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Second Hankel determinant obtained with New Integral Operator defined by Polylogarithm Function

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Abstract

By using polylogarithm function, a new integral operator is introduced. By using this operator a new subclass of analytic functions are introduced for these classes we obtained sharp upper bounds for functional $|a_2a_4 - a_3^2|$.

AMS subject classification:
Keywords: Integral Operator, Analytic Function, Hankel Determinant, Polylogarithm Function.

1. Introduction

In 1966, Pommerenke stated the qth Hankel determinant for q $\leq$ 1, & n $\leq$ 0 as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \vdots & \cdots & \vdots \\ a_{n+q-1} & \vdots & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.1}$$

where a$_{n'}$s are the coefficients of various power of z in f(z).

This determinant has also been considered by several authors. For example Noor determined the rate of growth of $H_q(n)$ an $n \to \infty$ for function f, with bounded boundary.

One can easily observe that Fekete and Szegö functional $H_2(1)$. Fekete and Szegö then further generalized and estimate $|a_3 - \mu a_2^2|$ where μ is real & $f \in \mathcal{S}$.

We consider the Hankel determinant for the case q = 2 and n = 2,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2| \tag{1.2}$$

We recall here the definition of well known generalization of the polylogarithm function $\phi(a, z)$ given by

$$\phi(a, z) = \sum_{k=1}^{\infty} \frac{z^k}{k^a} \qquad (a \in \mathbb{N}, z \in \mathbb{E}) \tag{1.3}$$

Let $\phi_\delta(a, z)$ denote the well known generalization of the Riemann Zeta & polylogarithm function or simply the δ^{th} order polylogarithm function given by,

$$\phi_\delta(a, z) = \sum_{k=1}^{\infty} \frac{z^k}{(k + a)^\delta} \tag{1.4}$$

where any term with k+a = 0 is excluded.

Using the definition of the Gamma function, a simply transformation produces the integral formula,

$$\phi_\delta(a, z) = \frac{1}{\Gamma(\delta)} \int_0^1 z\left(\log \frac{1}{t} \right)^{\delta - 1} \frac{t^a}{1 - tz} \delta t \qquad \Re a > -1, \Re \delta > 1 \tag{1.5}$$

Note that $\phi_{-1}(0, z) = \dfrac{z}{(1 - z)^2}$ is koebe function for more details about polylogarithms in theory of univalent functions see Punnusamy & Sabapathy . Recently Khalifa Alshaqsi introduced a certain Integral Operator I_a^δ defined by,

$$I_a^\delta f(z) = \frac{(1 + a)^\delta}{\Gamma(\delta)} \int_0^1 t^{a-1} \left(\log \frac{1}{t} \right)^{\delta - 1} f(tz) \delta t \qquad a > 0, \delta > 1.z \in \mathbb{E} \tag{1.6}$$

We also note that the operator $I_a^\delta f(z)$ defined by [1] can be expressed by the series expansion as follows,

$$I_a^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + a}{k + a} \right)^\delta a_k z^k \tag{1.7}$$

obviously, we have for $(\delta, \lambda \leq 0)$

$$I_a^\delta (I_a^\lambda f(z)) = I_a^{\delta + \lambda} f(z) \tag{1.8}$$

and

$$I_a^\delta(zf'(z)) = z(I_a^\delta f(z))' \tag{1.9}$$

Moreover from (1.7) if follows that,

$$z(I_a^{\delta+1} f(z)') = (a+1)I_a^\delta f(z) - aI_a^{\delta+1} f(z) \tag{1.10}$$

We note that,

- For a = 0 and δ = n (n is any integer) the multiplier transformation $I_0^n f(z) = I^n f(z)$ was studied by Flett and Salagean.
- For a = 0 and δ = -n (n$\in \mathbb{N}_0\{0, 1, 2, 3, \ldots\}$), the differential operator $I_0^{-n} f(z) = D^n f(z)$ was studied by Salagen.
- For a = 1 and δ = n (n is any integer) the operator $I_1^n f(z) = I^n f(z)$ was studied by Uralegaddi and Somanatha.
- For a = 1, the multiplier transformation $I_1^f f(z) = I^\delta f(z)$ was studied by Jung et al.
- For a = k - 1 (k>1), the integral operator $I_{k-1}^\delta f(z)$, $I_{k-1}^\delta f(z)$ was studied by Komatu using the operator I_a^δ, we now introduced the following cases.

Definition 1.1. We say that a function f $\in$ A is in the class $S_{a\delta}(b)$ if

$$\Re\left\{1 + \frac{1}{b}\left(\frac{z\left(I_a^\delta f(z)\right)'}{I_a^\delta f(z)} - 1\right)\right\} > 0 \qquad (a > 0, \delta \le 0, b \in c\backslash\{0\}; z \in \mathbb{E}) \tag{1.11}$$

Definition 1.2. We say that a function f $\in$ A is in class $C_{a\delta}(b)$ if

$$\Re\left\{1 + \frac{1}{p}\frac{z\left(I_a^\delta f(z)\right)''}{\left(I_a^\delta f(z)\right)'}\right\} > 0 \qquad (a > 0, \delta \le 0, b \in c\backslash\{0\}; z \in \mathbb{E}) \tag{1.12}$$

Note that

$$f \in C_{a\delta}(b) \Leftrightarrow zf' \in S_{a\delta}(b) \tag{1.13}$$

In particular, we have starlike & convex function classes $S_{a0}(1) = S^*$ and $C_{a0}(1)$ = C respective.

Lemma 1.3. Let $p \in \mathbb{P}$ then $|c_k| \le 2, k = 1, 2, \ldots$ and the inequality is sharp.

Lemma 1.4. Let $p \in \mathbb{P}$ then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4-c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2) \end{aligned} \tag{1.14}$$

for some x and y such that $|x| \leq 1, |y| \leq 1$.

Theorem 1.5. If $f \in S_{a\delta}$(b) then

$$|a_2a_4 - a_3^2| \leq b^2\left(\frac{a+3}{a+1}\right)^{2\delta}$$

Proof. By the definition of the class $S_{a\delta}$(b), there exist $p \in P$ such that,

$$\begin{aligned} 1 + \frac{1}{p}\left(\frac{z(I_a^\delta f(z))'}{I_a^\delta f(z)} - 1\right) &= p(z) \\ \therefore \frac{z(I_a^\delta f(z))'}{I_a^\delta f(z)} &= 1 - b + bp(z) \end{aligned} \tag{1.15}$$

Let $\mathrm{I}_a^\delta = z + A_2z^2 + a_3z^2 + \ldots$, where,

$$A_2 = \left(\frac{a+1}{a+2}\right)^\delta a_2$$

$$A_3 = \left(\frac{a+1}{a+2}\right)^\delta a_3$$

$$A_4 = \left(\frac{a+1}{a+2}\right)^\delta a_4$$

so that,

$$\frac{z\{1 + 2A_2z + 3A_3z^2 + 4A_3z^3 + \ldots\}}{z + A_2z^2 + A_3z^3 + \ldots} = 1 - b + b[1 + c_1z + c_2z^2 + c_3z^3] \tag{1.16}$$

Simplify and equating the coefficient of z^2 on both side

$$\begin{aligned} 2A_2 &= A_2 - A_2b + A_b + bc_1 \\ \therefore A_2 &= bc1 \\ A_2 &= \left(\frac{a+2}{a+1}\right)^\delta bc_1 \end{aligned} \tag{1.17}$$

Equating the coefficient of z^3 on both side

$$\begin{aligned} A_3 &= \frac{b}{2}[c_2 + 2bc_1^2] \\ \therefore a_3 &= \left[\frac{b}{2}(c_2 + 2bc_1^2)\left(\frac{a+2}{a+1}\right)^\delta\right] \end{aligned} \tag{1.18}$$

Equating the coefficients of z^4

$$\begin{aligned} 3A_4 &= \frac{b62c_1c_2}{2} + \frac{b^2c_1^3}{2}b^2c_1c_2 + bc_3 \\ \therefore A_4 &= \frac{3b^2c_1c_2 + b^2c_1^3 + 2bc_3}{6} \\ \therefore a_4 &= \left(\frac{a+4}{a+1}\right)^{\delta}\frac{(3b^2c_1c_2 + b^2c_1^3 + 2bc_3)}{6} \end{aligned} \tag{1.19}$$

It is establish that,

$$\begin{aligned} |a_2a_4 - a_3^2| = &\left|\left(\frac{a+2}{a+1}\right)^{\delta}\left(\frac{a+4}{a+1}\right)^{\delta}\frac{(3b^2c_1^2c_2 + b^2c_1^4 + 2bc_1c_3)}{6} - \right. \\ &\left.\left(\left(\frac{a+3^{\delta}}{a+1}\right)\frac{b}{2}(c_2 + bc_1^2)\right)^2\right| \end{aligned} \tag{1.20}$$

By using Lemma,

$$\begin{aligned} c_2 &= \frac{c_1^2 + x(4-c_1^2)}{2} \qquad for \quad some \quad |x| \le 1 \\ 4c_3 &= c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \\ &\quad \text{for some real value of } z \ with \ |z| \le 1 \end{aligned} \tag{1.21}$$

$$\begin{aligned} |a_2a_4 - a_3^2| = &\left|\left(\frac{a+2}{a+1}\right)^{\delta}\left(\frac{a+4}{a+1}\right)^{\delta}\frac{1}{6}\left[3b^2c_1^2\left(\frac{c_1^2 + x(4-c_1^2)}{2}\right) + b^2c_1^4 + 2bc_1\right]\right. \\ &\left[\frac{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2z)}{4}\right] - \\ &\left.\left\{\left(\frac{a+3}{a+1}\right)^{\delta}\frac{b}{2}\left[\frac{c_1^2 + x(4-c_1^2)}{2} + bc_1^2\right]\right\}^2\right| \end{aligned} \tag{1.22}$$

Since $|c_1| \le 2$, $c_1 = c$ assume without restriction, $c \in [0, 2]$, we obtain by using triangle inequality $|x| \le 1 = \rho$. $c_1 = c$.

$$\begin{aligned} |a_2a_4 - a_3^2| \le &\left[\left\{\left(\frac{a+2}{a+1}\right)^{\delta}\left(\frac{a+4}{a+1}\right)^{\delta}\frac{1}{6}\left[\frac{3b^2c^4 + 3b^2c^2(4-c^2)\rho}{2}\right] + b^2c^4 + \frac{2b}{c}\right.\right. \\ &\left.\left.\left[c^3 + 2c(4-c^2)\rho - \rho^2(4-c^2)\rho\right] + \left(\frac{a+3}{a+1}\right)^{2\delta}\frac{b^2}{4}\left[\frac{\rho(4-c^2)}{2}\right]^2\right\}\right] \\ &\le F(\rho) \end{aligned} \tag{1.23}$$

$$F'(\rho)=\left\{\left(\frac{a+2}{a+1}\right)^{\delta}\left(\frac{a+4}{a+1}\right)^{\delta}\frac{1}{6}\left(\frac{3b^2c^2(4-c^2)}{2}\right)+\frac{bc}{2}\left[2c(4-c^2)-2\rho(4-c^2)\rho\right]\right.$$
$$\left.+\left(\frac{a+3}{a+1}\right)^{2\delta}\frac{b^2}{4}\left[\frac{\rho(4-c^2)}{2}\right]^2\right)\Bigg\} \tag{1.24}$$

$F'(\rho) > 0$ for $\rho > 0$, implies that F is an increasing function. The upper bound for (5) are $c = 0$.

$$|a_2a_4 - a_3^2| \leq b^2\left(\frac{a+3}{a+1}\right)^{2\delta}$$

■

Corollary 1.6.

$$S_{a0}(1) = S^*$$

If we put $\delta = 0$, $b = 1$ we get,

$$|a_2a_4 - a_3^2| \leq 1$$

this result is coincide the result with Janteng.

Corollary 1.7. If f $\in c_{a\delta}$ (b) then,

$$\begin{aligned} 1+\frac{1}{b}\left(\frac{z(I_a^{\delta} f(z)'')}{(I_a^{\delta} f(z)')}\right) &= p(z) \\ \therefore \frac{z(I_a^{\delta} f(z)'')}{(I_a^{\delta} f(z)')} &= bp(z) - b \end{aligned} \tag{1.25}$$

Simplify & equating the coefficients we get,

$$\begin{aligned} a_2 &= \frac{bc_1}{2}\left(\frac{a+2}{a+1}\right)^{\delta} \\ a_3 &= \left(\frac{a+3}{a+1}\right)^{\delta}\left[\frac{b^2c_1^2+bc_2}{6}\right] \\ a_4 &= \left(\frac{a+4}{a+1}\right)^{\delta}\left[\frac{3b^2c_1c_2+b^2c_1^3+2bc_3}{24}\right] \end{aligned} \tag{1.26}$$

and calculate the same $|a_2a_4 - a_3^2|$ then we get the result of convex function. After simplification we put $\delta = 0$, $b = 1$ then we get $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ results is coincide the result with Janteng. ■

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