P_n$.

Now we have to make $|W_2|$ minimum; since $|W_1|$ containing one element is minimum.

For making $|W_2|$ minimum; we follow the steps given below:

(i) We omit one end vertex of P_n.

(ii) We select alternating vertices as the elements of W_2.

Thus; for P_n; $|W_2| = \dfrac{n-1}{2}$, $n \geq 7$.

Therefore minimum

$$|W_1 \cup W_2| = 1 + \frac{n-1}{2} \quad i.e. \quad \beta(P_m + P_n) = \frac{n+1}{2} \tag{8}$$

By (7) and (8) $\Rightarrow \beta(P_m + P_n) = \left\lceil \dfrac{n}{2} \right\rceil$; $2 \leq m \leq 3$ and $n \geq 7$.

***Case V*:** For $4 \leq m \leq 6$ and $n \geq 7$**;** then

(a) If n is even, then we take two ordered sets W_1 and W_2 from P_m and P_n respectively such that $W_1 = \{u_2, u_4\}$ and $W_2 = \{v_3, v_5, v_7 ... v_{n-1}\}$,then

$r(u_1 / W_1) = (1, 2)$ in the context of graph $P_m + P_n$

$r(u_3/W_1) = (1,1)$ in the context of graph $P_m + P_n$

$r(u_5/W_1) = (2,1)$ in the context of graph $P_m + P_n$

$r(u_6/W_1) = (2,2)$ in the context of graph $P_m + P_n$

Here we find that W_1 is also smallest ordered set such that

$r(u_i/W_1) \neq r(u_j/W_1)$; in the context of $P_m + P_n$, $\forall\, i \neq j$; because if we take $|W_1| = 1$; then there exist at least two vertices of P_m having same metric representation with respect to W_1 in the context of graph $P_m + P_n$.

And in same way as case IV (a); W_2 is a smallest ordered set such that

(i) $r(v_i/W_2) \neq r(v_j/W_2)$ in the context of graph $P_m + P_n$; $\forall\, i \neq j$.

(ii) $r(v_i/W_1) \neq (1,1,1,...,1)$ $\forall\, i$ in the context of graph $P_m + P_n$

Then $W = W_1 \cup W_2$ is a basis for $P_m + P_n$ and

$$\beta(P_m + P_n) = |W_1| + |W_2| \Rightarrow \beta(P_m + P_n) = 2 + \frac{n-2}{2} \Rightarrow \beta(P_m + P_n) = \frac{n}{2} + 1 \qquad (9)$$

(b) If n is odd, then similarly $W_1 = \{u_2, u_4\}$ and $W_2 = \{v_3, v_5, v_7, ...v_n\}$; are the smallest ordered sets in P_m and P_n respectively such that

(i) $r(u_i/W_1) \neq r(u_j/W_1)$ in the context of graph $P_m + P_n$; $\forall\, i \neq j$.

(ii) $r(v_i/W_2) \neq (1,1,1,...,1)$ $\forall\, i$ in the context of graph $P_m + P_n$.

(iii) $r(v_i/W_2) \neq r(v_j/W_2)$ in $P_m + P_n$ $\forall\, i \neq j$.

So $W = W_1 \cup W_2$ is a basis for $P_m + P_n$ and

$$\beta(P_m + P_n) = 2 + \frac{n-1}{2} \Rightarrow \beta(P_m + P_n) = \frac{n+1}{2} + 1 \qquad (10)$$

By (3) and (4) $\Rightarrow \beta(P_m + P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$; $4 \leq m \leq 6$ and $n \geq 7$.

Case VI: If $m, n \geq 7$

(a) If $m\ (or\, n)$ is even and $n\ (or\, m)$ is odd. Consider two ordered sets $W_1 = \{u_3, u_5, u_7, ..., u_{m-1}\}$ and $W_2 = \{v_3, v_5, v_7 ...v_n\}$, then in similar manner as above we can show that $W_1 \cup W_2$ is a basis for $P_m + P_n$. Therefore

$$\beta(P_m+P_n)=\frac{m-2}{2}+\frac{n-1}{2}\Rightarrow \beta(P_m+P_n)=\frac{m}{2}+\frac{n+1}{2}-2 \quad (11)$$

(b) If both m and n are even, then we can discuss as above that order sets $W_1=\{u_3,u_5,u_7,...,u_{m-1}\}$ and $W_2=\{v_3,v_5,v_7...v_{n-1}\}$ are such that

$$\beta(P_m+P_n)=|W_1\cup W_2|\Rightarrow \beta(P_m+P_n)=\frac{m-2}{2}+\frac{n-2}{2}\Rightarrow \beta(P_m+P_n)=\frac{m}{2}+\frac{n}{2}-2 \quad (12)$$

(c) If both m and n are odd, then we can discuss as above that the order sets $W_1=\{u_3,u_5,u_7,...,u_m\}$ and $W_2=\{v_3,v_5,v_7...v_n\}$ are such that

$$\beta(P_m+P_n)=|W_1\cup W_2|\Rightarrow \beta(P_m+P_n)=\frac{m-1}{2}+\frac{n-1}{2}\Rightarrow \beta(P_m+P_n)=\frac{m+1}{2}+\frac{n+1}{2}-2 \quad (13)$$

$(11),(12)$ and $(13)\Rightarrow \beta(P_m+P_n)=\left\lceil\frac{m}{2}\right\rceil+\left\lceil\frac{n}{2}\right\rceil-2.$

***Remark* 3.6:** According to Shahida, A. T. and M. S. Sunitha [11] the metric dimension of join of two paths P_m *and* P_n is $\beta(P_m+P_n)=\left\lfloor\frac{m}{2}\right\rfloor+n-1;\ n\geq 1, m\geq 4.$

Let us take $m=6$ *and* $n=4$, then $\beta(P_6+P_4)=\left\lfloor\frac{6}{2}\right\rfloor+4-1=6$.

Now we show that $\beta(P_6+P_4)=4$. Let $u_i\in V(P_4)$ *and* $v_j\in V(P_6); 1\leq i\leq 4, 1\leq j\leq 6$.

Consider $W=\{u_1,u_2,v_2,v_4\}\subset V(P_6+P_4)$; then metric representations of $V(P_6+P_4)\setminus W$ with respect to W are:

$r(u_3/W)=(2,1,1,1)$

$r(u_4/W)=(2,2,1,1)$

$r(v_1/W)=(1,1,1,2,)$

$r(v_3/W)=(1,1,1,1)$

$r(v_5/W)=(1,1,2,1)$

$r(v_6/W)=(1,1,2,2)$

Here we find that all these metric representations are different, therefore $\beta(P_6+P_4)\leq 4$.

In theorem 3.5 we have shown $\beta(P_m+P_n)=3$ *when* $2\leq m\leq 5$ and $2\leq n\leq 3$ and for all other values of m *and* n $\beta(P_m+P_n)\geq 4$. Therefore $\beta(P_6+P_4)=4$.

***Theorem* 3.7:** Let K_m be a complete graph with $m \geq 2$ vertices and G be any connected graph, then basis of graph $K_m + G$ must contains $m-1$ vertices of K_m.

Proof Let $\{v_1, v_2, v_3 ..., v_m\}$ be the vertex set of K_m and W be a basis for $K_m + G$. Suppose basis of $K_m + G$ contains at most $m-2$ vertices of K_m, then we have

$r(v_i / W) = r(v_j / W) = (1,1,1...,1);\ \ \forall\ v_i, v_j \notin W$

Which is a contradiction so any basis of $K_m + G$ must contains $m-1$ vertices of K_m.

***Lemma* 3.8:** Let K_m be a complete graph and G is a connected graph. If W is a basis for $K_m + G$, then W contains at least m vertices.

Proof Let W be a basis for $K_m + G$; then basis of $K_m + G$ must contain $m-1$ vertices of K_m and by lemma 3.1, basis of $K_m + G$ must contain at least one vertex of each graph. Therefore $|W| \geq m-1+1 \Rightarrow \beta(K_m + G) \geq m.$

Caceres et al. [5] have obtained the metric dimension for the graph $P_n + K_1;\ n \notin \{1,2,3,6\}$. We have generalized the result of [5] in the following theorem.

***Theorem* 3.9:** Let P_n be a path of n vertices and K_m be a complete graph with m vertices then

$$\beta(P_n + K_m) = \begin{cases} m+1; & 2 \leq n \leq 5 \\ m+2; & 6 \leq n \leq 8 \\ m + \left\lceil \frac{n}{2} \right\rceil - 2; & n \geq 9 \end{cases}$$

***Proof*:** Let $\{u_1, u_2, u_3, ..., u_n\}$ and $\{v_1, v_2, v_3, ..., v_m\}$ be the vertex sets of path P_n and complete graph K_m respectively. Following cases arise:

***Case I* (a):** If $2 \leq n < 5$, let us suppose $W = \{v_1, v_2, v_3, ... v_{m-1}, u_1, u_2\} \subset V(P_n + K_m)$. Now metric representation of any vertex of $V(P_n + K_m) \setminus W$ with respect to W are respectively:

$r(v_m / W) = (1,1,1,...,1,1)$
$r(u_3 / W) = (1,1,1,...,2,1)$
$r(u_4 / W) = (1,1,1,...,2,2)$

Obviously all these representations of vertices of graph $P_n + K_m$ are different. So W is a resolving set for the graph $P_n + K_m$. Therefore $\beta(P_n + K_m) \le m+1$

Now if possible, let $W = \{v_1, v_2, v_3, ...v_{m-1}, u_i\}$; vig $|W| = m$ then $\exists$ at least one vertex $u_j \in V(P_n)$ different from $u_i \in W$ such that $r(v_m / W) = (1,1,1,...,1) = r(u_j / W)$; a contradiction. So W cannot be resolving set and therefore it cannot be a basis for $P_n + K_m$ if $|W| \le m$ i.e. $\beta(P_n + K_m) \ge m+1$.

Therefore $\beta(P_n + K_m) = m+1$.

***Case I* (b):** If $n = 5$; then in similar manner $W = \{v_1, v_2, ...v_{m-1}, u_2, u_3\}$ can be proved a basis for the graph $P_n + K_m$. So $\beta(P_n + K_m) = m+1$; $2 \le n \le 5$.

***Case II* (a):** For $6 \le n < 8$.

Now if possible $W = \{v_1, v_2, v_3, ...v_{m-1}, u_i, u_j\}$; where $|W| = m+1$, then $\exists$ at least two vertices u_k, u_l in $V(P_n)$ having the same metric representation with respect to W as:

$r(u_k / W) = (1,1,...,2,2) = r(u_l / W)$

or

$r(u_k / W) = (1,1,...,2,1) = r(u_l / W)$

or

$r(u_k / W) = (1,1,...,1,2) = r(u_l / W)$

or

$r(u_k / W) = (1,1,...,1,1) = r(v_m / W)$

Which are contradictions; therefore $\beta(P_n + K_m) > m+1$.

Now let us take $W = \{v_1, v_2, v_3, ...v_{m-1}, u_3, u_5, u_6\} \subset V(P_n + K_m)$; then the metric representation of any vertex of $V(P_n + K_m) \setminus W$ with respect to W are:

$r(v_m / W) = (1,1,...,1,1,1)$
$r(u_1 / W) = (1,1,...,2,2,2)$
$r(u_2 / W) = (1,1,...,1,2,2)$
$r(u_4 / W) = (1,1,...,1,1,2)$
$r(u_7 / W) = (1,1,...,2,2,1)$

This shows that all vertices of $P_n + K_m$ have different metric representations with respect to W, so W is a resolving set and it is least resolving set containing $m-1+3$ *i.e.* $m+2$ elements $\beta(P_n + K_m) = m+2;\ 6 \le n < 8$

***Case II* (b):** If $n = 8$; then in similar manner $W = \{v_1, v_2, ...v_{m-1}, u_3, u_5, u_7\}$ can be proved basis for $P_n + K_m$. So $\beta(P_n + K_m) = m+2; \quad 6 \le n \le 8.$

***Case III* (a):** If $n \ge 9$ and n is even. Let $W = \{v_1, v_2, v_3, ...v_{m-1}, u_3, u_5, u_7 ... u_{n-1}\} \subset V(P_n + K_m)$; then the metric representation of vertices of $V(P_n + K_m) \setminus W$ with respect to W are:

$r(v_m / W) = (1,1,...,1,1,1,1)$
$r(u_1 / W) = (1,1,...,2,2,2,2)$
$r(u_2 / W) = (1,1,...,1,2,2,2)$
$r(u_4 / W) = (1,1,...,1,1,2,2)$
$r(u_6 / W) = (1,1,...,2,1,1,2)$
$r(u_8 / W) = (1,1,...,2,2,1,1)$
.
.
.
$r(u_n / W) = (1,1,...,2,2,2,1)$

These all the representations are distinct with respect to W. So W is a resolving set for $P_n + K_m$. $\beta(P_n + K_m) \le |W| = m - 1 + \frac{n-2}{2} \Rightarrow \beta(P_n + K_m) \le m + \frac{n}{2} - 2$.

Now if we consider an ordered set W' such that $\beta(P_n + K_m) \le |W'|$ and $|W'| < |W|$, then we find that there exist at least two vertices in $V(P_n + K_m) \setminus W'$ having the same metric representation with respect to W'.

For example if $W' = W - \{u_3\}$, then

$r(u_1 / W') = (1,1,...,1,2,2,...,2) = r(u_2 / W')$; a contradiction.

If $W' = W - \{u_5\}$, then

$r(u_1 / W) = (1,1,...,1,2,2,...,2) = r(u_5 / W)$; a contradiction.

If $W' = W - \{u_7\}$, then

$r(u_1 / W) = (1,1,...,1,2,2,...,2) = r(u_7 / W)$; a contradiction.

So on finally $W' = W - \{u_{n-1}\}$, then

$r(u_1 / W) = (1,1,...,1,2,2,...,2) = r(u_{n-1} / W);$ a contradiction again.

Similarly for all other cases we get a contradiction. It means that any ordered set W' such that $|W'| < |W| = m + \frac{n}{2} - 2$ can't be a resolving set for $P_n + K_m$.

It implies that W is a smallest resolving set for $P_n + K_m$ and

$$\beta(P_n + K_m) = m + \frac{n}{2} - 2;\ n \geq 9 \text{ and n is even} \qquad (1)$$

***Case III* (b):** If $n \geq 9$ and n is odd. Let $W = \{v_1, v_2, v_3, ...v_{m-1}, u_3, u_5, ..., u_n\} \subset V(P_n + K_m);$ then as above we can show that different vertices of $P_n + K_m$ have different metric representation with respect to W therefore

$$\beta(P_n + K_m) \leq |W| = m - 1 + \frac{n-1}{2} \Rightarrow \beta(P_n + K_m) \leq m + \frac{n+1}{2} - 2.$$

Now if we take any ordered set W' such that $|W'| < |W|$, then we observe that there exist at least two vertices in $V(P_n + K_m) \setminus W'$ having the same metric representation with respect to W'. So $\beta(P_n + K_m)$ can't less than $m - 2 + \frac{n+1}{2}$ i.e.

$\beta(P_n + K_m) \geq m - 2 + \frac{n+1}{2}$.Therefore

$$\beta(P_n + K_m) = m + \frac{n+1}{2} - 2;\ n \geq 9 \text{ and n is odd} \qquad (2)$$

Now by (1) and (2) implying together

$$\beta(P_n + K_m) = m + \left\lceil \frac{n}{2} \right\rceil - 2;\ n \geq 9.$$

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