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An Indefinite Kenmotsu Manifold Endowed With Quarter Symmetric Metric Connection[1]

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Abstract

The object of the present paper is to introduce an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection. In this paper, we study the existence of such connection and provide an example. We have established relation between Levi-Civita connection and the quarter symmetric metric connection on an indefinite Kenmotsu manifold. We also deduce curvature tensor and it's properties for such manifolds. We characterize locally ϕ-symmetric and ϕ-symmetric indefinite Kenmotsu manifold with respect to a quarter symmetric metric connection. Finally, we have proved that an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$ is a η-Einstein manifold.

AMS subject classification: 53C15, 53C25.
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[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Introduction

In 1972, K. Kenmotsu [11] introduced a new class of almost contact Riemannian manifold which are known nowadays as Kenmotsu manifold. Kenmotsu manifolds have been studied by many authors viz. Jun, De. and Pathak [25], G. Pitis [6], and C. Ozgur [3]. De. and Sarkar introduced and studied an indefinite Kenmotsu manifolds in [24]. Duggal and Sharma [10] studied a semi-symmetric metric connection in a semi-Riemannian manifold. They studied many properties of the Ricci tensor, affine conformal motions, geodesic and group manifolds with respect to the semi-symmetric metric connection. Many other authors also obtained interesting properties in [20], [9], [7], [12], [13]. In 1975, S. Golab [16] initiated the study of quarter symmetric metric connection on a differentiable manifold. In 1980, R.S. Mishra and S.N. Pandey [14] studied quarter symmetric connection and a particular Ricci quarter symmetric metric connection on a Riemannian, Sasakian and Kaelerian manifolds. A quarter symmetric metric connection is a Haydon connection with the torsion tensor of the form $(1, 2)$. The notion of locally symmetry of Riemannian manifold have been weakened by many authors in several ways to different extent. As a weaker version of local symmetry, T. Takahashi [21] introduced the notion of locally ϕ-symmetry on Sasakian manifolds. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke [5] with some examples. Quarter symmetric metric connection plays an important role in the study of Riemannian and semi-Riemannian manifold. There are various physical problems involving the quarter symmetric metric connection. Generalizing the notion of locally ϕ-symmetric Sasakian manifolds, De., Shaikh and Biswas [15] introduced the notion of ϕ-recurrent Sasakian manifolds. De. [22] introduced and studied ϕ-symmetric Kenmotsu manifolds and in [23] De., Yildiz and Yaliniz introduced and studied ϕ-recurrent Kenmotsu manifolds. Shaikh and Hui studied locally ϕ-symmetric β-Kenmotsu manifolds [17] and extended generalized ϕ-recurrent β-Kenmotsu manifolds [18] respectively. Also, in [1] Prakash studied concircularly ϕ-recurrent Kenmotsu manifolds. In [19] Shukla and Shukla studied ϕ-Ricci symmetric Kenmotsu manifolds. Quarter symmetric metric connection have been studied by many author in several ways to different extent such as [8], [26], [2], [4].

Motivated by the above studies, we obtain some interesting properties of an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection and provide an example of such manifold. In present paper, first we give some basic results and definition of indefinite Kenmotsu manifold which is needed for further work. We deduce the existence of quarter symmetric metric connection and find the relation between Levi-Civita connection and quarter symmetric metric connection on indefinite Kenmotsu manifold. We obtain relation between curvature tensor of Levi-Civita connection and quarter symmetric metric connection for such manifolds. Also, we study locally ϕ-symmetric and ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection. Finally, we prove that an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfying $\overline{P}.\overline{S} = 0$ is a η-Einstein manifold.

2. Preliminaries

An odd n dimensional smooth manifold (M, g) is called an indefinite almost contact metric manifold if there is an indefinite almost contact structure (M, ϕ, ξ, η) consisting of a (1, 1) tensor field ϕ, a structure vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1,\ \phi\xi = 0,\ \ \eta \circ \phi = 0\,. \tag{2.2}$$

Let g be a compatible indefinite metric with (M, ϕ, ξ, η) that is

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.3}$$

or equivalently

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } \eta(X) = \epsilon g(X, \xi), \tag{2.4}$$

for all vector fields $X, Y \in \Gamma TM$. Where ϵ is $+1$ or -1 according to which is ξ is space like or time like vector field and rank of ϕ is $(n-1)$. if

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.5}$$

$\forall X, Y \in \Gamma TM$. Then, we say that $(M, \phi, \xi, \eta, g, \epsilon)$ is an indefinite almost contact metric manifold.

If an indefinite almost contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X, \tag{2.6}$$

where ∇ denotes the Levi-Civita connection with respect to g, then $(M, \phi, \xi, \eta, g, \epsilon)$ is called an indefinite Kenmotsu manifold [6]. An indefinite almost contact metric manifold is an indefinite Kenmotsu iff

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi). \tag{2.7}$$

Moreover, the curvature R, the Ricci tensor S and the Ricci operator Q in an indefinite Kenmotsu manifold $(M, \phi, \xi, \eta, g, \epsilon)$ with respect to the Levi-Civita connection satisfies [6] the following:

$$(\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi, \tag{2.10}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.11}$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Y)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.12}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.13}$$

$$Q\xi = -\epsilon(n-1)\xi, \tag{2.14}$$

where $g(QX, Y) = S(X, Y)$. It yields

$$S(\phi X, \phi Y) = S(X, Y) + \epsilon(n-1)\eta(X)\eta(Y). \tag{2.15}$$

We note that if $\epsilon = 1$ and the structure vector field ξ is space like, then an indefinite Kenmotsu manifold is usual Kenmotsu manifold.

Definition 2.1. An indefinite Kenmotsu manifold M is said to be η-Einstein manifold if its Ricci tensor S of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.16}$$

where a and b are scalar function of ϵ.

Definition 2.2. An indefinite Kenmotsu manifold M is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_U R)(X, Y)Z = 0, \tag{2.17}$$

for all vector fields $X,\ Y,\ Z,\ U$ orthogonal to ξ.

Definition 2.3. An indefinite Kenmotsu manifold M is said to be ϕ-symmetric if

$$\phi^2(\nabla_U R)(X, Y)Z = 0, \tag{2.18}$$

for arbitrary vector fields $X,\ Y,\ Z,\ U$.

A linear connection $\bar{\nabla}$ in M is called quarter symmetric connection if its torsion tensor

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y], \tag{2.19}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{2.20}$$

Further a quarter symmetric connection is said to be quarter symmetric metric connection if

$$(\bar{\nabla}_X g)(Y, Z) = 0. \tag{2.21}$$

3. Existence of quarter symmetric metric connection on an indefinite Kenmotsu manifold

Let X and Y be any two vector fields on $(M, \phi, \xi, \eta, g, \epsilon)$. Let we define a connection $\bar{\nabla}_X Y$ by the following equation

$$\begin{aligned} 2g(\bar{\nabla}_X Y, Z) &= Xg(Y, Z) - Yg(Z, X) + Zg(X, Y) + g([X, Y], Z) \\ &\quad -g([Y, Z], X) - g([Z, X], Y) + g(\eta(Y)\phi X - \eta(X)\phi Y, Z) \\ &\quad +g(\eta(Y)\phi Z - \eta(Z)\phi Y, X) + g(\eta(X)\phi Z - \eta(Z)\phi X, Y), \end{aligned} \tag{3.1}$$

which holds $\forall X, Y, Z \in \Gamma TM$. The mapping $(X, Y) \to \bar{\nabla}_X Y$ satisfies the following conditions:

$$\begin{aligned} \bar{\nabla}_X (Y + Z) &= \bar{\nabla}_X Y + \bar{\nabla}_X Z, && (3.2) \\ \bar{\nabla}_{(X+Y)} Z &= \bar{\nabla}_X Z + \bar{\nabla}_Y Z, && (3.3) \\ \bar{\nabla}_{fX} Y &= f\bar{\nabla}_X Y, && (3.4) \\ \bar{\nabla}_X fY &= f\bar{\nabla}_X Y + (Xf)Y, && (3.5) \end{aligned}$$

$\forall X, Y, Z \in \Gamma TM$ and $f \in F(M)$, the set of all differentiable mapping over M.

From equations (3.2), (3.3), (3.4) and (3.5), we can conclude that $\bar{\nabla}$ determine a linear connection on $(M, \phi, \xi, \eta, g, \epsilon)$.

Now, using equation (3.1), we get

$$2g(\bar{\nabla}_X Y, Z) - 2g(\bar{\nabla}_Y X, Z) = 2g([X, Y], Z) + 2g(\eta(Y)\phi X - \eta(X)\phi Y, Z). \quad (3.6)$$

Hence

$$\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y. \quad (3.7)$$

Or

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \quad (3.8)$$

Also, we obtain

$$2g(\bar{\nabla}_X Y, Z) + 2g(\bar{\nabla}_X Z, Y) = 2Xg(Y, Z). \quad (3.9)$$

Or

$$(\bar{\nabla}_X g)(Y, Z) = 0. \quad (3.10)$$

That is

$$\bar{\nabla} g = 0. \quad (3.11)$$

From equations (3.8) and (3.11). It is shown that $\bar{\nabla}$ determines a quarter symmetric metric connection on $(M, \phi, \xi, \eta, g, \epsilon)$. It can be easily found that $\overline{\nabla}$ determines a unique quarter symmetric metric connection on $(M, \phi, \xi, \eta, g, \epsilon)$.

Theorem 3.1. Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an indefinite Kenmotsu manifold and η be a 1-form on it . Then there exist a unique linear connection $\bar{\nabla}$ satisfying

$$\begin{aligned} T(X, Y) &= \eta(Y)\phi X - \eta(X)\phi Y, \\ \bar{\nabla} g &= 0. \end{aligned}$$

The above theorem proves that the existence of quarter symmetric metric connection on an indefinite Kenmotsu manifold.

4. Relation between the Levi-Civita connection and the quarter symmetric metric connection on an indefinite Kenmotsu manifold

Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold and ∇ be the Levi-Civita connection on M. A quarter symmetric metric connection $\bar{\nabla}$ on an indefinite Kenmotsu manifold is

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{4.1}$$

where H is a tensor of type $(1, 2)$ such that

$$H(X, Y) = \frac{1}{2}\{T(X, Y) + T'(X, Y) + T'(Y, X)\}, \tag{4.2}$$

and

$$g(T'(X, Y), Z) = g(T(Z, X), Y), \tag{4.3}$$

From equations (3.10) and (4.3), we have

$$T'(X, Y) = -\epsilon g(\phi X, Y)\xi - \eta(X)\phi Y. \tag{4.2}$$

Using equations (3.10) , (4.4) and (4.2), we get

$$H(X, Y) = -\eta(X)\phi Y.$$

Hence, a quarter symmetric metric connection $\bar{\nabla}$ in an indefinite Kenmotsu manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{4.3}$$

Conversely, we show that a quarter symmetric linear connection $\bar{\nabla}$ on an indefinite Kenmotsu manifold defined by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{4.4}$$

Using (4.6), the torsion tensor of a quarter symmetric linear connection $\bar{\nabla}$ on an indefinite Kenmotsu manifold defined by

$$\begin{aligned} T(X, Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y. \end{aligned} \tag{4.5}$$

The above equation shows that the $\bar{\nabla}$ is a quarter symmetric connection on an indefinite Kenmotsu manifold. Also, we have

$$\begin{aligned} (\bar{\nabla}_X g)(Y, Z) &= Xg(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \\ &= \eta(X)\{g(\phi Y, Z) + g(\phi Z, Y)\} \\ &= 0. \end{aligned} \tag{4.6}$$

In virtue of equations (4.7) and (4.8), we deduce that $\bar{\nabla}$ is a quarter symmetric metric connection.

Therefore equation (4.5) is the relation between the Levi-Civita connection and the quarter symmetric metric connection on an indefinite Kenmotsu manifold.

5. Curvature tensor on an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection

Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold. Also, let ∇ be the Levi-Civita connection and $\bar{\nabla}$ be a quarter symmetric metric connection. The curvature tensor $\bar{R}$ of $(M, \phi, \xi, \eta, g, \epsilon)$ endowed with quarter symmetric symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{5.1}$$

By equation (4.5) and (5.1), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \{(\nabla_X \eta)(Z)Y - (\nabla_Y \eta)(Z)X\} \\ &\quad + \{g(X, Z)\nabla_Y \xi - g(Y, Z)\nabla_X \xi\} + \eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &\quad + \{g(X, Z)Y - g(Y, Z)X\} + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi. \end{aligned} \tag{5.2}$$

Now, using equation (2.6) , (2.7) and (2.8), we have

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - 2d\eta(X, Y)\phi Z + \{g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)\}\xi \\ &\quad + \epsilon\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z), \end{aligned} \tag{5.3}$$

where R defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

is the curvature tensor of Levi-Civita connection ∇.

Lemma 5.1. Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold endowed with quarter symmetric metric connection, then

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X. \tag{5.4}$$

$$\bar{\nabla}_X \xi = \nabla_X \xi = \epsilon(X - \eta(X)\xi) = -\epsilon\phi^2 X. \tag{5.5}$$

$$(\bar{\nabla}_X \eta)Y = (\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y). \tag{5.6}$$

Proof. Differentiating covariantly (ϕY) with respect to X, we have

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X(\phi Y) + \phi(\bar{\nabla}_X Y).$$

Using equations (2.6) and (4.5), we get

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y, \tag{5.7}$$

so by equation (5.7), we obtain

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi X. \tag{5.8}$$

Hence, we get equation (5.4).

To prove (5.5). Taking $Y = \xi$ in (4.5), we get

$$\bar{\nabla}_X \xi = \nabla_X \xi - \eta(X)\phi\xi. \tag{5.9}$$

Using equations (2.7) in (5.9), we have

$$\bar{\nabla}_X \xi = \nabla_X \xi = -\epsilon(X - \eta(X)\xi) = -\epsilon\phi^2 X. \tag{5.10}$$

Hence, we get equation (5.5).

To prove equation (5.6), differentiating covariantly (ηY) with respect to X, we have

$$(\bar{\nabla}_X \eta)Y = \bar{\nabla}_X \eta Y + \eta(\bar{\nabla}_X Y). \tag{5.11}$$

Using equation (4.5), we obtain

$$(\bar{\nabla}_X \eta)Y = (\nabla_X \eta)(Y). \tag{5.12}$$

Using equations (2.8) and (5.12), we have

$$(\bar{\nabla}_X \eta)Y = (\nabla_X \eta)(Y) = g(X, Y) - \epsilon\eta(X)\eta(Y). \tag{5.13}$$

Hence, we get equation (5.6). ■

Lemma 5.2. Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold endowed with quarter symmetric metric connection, then

$$\bar{R}(X, Y)\xi = \{\eta(X)Y - \eta(Y)X\} + \epsilon\{\eta(Y)\phi X - \eta(X)\phi Y\}. \tag{5.14}$$

$$\bar{R}(\xi, X)Y = \{g(\phi X, Y)\xi + \eta(Y)X\} - \epsilon\{g(X, Y)\xi + \eta(Y)(\phi X)\}. \tag{5.15}$$

$$\bar{R}(\xi, X)\xi = \epsilon(X - \eta(X)\xi) - \epsilon(\phi X) = -\epsilon\{\phi^2 X + (\phi X)\}. \tag{5.16}$$

Proof. To prove equation (5.14), replace $Z = \xi$ in equation (5.3), we have

$$\begin{aligned}\bar{R}(X,Y)\xi &= R(X,Y)\xi - 2d\eta(X,Y)\phi\xi + \{g(\phi Y,\xi)\eta(X) \\ &\quad - g(\phi X,\xi)\eta(Y)\}\xi + \epsilon\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(\xi).\end{aligned} \tag{5.17}$$

Using equations (2.2), (2.5) and (5.17), we get

$$\bar{R}(X,Y)\xi = R(X,Y)\xi + \epsilon\{\eta(Y)\phi X - \eta(X)\phi Y\}. \tag{5.18}$$

In view of equations (2.9) and (5.18), we have

$$\bar{R}(X,Y)\xi = \{\eta(X)Y - \eta(Y)X\} + \epsilon\{\eta(Y)\phi X - \eta(X)\phi Y\}. \tag{5.19}$$

Hence, we obtain equation (5.14).

To prove equation (5.15), replace $X = \xi$, $Y = X$, $Z = Y$ in equation (5.3), we get

$$\begin{aligned}\bar{R}(\xi,X)Y &= R(\xi,X)Y - 2d\eta(\xi,X)\phi Y + \{g(\phi X,Y)\eta(\xi) \\ &\quad - g(\phi\xi,Y)\eta(X)\}\xi + \epsilon\{\eta(X)\phi\xi - \eta(\xi)\phi X\}\eta(Y).\end{aligned} \tag{5.20}$$

Using equations (2.2) and (5.20), we get

$$\bar{R}(\xi,X)Y = R(\xi,X)Y + g(\phi X,Y)\xi - \epsilon\eta(Y)(\phi X). \tag{5.21}$$

In view of equations (2.11) and (5.21), we get

$$\bar{R}(\xi,X)Y = \{g(\phi X,Y)\xi + \eta(Y)X\} - \epsilon\{g(X,Y)\xi + \eta(Y)(\phi X)\}. \tag{5.22}$$

Hence, we obtain equation (5.15).

To prove equation (5.16), replace $X = \xi$, $Y = X$, $Z = \xi$ and using equations (2.2), (2.5) and (5.3), we have

$$\bar{R}(\xi,X)\xi = R(\xi,X)\xi - \epsilon(\phi X). \tag{5.23}$$

In view of equations (2.11) and (5.23), we get

$$\bar{R}(\xi,X)\xi = \epsilon(X - \eta(X)\xi) - \epsilon(\phi X) = -\epsilon\{\phi^2 X + (\phi X)\}. \tag{5.24}$$

Hence, we get equation (5.16). ■

Now, contracting X in equation (5.3), we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z,Y) + \epsilon g(\phi Y,Z). \tag{5.25}$$

Where $\overline{S}$ and S are respectively the Ricci tensor on an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection and Levi-Civita connection. From equation (5.25), it follows that the Ricci tensor endowed with quarter symmetric metric connection is not symmetric.

Also, from (5.25), we get

$$\overline{S}(Y, Z) = S(Y, Z) - 2g(Y, Z) + 2\epsilon\eta(Z)\eta(Y) + \epsilon g(\phi Y, Z), \tag{5.26}$$

where $\overline{S}(Y, Z)$ and $S(Y, Z)$ are respectively the curvature endowed with quarter symmetric metric connection and Levi-Civita connection. From (5.26), it follows that the Ricci tensor endowed with quarter symmetric metric connection is not invariant.

By equation (5.25) and using the fact $S(X, Y) = g(QX, Y)$ and $\overline{S}(X, Y) = g(\overline{Q}X, Y)$. We get

$$\overline{Q}Y = QY + 2\phi^2 Y + \epsilon\phi Y, \tag{5.27}$$

where Q and $\overline{Q}$ are respectively Ricci operators with respect to Levi-Civita connection and quarter symmetric metric connection on an indefinite Kenmotsu manifold.

Lemma 5.3. Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold endowed with quarter symmetric metric connection, then

$$\overline{S}(X, \xi) = S(X, \xi) = -(n-1)\eta(X). \tag{5.28}$$

$$\overline{Q}\xi = Q\xi = -\epsilon(n-1)\xi. \tag{5.29}$$

$$\begin{aligned}\overline{S}(\phi X, \phi Y) &= S(X, Y) + \epsilon g(\phi X, Y) - 2g(X, Y) \\ &\quad + \epsilon(n+3)\eta(X)\eta(Y).\end{aligned} \tag{5.30}$$

Proof. Using equation (5.25), and taking $X = Y$, $Y = \xi$. We get

$$\overline{S}(X, \xi) = S(X, \xi) - 2d\eta(\phi\xi, X) + \epsilon g(\phi X, \xi). \tag{5.31}$$

In view of equations (2.2), (2.5) and (2.13), we have

$$\overline{S}(X, \xi) = S(X, \xi) = -(n-1)\eta(X). \tag{5.32}$$

Hence, we get equation (5.28).

To prove equation (5.29), replace $Y = \xi$ in (5.27), we get

$$\overline{Q}\xi = Q\xi + 2\phi^2\xi + \epsilon\phi\xi. \tag{5.33}$$

Using equations (2.2) and (2.14), we have

$$\bar{Q}\xi = Q\xi = -\epsilon(n-1)\xi. \tag{5.34}$$

Hence, we get equation (5.29).

To prove equation (5.30), replace $Y = \phi X$ and $Z = \phi Y$ in equation (5.25), we get

$$\bar{S}(\phi X, \phi Y) = S(\phi X, \phi Y) - 2d\eta(\phi^2 Y, \phi X) + \epsilon g(\phi^2 X, \phi Y). \tag{5.35}$$

Using equations (2.1), (2.2), (2.3), (2.4), (2.5) and (2.15), we have

$$\begin{aligned}\bar{S}(\phi X, \phi Y) &= S(X, Y) + \epsilon g(\phi X, Y) - 2g(X, Y) \\ &\quad + \epsilon(n+3)\eta(X)\eta(Y).\end{aligned} \tag{5.36}$$

Hence, we get equation (5.30). ■

6. Locally ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection

In this section, we obtain the result for locally ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection.

In view of the definition of locally ϕ-symmetric indefinite Kenmotsu manifold with respect to Levi-Civita connection. We define locally ϕ-symmetric an indefinite Kenmotsu manifold with respect to quarter symmetric metric connection $\bar{\nabla}$ as

$$\phi^2((\bar{\nabla}_U \bar{R})(X, Y)Z) = 0, \tag{6.1}$$

for all vector fields X, Y, Z, U orthogonal to ξ.

Using equation (4.5), we get

$$(\bar{\nabla}_U \bar{R})(X, Y)Z = (\nabla_U \bar{R})(X, Y)Z - \eta(U)\phi\bar{R}(X, Y)Z. \tag{6.2}$$

Now, differentiating covariantly (5.3) with respect to U. We have

$$\begin{aligned}(\nabla_U \bar{R})(X, Y)Z &= (\nabla_U R)(X, Y)Z - 2d\eta(X, Y)(\nabla_U \phi)Z + \{(\nabla_U \eta) \\ &\quad (X)g(\phi Y, Z) - (\nabla_U \eta)(X)g(\phi X, Z)\}\xi + \{g(\phi Y, Z) \\ &\quad \eta(X) - g(\phi X, Z)\eta(Y)\}(\nabla_U \xi) + \epsilon[\{\eta(Y)\phi X - \eta(X) \\ &\quad \phi Y\}(\nabla_U \eta)(Z) + (\nabla_U \eta)(Y)\phi X + \eta(Y)(\nabla_U \phi)(X) - \\ &\quad (\nabla_U \eta)(X)\phi Y - \eta(X)(\nabla_U \phi)(Y)]\eta(Z).\end{aligned} \tag{6.3}$$

Using equations (2.2), (2.4), (2.6), (2.7), (2.8) and (6.3), we get

$$\begin{aligned}(\bar{\nabla}_U \bar{R})(X, Y)Z &= (\nabla_U R)(X, Y)Z - 2d\eta(X, Y)\{g(\phi U, Z)\xi - \epsilon\eta(Z)\phi U\} \\ &+\{g(X, U)g(\phi Y, Z) - g(Y, U)g(\phi X, Z)\}\xi - 2\epsilon\{\eta(X) \\ &\eta(U)g(\phi Y, Z) - \eta(Y)\eta(U)g(\phi X, Z)\}\xi + \{\eta(X)g(\phi Y, Z) \\ &-\eta(Y)g(\phi X, Z)\}(W) + \epsilon\{g(U, Z) - \epsilon\eta(U)\eta(Z)\} \times \\ &\{\eta(Y)\phi X - \eta(X)\phi Y\} + \epsilon\{g(Y, U)\phi X - g(X, U)\phi Y + \\ &g(\phi U, X)\eta(Y)\xi - g(\phi U, Y)\eta(X)\xi - \epsilon\eta(Y)\eta(U)\phi X - \\ &\epsilon\eta(X)\eta(U)\phi Y - 2\epsilon\eta(X)\eta(Y)\phi W\}\eta(Z). \end{aligned} \tag{6.4}$$

In view of equation (2.2), (6.4) and (6.2), we have

$$\begin{aligned}\phi^2(\bar{\nabla}_U \bar{R})(X, Y)Z &= \phi^2(\nabla_U R)(X, Y)Z + \epsilon 2d\eta(X, Y)\eta(Z)\phi^2(\phi U) + \{\eta(X) \\ &g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\}(\phi^2 W) + \epsilon\{g(U, Z) - \epsilon\eta(U) \\ &\eta(Z)\} \times \{\eta(Y)\phi^2(\phi X) - \eta(X)\phi^2(\phi Y)\} + \epsilon\{g(Y, U) \\ &\phi^2(\phi X) - g(X, U)\phi^2(\phi Y) - \epsilon\eta(Y)\eta(U)\phi^2(\phi X) \\ &-\epsilon\eta(X)\eta(U)\phi^2(\phi Y) - 2\epsilon\eta(X)\eta(Y)\phi^2(\phi W)\}\eta(Z) \\ &-\eta(U)\phi^2(\phi\bar{R})(X, Y)Z. \end{aligned} \tag{6.5}$$

Since, we assume $X,\ Y,\ Z$ and U are orthogonal to ξ, then (6.5) gives

$$\phi^2(\bar{\nabla}_U \bar{R})(X, Y)Z = \phi^2(\nabla_U R)(X, Y)Z. \tag{6.6}$$

Therefore, we can deduce the following:

Theorem 6.1. An n-dimensional indefinite Kenmotsu manifold with respect to a quarter symmetric metric connection $\bar{\nabla}$ is locally ϕ-symmetric if and only if it is locally ϕ-symmetric with respect to Levi-Civita connection.

7. ϕ-Symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection

In this section, we obtain the result for ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection.

In view of the definition of ϕ-symmetric indefinite Kenmotsu manifold with respect to Levi-Civita connection, we define a ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection $\bar{\nabla}$ as

$$\phi^2((\bar{\nabla}_U \bar{R})(X, Y)Z) = 0, \tag{7.1}$$

for arbitrary vector fields $X,\ Y,\ Z,\ U \in \Gamma TM$.

Let us consider a ϕ-symmetric indefinite Kenmotsu manifold with respect to quarter symmetric metric connection. Then by equations (2.1) and (7.1), we get

$$-(\bar{\nabla}_U \bar{R})(X, Y)Z + \eta((\bar{\nabla}_U \bar{R})(X, Y)Z)\xi = 0. \tag{7.2}$$

Taking inner product with vector field W in (7.2), we get

$$-g((\bar{\nabla}_U \bar{R})(X, Y)Z, W) + \eta((\bar{\nabla}_U \bar{R})(X, Y)Z)g(W, \xi) = 0. \tag{7.3}$$

Let $\{e_1,\ e_2, \ldots, e_{n-1,}\ \xi\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = W = e_i$ in equation (7.3) and taking summation over i, $1 \leq i \leq n$. We have

$$-(\bar{\nabla}_U \bar{S})(Y, Z) + \sum \eta((\bar{\nabla}_U \bar{R})(e_i, Y)Z)g(e_i, \xi) = 0. \tag{7.4}$$

Replace $Z = \xi$ in second term of equation (7.4). Then it takes the form

$$\eta((\bar{\nabla}_U \bar{R})(e_i, Y)Z)\eta(e_i) = g((\bar{\nabla}_U \bar{R})(e_i, Y)\xi, \xi)g(e_i, \xi). \tag{7.5}$$

Using equation (4.5), we get

$$g((\bar{\nabla}_U \bar{R})(e_i, Y)\xi, \xi) = g((\nabla_U \bar{R})(e_i, Y)\xi, \xi) - \eta(U)\eta(\bar{R}(e_i, Y)\xi). \tag{7.6}$$

In view of equations (2.2) and (5.3), we have from equation (7.6)

$$g((\bar{\nabla}_U \bar{R})(e_i, Y)\xi, \xi) = g((\nabla_U R)(e_i, Y)\xi, \xi). \tag{7.7}$$

Also, in an indefinite Kenmotsu manifold, we have similar result as [9]

$$g((\nabla_U R)(e_i, Y)\xi, \xi) = 0. \tag{7.8}$$

Using equations (7.8) and (7.7), we can write

$$g((\bar{\nabla}_U \bar{R})(e_i, Y)\xi, \xi) = 0. \tag{7.9}$$

Replace $Z = \xi$ in equation (7.4) and using equation (7.8). Then equation (7.4) takes the form

$$(\bar{\nabla}_U \bar{S})(Y, \xi) = 0. \tag{7.10}$$

Now, we have

$$(\bar{\nabla}_U \bar{S})(Y, \xi) = \bar{\nabla}_U \bar{S}(Y, \xi) - \bar{S}(\bar{\nabla}_U Y, \xi) - \bar{S}(Y, \bar{\nabla}_U \xi). \tag{7.11}$$

Using equations (2.7), (2.13), (4.5) and (5.28) in equation (7.11), we get

$$(\bar{\nabla}_U \bar{S})(Y, \xi) = -\epsilon S(Y, U) - (n-1)g(Y, U) - 2\epsilon d\eta(\phi U, Y) + \epsilon g(\phi Y, U). \quad (7.12)$$

Using equation (7.10) in equation (7.12), we obtain

$$S(Y, U) = -\epsilon(n-1)g(Y, U) - 2d\eta(\phi U, Y) + g(\phi Y, U). \quad (7.13)$$

Contracting equation (7.13), we have

$$r = -\epsilon(n - 1 + \epsilon)(n - 1). \quad (7.14)$$

From above, we can state to the following theorem:

Theorem 7.1. Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an n-dimensional indefinite Kenmotsu manifold endowed with quarter symmetric metric connection $\bar{\nabla}$. Then the manifold has constant negative scalar curvature r with respect to Levi-Civita connection ∇ of M.

8. An n-dimensional indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$

In this section, we obtain the result on an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$.

The projective curvature tensor is an important tensor from the differential point of view. Let M^n be an n-dimensional Riemannian manifold. If there exist a one to one correspondence between each coordinate neighborhoods of M^n and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M^n is said to be locally projectively flat. The projective tensor with respect to quarter symmetric metric connection is defined by

$$\bar{P}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-1)}\{\bar{S}(Y, Z)X - \bar{S}(X, Z)Y\}. \quad (8.1)$$

Assume that an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfies

$$(P(\bar{X}, Y).\bar{S})(Z, U) = 0, \quad (8.2)$$

where $\bar{S}$ is the Ricci tensor with respect to a quarter symmetric metric connection. Then, we get

$$\bar{S}(\bar{P}(X, Y)Z, U) + \bar{S}(Z, \bar{P}(X, Y)U) = 0. \quad (8.3)$$

Set $X = \xi$ in equation (8.3), we get

$$\bar{S}(\bar{P}(\xi, Y)Z, U) + \bar{S}(Z, \bar{P}(\xi, Y)U) = 0. \quad (8.4)$$

In view of equation (8.1), we get

$$\bar{P}(\xi, Y)Z = \bar{R}(\xi, Y)Z - \frac{1}{(n-1)}\{\bar{S}(Y, Z)\xi - \bar{S}(\xi, Z)Y\}. \tag{8.5}$$

Using equation (5.15) in (8.5). We have

$$\begin{aligned}\bar{P}(\xi, Y)Z &= g(\phi Y, Z)\xi - \epsilon g(Y, Z)\xi + \eta(Z)Y - \epsilon(\phi Y)\eta(Z) \\ &\quad - \frac{1}{(n-1)}\{\bar{S}(Y, Z)\xi - \bar{S}(\xi, Z)Y\}.\end{aligned} \tag{8.6}$$

Replace Z by U in equation (8.6), we get

$$\begin{aligned}\bar{P}(\xi, Y)U &= g(\phi Y, U)\xi - \epsilon g(Y, U)\xi + \eta(U)Y - \epsilon(\phi Y)\eta(U) \\ &\quad - \frac{1}{(n-1)}\{\bar{S}(Y, U)\xi - \bar{S}(\xi, U)Y\}.\end{aligned} \tag{8.7}$$

Now, using equations (8.6), (8.7) and taking $U = \xi$ in equation (8.4), we have

$$\bar{S}(Y, Z) - \epsilon\bar{S}(\phi Y, Z) = (n-1)g(\phi Y, Z) - \epsilon(n-1)g(Y, Z). \tag{8.8}$$

Using equations (5.25) and (8.8), we obtain

$$\begin{aligned}S(Y, Z) - \epsilon S(\phi Y, Z) &= (2+\epsilon)g(Y, Z) + (1-\epsilon+n)g(\phi Y, Z) \\ &\quad -(1+\epsilon(n+1))\eta(Y)\eta(Z).\end{aligned} \tag{8.9}$$

Replacing Y by Z and Z by Y with using equations (2.4) and (2.15), we obtain

$$S(Y, Z) = (2+\epsilon)g(Y, Z) - (1+\epsilon(n+1))\eta(Y)\eta(Z), \tag{8.10}$$

which is of the form

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \tag{8.11}$$

where $a = (2+\epsilon)$ and $b = -(1+\epsilon(n+1))$.

Which shows that it is a η-Einstein manifold. Hence, we can state the following theorem:

Theorem 8.1. An indefinite Kenmotsu manifold endowed with quarter symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$,is a η-Einstein manifold.

9. Example of an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection $\bar{\nabla}$

We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are standard coordinate of R^3

The vector field

$$e_1 = \epsilon z \frac{\partial}{\partial x},\ e_2 = \epsilon z \frac{\partial}{\partial y},\ e_3 = -\epsilon z \frac{\partial}{\partial z},$$

are linearly independent at each point of M.

Let g be the indefinite metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_1, e_2) = g(e_2, e_3) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = \epsilon. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z, e_3)$ for any $Z \in \Gamma TM$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \ \phi e_3 = 0$$

Then using the linearity of ϕ and g. We have

$$\phi^2 Z = -Z + \eta(Z)\xi,\ \eta(\xi) = 1,$$

$$g(\phi Z, \phi W) = g(Z, W) - \epsilon\eta(Z)\eta(W).$$

For any $Z, W \in \Gamma TM$. Then for $\xi = e_3$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then, we get

$$[e_1, e_3] = \epsilon e_1,\ \ [e_2, e_3] = \epsilon e_2,\ \ [e_1, e_2] = 0.$$

The connection ∇ of the metric g is given by Koszul' formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{9.1}$$

Using this formula, we have

$$\begin{aligned} \nabla_{e_1} e_3 &= \epsilon e_1,\ \ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_1 = -\epsilon e_3, \\ \nabla_{e_2} e_3 &= \epsilon e_2,\ \ \nabla_{e_2} e_2 = \epsilon e_3,\ \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 &= 0,\quad \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_1 = 0. \end{aligned}$$

From above, it follows that the manifold satisfies $\nabla_X \xi = \epsilon(X - \eta(X)\xi)$, for $\xi = e_3$. Hence, the manifold is an indefinite Kenmotsu manifold.

Using Koszul formula for $\bar{\nabla}$, we get

$$\begin{aligned}
\bar{\nabla}_{e_1}e_3 &= \epsilon e_1,\ \ \bar{\nabla}_{e_1}e_2 = 0,\ \bar{\nabla}_{e_1}e_1 = -\epsilon e_{3,} \\
\bar{\nabla}_{e_2}e_3 &= \epsilon e_2,\ \ \bar{\nabla}_{e_2}e_2 = \epsilon e_3,\ \bar{\nabla}_{e_2}e_1 = 0, \\
\bar{\nabla}_{e_3}e_3 &= 0,\quad \bar{\nabla}_{e_3}e_2 = -\epsilon e_1,\ \bar{\nabla}_{e_3}e_1 = \epsilon e_2.
\end{aligned}$$

We obtain that the manifold satisfies $\bar{\nabla}_X\xi = \epsilon(X - \eta(X)\xi)$, for $\xi = e_3$.
The torsion tensor is defined as

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y \tag{9.2}$$

Using equation (9.2), we get

$$\begin{aligned}
T(e_1,e_1) &= 0,\ T(e_2,e_2) = 0,\ T(e_3,e_3) = 0, \\
T(e_1,e_2) &= 0,\ T(e_2,e_3) = e_1,\ T(e_1,e_3) = -e_2.
\end{aligned}$$

Now, differentiating g covariantly, we have

$$(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \tag{9.3}$$

Taking $X = e_1,\ Y = e_2,\ Z = e_3$ in equation (9.3). We get

$$\begin{aligned}
(\bar{\nabla}_{e_1}g)(e_2,e_3) &= \bar{\nabla}_{e_1}g(e_2,e_3) - g(\bar{\nabla}_{e_1}e_2,\ e_3) - g(e_2, \bar{\nabla}_{e_1}e_3) \\
&= 0 - \epsilon g(0,e_3) - g(e_2,e_1) \\
&= 0 - 0 - 0 = 0
\end{aligned}$$

Similarly, we can get easily

$$\begin{aligned}
(\bar{\nabla}_{e_1}g)(e_3,e_2) &= 0,\ (\bar{\nabla}_{e_2}g)(e_1,e_3) = 0,\ (\bar{\nabla}_{e_2}g)(e_3,e_1) = 0 \\
(\bar{\nabla}_{e_3}g)(e_1,e_2) &= 0,\ (\bar{\nabla}_{e_3}g)(e_2,e_1) = 0,\ (\bar{\nabla}_{e_1}g)(e_1,e_1) = 0 \\
(\bar{\nabla}_{e_2}g)(e_2,e_2) &= 0,\ (\bar{\nabla}_{e_3}g)(e_3,e_3) = 0.
\end{aligned}$$

So, we conclude that for all $X, Y, Z \in \Gamma TM$. We get

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{9.4}$$

Hence, the manifold is an indefinite Kenmotsu manifold endowed with quarter symmetric metric connection. By direct calculation, we get

$$\phi^2((\bar{\nabla}_U \bar{R})(X,Y)Z) = 0 \tag{9.5}$$

for all $X, Y, Z, U \in \Gamma TM$

From above, we see that the theorem (2) is also verified.

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