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Some Regularity Properties of Three Dimensional Incompressible Magnetohydrodynamic Flows

R.V. Saraykar[1] and Swapna V. Uddhao[2]

Department of Mathematics, RTM Nagpur University,
University Campus, Nagpur-440033, India

Abstract

In this paper, we establish Serrin-type regularity and global regularity of strong solutions to three dimensional incompressible magnetohydrodynamic equations. Global regularity is proved by assuming certain sufficient condition involving only one velocity component.

Keywords: Incompressible magnetohydrodynamic equations, Sobolev spaces, Serrin type regularity, global regularity.

AMS subject classifications: 35B65, 35G60, 46E35, 76D03, 76D05, 76W05.

1. INTRODUCTION

Before discussing regularity problem for solutions of three dimensional incompressible magnetohydromagnetic (MHD) equations, we first look into similar problem for Navier-Stokes equations. It is well-known that the global regularity problem for three dimensional Navier-Stokes equations is a Clay Millennium Prize problem and that it asks for existence of global smooth solutions to a Cauchy problem for a nonlinear partial differential equation describing motion of three dimensional viscous incompressible fluid. There are many global regularity results of this type for other nonlinear partial differential equations. For example, global regularity is known for Navier-Stokes equations in two spatial dimensions rather than three and this result essentially dates back to Jean Leray's thesis in 1933 ! Why is then the three-dimensional Navier-Stokes global regularity problem considered so hard, when global regularity for so many other equations is easy, or at least achievable? The detailed answer to third question was described by Terence Tao [1] in his 2007 article and we briefly discuss it here. According to Tao, the standard response to this question is turbulence – the behaviour of three-dimensional Navier-Stokes equations at fine scales is much more nonlinear (and hence unstable) than at coarse scales. Tao describes the obstruction slightly differently, as supercriticality. Or more precisely, all of the globally controlled quantities for Navier-Stokes evolution which we are aware of are either supercritical with respect to scaling, which means that they are much weaker at controlling fine-scale behaviour than controlling coarse-scale behaviour, or they are non-coercive, which means that they do not really control the solution at all, either at coarse scales or at fine. At present, all known methods for obtaining global smooth solutions to a (deterministic) nonlinear partial differential equation as a Cauchy problem require either

1. Exact and explicit solutions or at least an exact, explicit transformation to a significantly simpler partial differential equation or ordinary differential equation;
2. Perturbative hypotheses, for example small data, data close to a special solution, or more generally a hypothesis which involves an somewhere); or
3. One or more globally controlled quantities (such as the total energy) which are both coercive and either critical or subcritical.

We note here that the presence of (1), (2) or (3) are currently necessary conditions for a global regularity result, but far from sufficient. In particular, there have been many good, deep, and highly non-trivial papers recently on global regularity for Navier-Stokes, but they all assume either (1), (2) or (3) via additional hypotheses on the data or solution. For instance, in recent years we have seen good results on global regularity assuming (2), as well as good results on global regularity assuming (3). The papers by Cao and Titi [3,4] and other references cited there in are a proof of this. Tao further remarks that the Navier-Stokes global regularity problem for arbitrary large smooth data lacks all of these three ingredients. Reinstating (2) is impossible without changing the statement of the problem, or adding some additional hypotheses; also, in perturbative situations the Navier-Stokes equation evolves almost linearly, while in the non-perturbative setting it behaves very nonlinearly, so there is basically no chance of a reduction of the non-perturbative case to the perturbative one unless one comes up with a highly nonlinear transform to achieve this (e.g. a naive scaling argument cannot possibly work). Thus, one is left with only three possible strategies if one wants to solve the full problem:

1. Solve the Navier-Stokes equation exactly and explicitly (or at least transform this equation exactly and explicitly to a simpler equation);
2. Discover a new globally controlled quantity which is both coercive and either critical or subcritical; or

3. Discover a new method which yields global smooth solutions even in the absence of the ingredients (1), (2), and (3) above.

Tao himself and other researchers are working towards settling this important unresolved problem. Similar situation holds for three dimensional incompressible magnetohydrodynamic (MHD) flows, and global regularity problem is unresolved there also. We briefly describe, below, the situation regarding MHD flows.

To begin with, the incompressible MHD equations describe the motion of an electrical conducting fluid in the presence of a magnetic field and are of importance in physics and other applied areas. Hence the study of the MHD equations has aroused a lot of interest during past four decades. The first qualitative results proving existence and uniqueness of solutions for MHD equations were derived by E. Sanchez-Palencia in 1969 [5]. A very useful and elegant review of results till 1980 has been given by Sermange and Temam [6]. During past two decades, fresh interest aroused among applied mathematicians in the pursuit of proving global in time regularity, which is one of the major questions for solution to the three dimensional MHD equations. Thus, in particular, there have been extensive mathematical discussions on the regularity of the weak solution to the MHD equations in three space dimensions. For two dimensional MHD equations, it is well known that there exists a unique global classical solution for every initial data $(u_0, B_0) \in H_m, m > 2$ (see Cao and Wu [7], C´ordoba D and Marliani [9], and Sermange and Temam [6] and references therein). Local existence of the solutions for three dimensional MHD equations was also proved in [3]. However, global regularity is still an open problem. Therefore, many researchers seem to discuss this subject by addressing the sufficient conditions which would guarantee the global regularity of the weak solution. Different criteria for regularity in terms of the velocity field, the magnetic field, the pressure and their derivatives have been proposed in the work of Cao C and Wu J [8], Chen Q, Miao C and Zhang Z [10], Luo L, Zhao Y and Yang Q [11], and Zhou Y and Gala S [12] and references cited in these papers. **C.** He and Z. Xin [13,14] realized, for the first time, that the velocity fields play a dominate role in the regularity of the solution to three dimensional incompressible MHD equations. More precisely, they proved the global regularity of the strong solution in terms of only the velocity field for the first time. Motivated by this work, Zhou Y [15] and Zhou Y and Gala S[12] established the global regularity criteria by providing sufficient conditions on one of the components u, B and p independently. In an elegant work, Cao and Wu [8] provided two regularity criteria in terms of the derivatives of the velocity or the pressure in one direction. In particular, they showed that any suitable weak solution (u, B) to the three dimensional MHD equations is regular with suitable conditions on ${\partial u}/{\partial x_3}$. Lin and Du [16] generalized the results in [7] and established some general sufficient conditions for the global regularity of strong solutions to the three dimensional MHD equations. Bie, Wang and Yao [17] (2013) consider three-dimensional incompressible magnetohydrodynamics equations. By using interpolation inequalities in anisotropic Lebesgue space, they prove regularity criteria involving the velocity or alternatively involving the fractional derivative of velocity in one direction. This generalizes some known results. In the present paper, we improve upon these results and prove global regularity of weak solutions under suitable conditions on one of the components of velocity field. Thus, in Section 2, we prove Serrin-type regularity result and in Section 3, we prove global regularity of weak solutions. We end the paper with concluding remarks by comparing these two results. Our work is based upon a paper by Zhang [2] and two papers by Cao and Titi [3,4] including a paper by Lin and Du [16].

2. SERRIN-TYPE REGULARITY

In this section, we prove the Serrin-type regularity of the 3D MHD system assuming condition on only one velocity component. The domain that we consider is R^3. Thus, the equations describing viscous incompressible three-dimensional magnetohydrodynamic (MHD) flow are :

$$\partial_t u - \nu\Delta u + u\,.\nabla \mathrm{u} - \mathrm{B}\,.\nabla \mathrm{B} + \nabla \mathrm{p} = \mathrm{f} \qquad (1)$$

$$\partial_t B - \lambda\Delta B + u\,.\nabla \mathrm{B} - \mathrm{B}\,.\nabla \mathrm{u} = 0 \qquad (2)$$

$$\nabla\,.u = 0 \; and \; \nabla\,.B = 0 \qquad (3)$$

$$(u, B)|_{t=0} = (u_0, B_0) \qquad (4)$$

Where, $u = u(x,t)$ is the velocity field, $B = B(x,t)$ is the magnetic field, $\nu > 0$ is the kinematic coefficient of viscosity, $\lambda > 0$ is the coefficient of magnetic diffusivity, $p = p(x,t)$ is the pressure and f is the external force term.

Before we state the main theorem, we recall some standard definitions:

The Lebesgue space $L^p(R^3)$ is defined by

$L^p(R^3) = \{u : \int |u(x)|^p dx < \infty\}, p \in [1, \infty)$, which is endowed with a norm $\|\,.\,\|_p$

We denote by $\|\,.\,\|_{p,q}$ the norm for anisotropic Lebesgue spaces $L^p(0,T; L^q(R^3))$, the space of all L^p-functions defined a.e. on $(0,T)$, for some $T > 0$, with values in $L^q(R^3)$. The Sobolev spaces $w^{m,p}(R^3)$ is the collection of all functions in $L^p(R^3)$ such that all weak derivatives upto order m are also in $L^p(R^3)$. It is equipped with the norm $\|\,.\,\|_{m,p}$. When p = 2, the Sobolev space $w^{m,2}(R^3)$ becomes a Hilbert space $w^{m,2}(R^3) = H^m(R^3)$, equipped with the norm $\|\,.\,\|_{m,2}$.

$C^k(R^3)$ is the space of k-times continuously differentiable functions in R^3.

$C_0^\infty(R^3)$ denotes the space of all infinitely differentiable functions defined on R^3 with compact support in R^3.

We set, $\mathcal{V} = \{u \in C_0^\infty(R^3): div\, u = 0\}$ which will form the space of test functions. Let H and V be the closure spaces of $\mathcal{V}$ in $L^2(R^3)$ under L^2- topology, and in $H^1(R^3)$ under H^1- topology respectively.

We set $\nabla_h = (\partial x_1, \partial x_2)$ to be the horizontal gradient operator and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ the horizontal Laplacian, while ∇ and Δ are the usual gradient and the Laplacian operators respectively.

Definition 1: Let $(u_0, B_0) \in H \times H,\ T > 0$. A pair (u, B) of measurable functions defined in $[0, T] \times R^3$ is called a weak solution of the system (1)-(4) if

(1) $(u, B) \in L^\infty(0, T; H \times H) \cap L^2(0, T; V \times V)$ and $(\partial_t u, \partial_t B) \in L^1(0, T; V' \times V')$, where V' is the dual space of V.

(2) the 3D MHD system holds in the sense of distribution:

$$(u(t), \varphi(t)) + \int_0^t \{-(u, \partial_t \varphi) + \nu(\nabla u, \nabla \varphi) + (u.\nabla u, \varphi) - (B.\nabla B, \varphi)\}\, ds = \big(u_0, \varphi(0)\big)$$

$$(B(t), \varphi(t)) + \int_0^t \{-(B, \partial_t \varphi) + \lambda(\nabla B, \nabla \varphi) + (u.\nabla B, \varphi) - (B.\nabla u, \varphi)\}\, ds = \big(B_0, \varphi(0)\big)$$

for all $\varphi \in C_0^\infty([0, T) \times R^3)$ such that $\nabla.\varphi = 0$.

Here, $(\cdot,\cdot)$ is the scalar product in $L^2(R^3)$.

Definition 2: Suppose $(u_0, B_0) \in V \times V$, a weak solution is said to be a strong solution of the system (1)-(4) if, in addition, it satisfies

$(u, B) \in C(0, T; V \times V) \cap L^2(0, T; H^2 \times H^2)$ and $(\partial_t u, \partial_t B) \in L^2(0, T; H \times H)$.

We now state and prove the main theorem of this section.

Theorem 1:

Let $(u_0, B_0) \in V \times V$ and (u, B) be a weak solution to the system (1)-(4) in $[0, T]$ with initial data (u_0, B_0).

If $(u_3, B) \in L^p(0, T; L^q(R^3) \times L^q(R^3))$, $(\partial_3 u_3, \partial_3 B) \in L^r(0, T; L^s(R^3) \times L^s(R^3))$, (5)

with $1 \le p, q, r, s \le \infty$, $0 \le \beta, \gamma < \infty$ satisfying

$$\left.\begin{array}{c} {}^{2}/_{p} + {}^{3}/_{q} = \beta,\ {}^{2}/_{r} + {}^{3}/_{s} = \gamma \\ \left(1-\frac{1}{s}\right) q = \frac{{}^{1}/_{r} + {}^{3}/_{8}}{{}^{3}/_{8} - {}^{1}/_{p}} = \frac{{}^{9}/_{4} - \gamma}{\beta - {}^{3}/_{4}} > 1 \\ p < \infty \text{ or } r < \infty \end{array}\right\} \quad (6)$$

then (u, B) is smooth in $[0, T] \times R^3$.

(Here we take $p = 6,\ q = 4,\ r = 4,\ s = 4$. $so\ \beta = {}^{13}/_{12},\ \gamma = {}^{5}/_{4}$).

We need following lemma (see [18]) to prove the theorem 1.

Lemma 1: For $f, g, h \in C_0^\infty(R^3)$, we have

$$\left|\int f\ g\ h\ dx_1 dx_2 dx_3\right| \le C\ \|f\|_q^{\frac{\alpha-1}{\alpha}}\ \|\partial_3 f\|_s^{\frac{1}{\alpha}}\ \|g\|_2^{\frac{\alpha-2}{\alpha}}\ \|\partial_1 g\|_2^{\frac{1}{\alpha}}\ \|\partial_2 g\|_2^{\frac{1}{\alpha}}\ \|h\|_2\ , \quad (7)$$

Where

$\alpha > 2, 1 \le q, s \le \infty, \frac{\alpha-1}{q} + \frac{1}{s} = 1$. (We can take $\alpha = 4$).

Also, We have the Sobolev imbedding inequality:

$$\|f\|_6 \le C\ \|\nabla_h f\|_2^{{}^{2}/_{3}}\ \|\partial_3 f\|_2^{{}^{1}/_{3}}\ , \quad (8)$$

Proof of the Theorem 1

Step-I : $\|(\nabla_h \boldsymbol{u}, \nabla_h \boldsymbol{B})\|_2$ estimates:

By definition 1, $(u_3, B) \in L^\infty(0, T; L^2(R^3) \times L^2(R^3))$, we may take $p = \infty$ and $q = 2$, $\gamma = {}^{3}/_{4} + {}^{3}/_{2s}$ in Theorem 1 so that the condition $(\partial_3 u_3, \partial_3 B) \in L^r(0, T; L^s(R^3) \times L^s(R^3))$, ${}^{2}/_{r} + {}^{3}/_{s} = {}^{3}/_{4} + {}^{3}/_{2s}$, $1 \le r < \infty$, ensures that (u, B) is smooth in $[0, T] \times R^3$. See, for example, references [2,4].

We prove that, $\|(\nabla u, \nabla B)\|_{\infty,2} < \infty$. (9)

Here, $\|(u, B)\|$ denotes the product norm, which is usually defined as :

$\|(u, B)\|^2 = \|u\|^2 + \|B\|^2$.

By (7), we can take $\alpha = 4$ such that

$$\alpha - 1 = \left(1 - \frac{1}{s}\right) q = \frac{{}^{1}/_{r} + {}^{3}/_{8}}{{}^{3}/_{8} - {}^{1}/_{p}} = \frac{{}^{9}/_{4} - \gamma}{\beta - {}^{3}/_{4}},$$

$$\left.\begin{aligned} \frac{\alpha-1}{p}+\frac{1}{r}&=\frac{3(\alpha-2)}{8}\\ \frac{\alpha-1}{q}+\frac{1}{s}&=1\\ (\alpha-1)\beta+\gamma&=\frac{3(\alpha-2)}{4}+3 \end{aligned}\right\} \tag{10}$$

Let us take the inner product of (1) with $-\Delta_h u$ and (2) with $-\Delta_h B$ in $L^2(R^3)$, and obtain

$$\frac{1}{2}\frac{d}{dt}(\|(\nabla_h u,\nabla_h B)(t)\|_2^2)+\ \nu(\|(\nabla_h\nabla u,\nabla_h\nabla B)\|_2^2)=\int(u.\nabla u)\Delta_h u\ dx-\int(B.\nabla B)\Delta_h u\ dx+\int(u.\nabla B)\Delta_h B\ dx-\int(B.\nabla u)\Delta_h B\ dx$$

For simplicity, we have chosen $\lambda=\nu$.

Solving the integrals we obtain,

$$\frac{1}{2}\frac{d}{dt}(\|(\nabla_h u,\nabla_h B)(t)\|_2^2)+\ \nu(\|(\nabla_h\nabla u,\nabla_h\nabla B)\|_2^2)\le C\int|u_3||\nabla u||\nabla_h\nabla u|\ dx+C\int|B||\nabla B||\nabla_h\nabla u|\ dx+C\int|u_3||\nabla B||\nabla_h\nabla B|\ dx+C\int|B||\nabla u||\nabla_h\nabla B|\ dx$$

$$\le I_1+I_2+I_3+I_4 \tag{11}$$

Using (7), 2nd equality in (10) and Young's inequality we get,

$$I_1=C\int|u_3||\nabla u||\nabla_h\nabla u|\ dx$$

$$\le C\|u_3\|_4^{3/4}\|\partial_3 u_3\|_4^{1/4}\|\nabla u\|_2^{1/2}\|\nabla_h\nabla u\|_2^{3/2}$$

$$\le C\|u_3\|_4^3\|\partial_3 u_3\|_4\|\nabla u\|_2^2+\frac{\nu}{2}\|\nabla_h\nabla u\|_2^2 \tag{12}$$

Similarly,

$$I_2\le C\|B\|_4^3\|\partial_3 B\|_4\|\nabla B\|_2^2+\frac{\nu}{2}\|\nabla_h\nabla u\|_2^2 \tag{13}$$

$$I_3\le C\|u_3\|_4^3\|\partial_3 u_3\|_4\|\nabla B\|_2^2+\frac{\nu}{2}\|\nabla_h\nabla B\|_2^2 \tag{14}$$

$$I_4\le C\|B\|_4^3\|\partial_3 B\|_4\|\nabla u\|_2^2+\frac{\nu}{2}\|\nabla_h\nabla B\|_2^2 \tag{15}$$

Using (12)-(15) in (11) we obtain,

$$\frac{d}{dt}(\|(\nabla_h u,\nabla_h B)(t)\|_2^2)+\ \nu(\|(\nabla_h\nabla u,\nabla_h\nabla B)\|_2^2)\le C\|(u_3,B)\|_4^3\|(\partial_3 u_3,\partial_3 B)\|_4\|(\nabla u,\nabla B)\|_2^2 \tag{16}$$

Now, integrating (16) we get,

$$\|(\nabla_h u,\nabla_h B)(t)\|_2^2+\nu\int_0^t\|(\nabla_h\nabla u,\nabla_h\nabla B)(s)\|_2^2\ ds\le\|(\nabla_h u_0,\nabla_h B_0)\|_2^2+C\int_0^t[\|(u_3,B)\|_4^3\|(\partial_3 u_3,\partial_3 B)\|_4\|(\nabla u,\nabla B)(s)\|_2^2]\ ds \tag{17}$$

for all $t \in [0, T]$.

Step-II : $\|(\nabla u, \nabla B)\|_2$ estimates:

Let us take the inner product of (1) with $-\Delta u$ and (2) with $-\Delta B$ in $L^2(R^3)$, and using (3) we obtain,

$$\frac{1}{2}\frac{d}{dt}(\|(\nabla u, \nabla B)(t)\|_2^2) + \nu(\|(\Delta u, \Delta B)\|_2^2) = \int (u.\nabla u)\Delta_h u \; dx + \int (u.\nabla u)\partial_{33}^2 u \, dx - \int (B.\nabla B)\Delta_h u \; dx - \int (B.\nabla B)\,\partial_{33}^2 u \, dx + \int (u.\nabla B)\Delta_h B \; dx + \int (u.\nabla B)\partial_{33}^2 B \, dx - \int (B.\nabla u)\Delta_h B \; dx - \int (B.\nabla u)\partial_{33}^2 B \, dx$$

$$\leq C\int [|u_3||\nabla u||\nabla_h \nabla u| + |\nabla_h u||\partial_3 u|^2] \; dx + C\int [|B||\nabla B||\nabla_h \nabla u| + |\nabla_h B||\partial_3 u|^2] \; dx + C\int [|u_3||\nabla B||\nabla_h \nabla B| + |\nabla_h B||\partial_3 B|^2] \; dx + C\int [|B||\nabla u||\nabla_h \nabla B| + |\nabla_h u||\partial_3 B|^2] \; dx$$

$$\equiv (I_1 + J_1) + (I_2 + J_2) + (I_3 + J_3) + (I_4 + J_4) \quad (18)$$

Using Holder, interpolation inequalities and (8), we get

$$J_1 \leq C\int |\nabla_h u||\partial_3 u|^2 \, dx$$

$$\leq C\|\nabla_h u\|_2 \|\nabla u\|_4^2$$

$$\leq C\|\nabla_h u\|_2 \|\nabla u\|_2^{1/2} \|\nabla u\|_6^{3/2}$$

$$\leq C\|\nabla_h u\|_2 \|\nabla u\|_2^{1/2} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{1/2} \quad (19)$$

Similarly,

$$J_2 \leq C\|\nabla_h B\|_2 \|\nabla u\|_2^{1/2} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{1/2} \quad (20)$$

$$J_3 \leq C\|\nabla_h B\|_2 \|\nabla B\|_2^{1/2} \|\nabla_h \nabla B\|_2 \|\Delta B\|_2^{1/2} \quad (21)$$

$$J_4 \leq C\|\nabla_h u\|_2 \|\nabla B\|_2^{1/2} \|\nabla_h \nabla B\|_2 \|\Delta B\|_2^{1/2} \quad (22)$$

Thus, using (12)-(15) and (19)-(22) in (18), we obtain after some manipulations, by using the product norm,

$$\frac{d}{dt}(\|(\nabla u, \nabla B)(t)\|_2^2) + \nu(\|(\Delta u, \Delta B)\|_2^2) \leq C\|(u_3, B)\|_4^3 \|(\partial_3 u_3, \partial_3 B)\|_4 \|(\nabla u, \nabla B)\|_2^2 + C\|(\nabla_h u, \nabla_h B)\|_2 \|(\nabla u, \nabla B)\|_2^{1/2} \|(\nabla_h \nabla u, \nabla_h \nabla B)\|_2 \|(\Delta u, \Delta B)\|_2^{1/2} \quad (23)$$

Integrating (23) and using Holder inequality, we get

$$\|(\nabla u,\nabla B)(t)\|_2^2+\nu\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\ ds\le\|(\nabla u_0,\nabla B_0)\|_2^2+$$
$$C\int_0^t[\|(u_3,B)\|_4^3\|(\partial_3u_3,\partial_3B)\|_4\|(\nabla u,\nabla B)(s)\|_2^2]\ ds+$$
$$C\sup\|(\nabla_h u,\nabla_h B)(s)\|_2\left(\int_0^t\|(\nabla u,\nabla B)(s)\|_2^2\,ds\right)^{1/4}\left(\int_0^t\|(\nabla_h\nabla u,\nabla_h\nabla B)(s)\|_2^2\,ds\right)^{1/2}$$
$$\left(\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\,ds\right)^{1/4}\qquad(24)$$

for all $t\in[0,T]$.

Using (17), we get

$$\|(\nabla u,\nabla B)(t)\|_2^2+\nu\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\ ds\le\|(\nabla u_0,\nabla B_0)\|_2^2+$$
$$C\int_0^t\|(u_3,B)\|_4^3\|(\partial_3u_3,\partial_3B)\|_4\|(\nabla u,\nabla B)(s)\|_2^2\ ds+C\|(\nabla_h u_0,\nabla_h B_0)\|_2^2+$$
$$C\int_0^t\|(u_3,B)\|_4^3\ \|(\partial_3u_3,\partial_3B)\|_4\|(\nabla u,\nabla B)(s)\|_2^2\ ds\left(\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\,ds\right)^{1/4}$$

Using Young's and Holder inequality and as $(u,B)\in(0,T;V\times V)$ we obtain

$$\|(\nabla u,\nabla B)(t)\|_2^2+\frac{\nu}{2}\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\ ds\le C\|(\nabla u_0,\nabla B_0)\|_2^2+$$
$$C\int_0^t[\|(u_3,B)\|_4^3\|(\partial_3u_3,\partial_3B)\|_4\|(\nabla u,\nabla B)(s)\|_2^2]\ ds+$$
$$C\int_0^t[\|(u_3,B)\|_4^4\ \|(\partial_3u_3,\partial_3B)\|_4^{4/3}\|(\nabla u,\nabla B)(s)\|_2^2]\ ds\qquad(25)$$

Again using Young's inequality, we get

$$\|(\nabla u,\nabla B)(t)\|_2^2+\frac{\nu}{2}\int_0^t\|(\Delta u,\Delta B)(s)\|_2^2\ ds\le C\|(\nabla u_0,\nabla B_0)\|_2^2+C\int_0^t(1+$$
$$\|(u_3,B)\|_4^6+\|(\partial_3u_3,\partial_3B)\|_4^4)\ \|(\nabla u,\nabla B)(s)\|_2^2\,ds\qquad(26)$$

Finally by using Gronwall inequality and (5), we get (9) as required.
This completes the proof of the theorem 1.

3. GLOBAL REGULARITY

In this section, assuming sufficient conditions in terms of only one component of the gradient of velocity field, we prove global regularity of the 3D MHD system. We also give argument for existence of maximal time for which solution exists.

The main theorem is as follows :

Theorem 2:

Let $(u_0,B_0)\in V\times V$ and (u,B) be a weak solution to 3D MHD system (1)-(4) with initial value (u_0,B_0). Let $T>0$, and suppose that, for some k,j with $1\le k,j\le 3$, (u,B) satisfies the condition

$\int_0^T \left\|\frac{(\partial u_j)(s)}{\partial x_k}\right\|_\alpha^\beta ds \leq M$; when $k \neq j$, and where $\alpha > 3, 1 \leq \beta < \infty$, and $\frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha+3}{2\alpha}$ (27)

or

$\int_0^T \left\|\frac{(\partial u_j)(s)}{\partial x_j}\right\|_\alpha^\beta ds \leq M$; where $\alpha > 2, 1 \leq \beta < \infty$, and $\frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha+2)}{4\alpha}$ (28)

for some $M > 0$. Then (u, B) is a strong solution of the system (1)-(4), on the interval $[0, T]$. Moreover, it is the only weak solution on $[0, T]$ with the initial data (u_0, B_0).

(Here we take $\alpha = 4,\ \beta = 16$ for (27) and $\alpha = 3,\ \beta = 8$ for (28)).

We need following lemmas (see [18]) to prove the above theorem.

Lemma 2: For $\varphi, f, g \in C_0^\infty(R^3)$, we have

$$|\int \varphi\, f\, g\ dx_1 dx_2 dx_3| \leq C\ \|\varphi\|_2^{\frac{r-1}{r}}\ \|\partial x_1 \varphi\|_{\frac{2}{3-r}}^{\frac{1}{r}}\ \|f\|_2^{\frac{r-2}{r}}\ \|\partial x_2 f\|_2^{\frac{1}{r}}\ \|\partial x_3 f\|_2^{\frac{1}{r}}\ \|g\|_2\ , \quad (29)$$

where $2 < r < 3$.

Lemma 3: For $\varphi, f, g \in C_0^\infty(R^3)$, we have

$$|\int \varphi\, f\, g\ dx_1 dx_2 dx_3| \leq C\ \|\varphi\|_2^{\frac{r-1}{r}}\ \|\partial x_3 \varphi\|_{\frac{2}{3-r}}^{\frac{1}{r}}\ \|f\|_2^{\frac{r-2}{r}}\ \|\partial x_1 f\|_2^{\frac{1}{r}}\ \|\partial x_2 f\|_2^{\frac{1}{r}}\ \|g\|_2\ , \quad (30)$$

where $2 < r < 3$.

We recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space R^3. There exists a positive constant C_s such that,

$$\|\psi\|_s \leq C_s\ \|\psi\|_2^{\frac{6-s}{2s}}\ \|\partial x_1 \psi\|_2^{\frac{s-2}{2s}}\ \|\partial x_2 \psi\|_2^{\frac{s-2}{2s}}\ \|\partial x_3 \psi\|_2^{\frac{s-2}{2s}}$$

$$\leq C_s\ \|\psi\|_2^{\frac{6-s}{2s}}\ \|\psi\|_{H^1(R^3)}^{\frac{3(s-2)}{2s}} \quad (31)$$

for every $\psi \in H^1(R^3)$ and every $s \in [2,6]$.

Proof of the Theorem 2

Without loss of generality, let us assume that $j = 3$ and $k = 1$ in (27) and (28), namely

$\int_0^T \left\|\frac{(\partial u_3)(s)}{\partial x_1}\right\|_\alpha^\beta ds \leq M$; where $\alpha > 3, 1 \leq \beta < \infty$, and $\frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha+3}{2\alpha}$ (32)

or

$\int_0^T \left\|\frac{(\partial u_3)(s)}{\partial x_3}\right\|_\alpha^\beta ds \leq M$; where $\alpha > 2, 1 \leq \beta < \infty$, and $\frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha+2)}{4\alpha}$ (33)

It is well-known that, if $(u_0, B_0) \in H \times H$, there exists a global in time weak solution for the 3D MHD equations. Also, if $(u_0, B_0) \in \mathrm{V} \times \mathrm{V}$, then there exists a unique strong solution for a short time interval $[0, T^*)$, where $[0, T^*)$ is the maximal interval of existence of the unique strong solution.

For the proof of the above theorem, we suppose that (u, B) is the strong solution of the 3D MHD system with the initial value $(u_0, B_0) \in \mathrm{V} \times \mathrm{V}$ on the maximal time interval $[0, T^*)$. The argument for the proof of maximality of the time of existence of the unique strong solution can be given as follows:

Let $[0, T]$ denote the interval for which solution exists. If $T \le T^*$ then there is nothing to prove. On the other hand, for $T > T^*$ we can prove the boundedness of the H^1 norm of the strong solution on interval $[0, T^*)$ provided conditions (32) and (33) are valid. This will then contradict the fact that $[0, T^*)$ is the maximal existence time interval and we can conclude our proof.

Thus, from now on we assume that (u, B) is the strong solution on its maximal interval of existence $[0, T^*)$, where we suppose that $T^* < T$. We note here that the strong solution (u, B) is the only weak solution on the interval $[0, T^*)$.

Hence, (u, B) satisfies the following energy inequality:

$$\|(u, B)(t)\|_2^2 + \nu \int_0^t \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \le K_1, \tag{34}$$

for all $t \ge 0$, where

$$K_1 = \|u_0, B_0\|_2^2 \tag{35}$$

We now show that the H^1 norm of the strong solution (u, B) is bounded on interval $[0, T^*)$.

Step-I : $\|(\nabla_h u, \nabla_h B)\|_2$ estimates:

Now, we will estimate the RHS of (11) using either lemma 2 or lemma 3. We can use each of them to deal with either one of the condition (32) or (33).

Using (30) with $\varphi = |u_3|\,, f = |\nabla u|\,, g = |\nabla_h \nabla u|,\ r = \frac{5}{2}$ and Young's inequality, we get

$$I_1 = C \int |u_3||\nabla u||\nabla_h \nabla u| \, dx$$

$$\le C\, \|u_3\|_2^{3/5} \|\partial x_1 u_3\|_4^{2/5} \|\nabla u\|_2^{1/5} \|\partial x_2 \nabla u\|_2^{2/5} \|\partial x_3 \nabla u\|_2^{2/5} \|\nabla_{\mathrm{h}} \nabla u\|_2$$

$$\le C\, \|u_3\|_2^2\, \|\partial x_1 u_3\|_4^{4/3}\, \|\nabla u\|_2^{2/3}\, \|\partial x_3 \nabla u\|_2^{4/3} + \frac{\nu}{2} \|\nabla_{\mathrm{h}} \nabla u\|_2^2 \tag{36}$$

Similarly,

$$I_2 \leq C\ \|B\|_2^2\ \|\partial x_1 B\|_4^{4/3}\ \|\nabla B\|_2^{2/3}\ \|\partial x_3 \nabla B\|_2^{4/3} + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla u\|_2^2 \quad (37)$$

$$I_3 \leq C\ \|u_3\|_2^2\ \|\partial x_1 u_3\|_4^{4/3}\ \|\nabla B\|_2^{2/3}\ \|\partial x_3 \nabla B\|_2^{4/3} + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla B\|_2^2 \quad (38)$$

$$I_4 \leq C\ \|B\|_2^2\ \|\partial x_1 B\|_4^{4/3}\ \|\nabla u\|_2^{2/3}\ \|\partial x_3 \nabla u\|_2^{4/3} + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla B\|_2^2 \quad (39)$$

Using (36)-(39) in (11) we obtain,

$$\frac{d}{dt}(\|(\nabla_h u, \nabla_h B)\|_2^2) + \nu(\|(\nabla_h \nabla u, \nabla_h \nabla B)\|_2^2) \leq$$
$$C\|(u_3, B)\|_2^2\ \|(\partial x_1 u_3, \partial x_1 B)\|_4^{4/3}\ \|(\nabla u, \nabla B)\|_2^{2/3}\ \|(\partial x_3 \nabla u, \partial x_3 \nabla B)\|_2^{4/3} \quad (40)$$

Now, integrating (40), using Holder's inequality and applying (34) we get

$$\|(\nabla_h u, \nabla_h B)(t)\|_2^2 + \nu \int_0^t \|(\nabla_h \nabla u, \nabla_h \nabla B)(s)\|_2^2\ ds \leq \|(\nabla_h u_0, \nabla_h B_0)\|_2^2 +$$
$$C\ \left(\int_0^t \|(\partial x_1 u_3, \partial x_1 B)(s)\|_4^4\ \|(\nabla u, \nabla B)(s)\|_2^2\ ds\right)^{1/3} \left(\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\ ds\right)^{2/3} \quad (41)$$

for all $t \in [0, T^*)$.

On the other hand, by using (30) in RHS of (11) and young's inequality, we get

$$I_1 = C \int |u_3||\nabla u||\nabla_h \nabla u|\ dx$$
$$\leq C\ \|u_3\|_2^{3/5} \|\partial x_3 u_3\|_4^{2/5} \|\nabla u\|_2^{1/5} \|\partial x_1 \nabla u\|_2^{2/5} \|\partial x_2 \nabla u\|_2^{2/5}\ \|\nabla_{\mathrm{h}} \nabla u\|_2$$
$$\leq C\ \|u_3\|_2^6\ \|\partial x_3 u_3\|_4^4\ \|\nabla u\|_2^2 + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla u\|_2^2 \quad (42)$$

Similarly,

$$I_2 \leq C\ \|B\|_2^6\ \|\partial x_3 B\|_4^4\ \|\nabla B\|_2^2 + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla u\|_2^2 \quad (43)$$

$$I_3 \leq C\ \|u_3\|_2^6\ \|\partial x_3 u_3\|_4^4\ \|\nabla B\|_2^2 + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla B\|_2^2 \quad (44)$$

$$I_4 \leq C\ \|B\|_2^6\ \|\partial x_3 B\|_4^4\ \|\nabla u\|_2^2 + \frac{\nu}{2}\|\nabla_{\mathrm{h}} \nabla B\|_2^2 \quad (45)$$

Using (42)-(45) in (11) we obtain,

$$\frac{d}{dt}(\|(\nabla_h u, \nabla_h B)\|_2^2) + \nu(\|(\nabla_h \nabla u, \nabla_h \nabla B)\|_2^2) \leq$$
$$C\|(u_3, B)\|_2^6\ \|(\partial x_3 u_3, \partial x_3 B)\|_4^4\ \|(\nabla u, \nabla B)\|_2^2 \quad (46)$$

Now, integrating (46), using Holder's inequality and applying (34) we get

$$\|(\nabla_h u, \nabla_h B)(t)\|_2^2 + \nu \int_0^t \|(\nabla_h \nabla u, \nabla_h \nabla B)(s)\|_2^2 \; ds \leq \|(\nabla_h u_0, \nabla_h B_0)\|_2^2 + C \int_0^t (\|(\partial x_3 u_3, \partial x_3 B)(s)\|_4^4 \, \|(\nabla u, \nabla B)(s)\|_2^2) \, ds \quad (47)$$

for all $t \in [0, T^*)$.

Step-II : $\|(\nabla u, \nabla B)\|_2$ estimates:

Here, we will estimate the RHS of (18) to deal with either one of the condition (32) or (33).

Now, using Cauchy-Schwartz inequality and (31) with $s = 4$, we obtain (19)-(22).

By (36)-(39) and using young's inequality, we get

$$I_1 = C \int |u_3||\nabla u||\nabla_h \nabla u| \, dx$$

$$\leq C \, \|u_3\|_2^2 \, \|\partial x_1 u_3\|_4^{4/3} \, \|\nabla u\|_2^{2/3} \, \|\partial x_3 \nabla u\|_2^{4/3} + \frac{\nu}{2} \|\nabla_{\mathrm{h}} \nabla u\|_2^2$$

$$\leq C \, \|u_3\|_2^6 \, \|\partial x_1 u_3\|_4^4 \, \|\nabla u\|_2^2 + \frac{3\nu}{4} \|\Delta u\|_2^2 \quad (48)$$

Similarly,

$$I_2 \leq C \, \|B\|_2^6 \, \|\partial x_1 B\|_4^4 \, \|\nabla B\|_2^2 + \frac{3\nu}{4} \|\Delta B\|_2^2 \quad (49)$$

$$I_3 \leq C \, \|u_3\|_2^6 \, \|\partial x_1 u_3\|_4^4 \, \|\nabla B\|_2^2 + \frac{3\nu}{4} \|\Delta B\|_2^2 \quad (50)$$

$$I_4 \leq C \, \|B\|_2^6 \, \|\partial x_1 B\|_4^4 \, \|\nabla u\|_2^2 + \frac{3\nu}{4} \|\Delta u\|_2^2 \quad (51)$$

Thus, using (19)-(22) and (48)-(51) in (18), we obtain

$$\frac{d}{dt}(\|(\nabla u, \nabla B)\|_2^2) + \frac{\nu}{2}(\|(\Delta u, \Delta B)\|_2^2) \leq C\|(\nabla_h u, \nabla_h B)\|_2 \|(\nabla u, \nabla B)\|_2^{1/2} \|(\nabla_h \nabla u, \nabla_h \nabla B)\|_2 \|(\Delta u, \Delta B)\|_2^{1/2} + C \, \|(u_3, B)\|_2^6 \, \left\|(\partial_{x_1} u_3, \partial_{x_1} B)\right\|_4^4 \, \|(\nabla u, \nabla B)\|_2^2 \quad (52)$$

Integrating (52) and using Holder inequality, we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{2} \int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \; ds \leq \|(\nabla u_0, \nabla B_0)\|_2^2 + C \, (\sup \|(\nabla_h u, \nabla_h B)(s)\|_2) \left(\int_0^t \|(\nabla u, \nabla B)(s)\|_2^2 \, ds\right)^{1/4} \left(\int_0^t \|(\nabla_h \nabla u, \nabla_h \nabla B)(s)\|_2^2 \, ds\right)^{1/2} \left(\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds\right)^{1/4} + C \, (\sup \|(u, B)(s)\|_2^6) \, \left(\int_0^t \, \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^4 \, \|(\nabla u, \nabla B)(s)\|_2^2 \; ds \right) \quad (53)$$

Now, using (34) and (41), we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{2}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds \le \|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C\, K_1^{1/4} \Bigg[\|(\nabla_h u_0, \nabla_h B_0)\|_2^2 +$$

$$C \left(\int_0^t \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \right)^{\frac{1}{3}} \left(\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds\right)^{\frac{2}{3}+\frac{1}{4}} \Bigg] +$$

$$C\, K_1^3 \left(\int_0^t \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \right)$$

Using Young's and Holder inequalities, we obtain

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{4}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds \le C\, \|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C \left(\int_0^t \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^{16} \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \right) \left(\int_0^t \|(\nabla u, \nabla B)(s)\|_2^2 \, ds\right)^{\frac{3}{4}} +$$

$$C\, K_1^3 \left(\int_0^t \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \right) \tag{54}$$

Using (34) again, we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{4}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds \le C\, \|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C \int_0^t \left\|(\partial_{x_1} u_3, \partial_{x_1} B)(s)\right\|_4^{16} \|(\nabla u, \nabla B)(s)\|_2^2 \, ds \tag{55}$$

Thus, in case (27) holds, we apply Gronwall inequality to obtain

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \nu \int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds \le C\, (1 + \|(\nabla u(0), \nabla B(0))\|_2^2) e^{C\, M}$$

for all $t \in [0, T^*)$.

Thus, we have proved that if the condition (32) holds then the H^1 norm of the solution (u, B) is bounded. This completes the proof for first case.

Now, we complete the proof when (u_3, B) satisfies (28).

Using (42)-(45) and (19)-(22) in (18), we obtain

$$\frac{d}{dt}(\|(\nabla u, \nabla B)\|_2^2) + \frac{\nu}{2}(\|(\Delta u, \Delta B)\|_2^2) \le$$

$$C\|(\nabla_h u, \nabla_h B)\|_2 \|(\nabla u, \nabla B)\|_2^{1/2} \|(\nabla_h \nabla u, \nabla_h \nabla B)\|_2 \|(\Delta u, \Delta B)\|_2^{1/2} +$$

$$C\, \|(u_3, B)\|_2^6 \left\|(\partial_{x_3} u_3, \partial_{x_3} B)\right\|_4^4 \|(\nabla u, \nabla B)\|_2^2 \tag{56}$$

Integrating (56) and using Holder inequality, we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{2}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2 \, ds \le \|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C\, (\sup\|(\nabla_h u, \nabla_h B)(s)\|_2) \left(\int_0^t \|(\nabla u, \nabla B)(s)\|_2^2 \, ds\right)^{1/4} \left(\int_0^t \|(\nabla_h \nabla u, \nabla_h \nabla B)(s)\|_2^2 \, ds\right)^{1/2}$$

$$\left(\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds\right)^{1/4} +$$

$$C\,(\sup\|(u,B)(s)\|_2^6)\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right) \tag{57}$$

Now, using (34) and (47), we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{2}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds \le \|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C\,K_1^{1/4}\Bigg[\,\|(\nabla_h u_0, \nabla_h B_0)\|_2^2 +$$

$$C\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^4\, \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right)\left(\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds\right)^{\frac{1}{4}}\Bigg] +$$

$$C\,K_1^3\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right)$$

Using Young's and Holder inequalities, we obtain

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{4}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds \le C\,\|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^{\frac{16}{3}}\, \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right)\left(\int_0^t \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right)^{\frac{1}{4}} +$$

$$C\,K_1^3\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^4 \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right) \tag{58}$$

Using (34) again, we get

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \frac{\nu}{4}\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds \le C\,\|(\nabla u_0, \nabla B_0)\|_2^2 +$$

$$C\,\left(\int_0^t \left\|(\partial_{x_3}u_3, \partial_{x_3}B)(s)\right\|_4^{\frac{16}{3}}\, \|(\nabla u, \nabla B)(s)\|_2^2\, ds\right) \tag{59}$$

Thus, in case (28) holds, we apply Gronwall inequality to obtain

$$\|(\nabla u, \nabla B)(t)\|_2^2 + \nu\int_0^t \|(\Delta u, \Delta B)(s)\|_2^2\, ds \le C\,(1 + \|(\nabla u(0), \nabla B(0))\|_2^2)e^{C\,M}$$

for all $t \in [0, T^*)$.

Therefore, the H^1 norm of the strong solution (u, B) is bounded on the maximal interval of existence $[0, T^*)$.

This completes the proof of the Theorem 2.

CONCLUDING REMARKS

In this paper, the spaces used to prove the Serrin-type regularity in section 2 are anisotropic Lebesgue spaces. The norm in the inequality (9) defined on this space shows that the solution $(u, B) \in H^1 \times H^1$ with respect to the space coordinates. Similarly, the global regularity in section 3 is also proved for the H^1 norm.

The condition assumed to prove Serrin-type regularity was condition (5) in section 2 , whereas the sufficient conditions assumed to prove global regularity were the conditions (32), (33) in section 3. These conditions are independent of each other, and hence these two regularity results are also independent of each other. As mentioned in the Introduction, there are many regularity results that are available for solutions of incompressible MHD equations, but none of them provide the general proof of global regularity. We also note that in all the regularity proofs, available so far, sufficient conditions assumed are only on velocity and its gradient or pressure. Magnetic field plays no significant role in these proofs ! However, as in the case of three dimensional Navier-Stokes equations, proof of general global regularity still remains an open issue.

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