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An Embedding Theorem for Locales

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Abstract

If X is a topological space,the collection of sets of the form $\{f \in C(X) : f(x) \in V\}$, which depends on both points of X and topology of X,forms sub base for a point-open topology on C(X), where C(X) is the collection of all continuous real-valued functions on X. This situation in the background of point free topology gives rise to new spatial locales from a given non spatial locale L. We claim that for each $a \in L$, the collection $J_a = \{[a, \Sigma_b] : b \in L\}$ is a spatial locale of pseudo subframes of $O(L)$. Defining proper congruences on L and O(L), we have derived an embedding theorem for locale L. Finally the collection $B = \{J_a, a \in L\}$ forms a full subcategory of the category **Loc**. The coproduct $J = \coprod J_a$ satisfies the separation axioms subfit and normal if and only if each J_a is subfit and normal respectively.

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1. Introduction

Among many introductions to topology, a particular view that has arisen in Theoretical Computer Science starts with the theory of domains as defined by Scott and Strachey [8] to provide a mathematical foundation for semantics of programming languages, establishing that domains could be put into a topological setting. Mike Smyth [9] has developed the idea further. The topology describes an essential computational notion that provide them an independence from the points of topological spaces and this has led to the branch of mathematics called locale theory [7].

The work of Marshall Stone on the topological representation of Boolean algebras and distributive lattices describes that topology, in addition to geometrical inputs, has an algebraic (lattice theoretic) content. The fact that a lattice with infinite distributive property could be studied as a ”generalized topological space”, was seen in late fifties with the work of Charles Ehresmann [3]. Subsequently, results from topology were extended to these ”generalized spaces” which were called locales by Hugh Dowker and Dona Papert Strauss. It was John Isbell [5] who proved that the product of generalized spaces were better behaved than Tychenoff product of spaces. The term frame is given to the opposite of category of locales.

If X is a topological space,the collection of sets of the form $\{f \in C(X) : f(x) \in V\}$, which depends on both points of X and topology of X, forms a sub base for a point-open topology on C(X). In the context of point free topology, as a generalization we have defined the collection $[a, \Sigma_b] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_b\}$ which are ideals in $O(L)$,where $O(L)$ is the locale of order preserving maps on a given locale L. Consequently some algebraic properties of these ideals $[a, \Sigma_b]$ have been established. For each $a \in L$,ideals of the form $[a, \Sigma_b]$ generates a spatial locale $J_a = \{[a, \Sigma_b], b \in L\}$. Seperation properties such as subfit, S_2', regularity and normal pertaining to locale J_a have been established. Using the ideals $[a, \Sigma_b]$, a congruence $\sim_a$ is defined in L and we find that $L/\sim_a$ is isomorphic to J_a. For each $a \in L$,congruences R_a is defined in $O(L)$ and it is observed that the quotient locale $O(L)/R_a$ of $O(L)$ is isomorphic to the quotient locale $L/\sim_a$ of L. Using the congruences on L and $O(L)$, we have proved an embedding theorem for locale L. Setting the coproduct $J = \coprod J_a$, the productive properties of J have been proved. The category $B = \{J_a : a \in L\}$ forms a full subcategory of the category **Loc**.

Throughout this paper the lattice of open sets of a topological space X will be denoted by $\Omega(X)$ and L is a locale.

2. Preliminaries

A frame (or a locale) is a is a complete lattice L satisfying the infinite distributivity law $a \wedge \bigvee B = \bigvee\{a \wedge b; b \in B\}$ for all $a \in L$ and $B \subseteq L$ [6]. Given the frames L, M, a frame homomorphism is a map h:$L \to M$ preserving all finite meets (including the top 1) and all joins (including the bottom 0). The category of frames is denoted by **Frm**. The opposite of category **Frm** is the category **Loc** of locales.

Example 2.1. The lattice of open subsets of topological space.

Definition 2.2. [7] A subset I of a locale L is said to be an ideal if

i) I is a sub-join-semi lattice of L; that is $0 \in I$ and $a \in I, b \in I$ implies $a \vee b \in I$; and

ii) I is a lower set; that is $a \in I$ and $b \leq a$ imply $b \in I$.

If $a \in L$, the set $\downarrow (a) = \{x \in L; x \leq a\}$ is an ideal of L. $\downarrow (a)$ is the smallest ideal containing a and is called the principal ideal generated by a.

Defintion 2.3. [6] A subset F of locale L is said to be a filter if

i) F is a sub-meet-semi lattice of L; that is $1 \in F$ and $a \in F, b \in F$ imply $a \wedge b \in F$.

ii) F is an upper set;that is $a \in F$ and $a \leq b$ imply $b \in F$.

Definition 2.4. [6] A filter F is proper if $F \neq L$,that is if $0 \notin F$.
A proper filter F in a locale L is prime if $a_1 \vee a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.

Definition 2.5. [6] A proper filter F in a locale L is a completely prime filter if for any J and $a_i \in L$,$i \in J$, $\bigvee a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$. Completely prime filters are denoted by c.p filters.

Example 2.6. $U(x) = \{V \in \Omega(X); x \in V\}$ is a completely prime filter in the locale $\Omega(X)$.

For $a \in L$, set $\Sigma_a = \{F \subseteq L; F \neq \phi, F$ is c.p filters; $a \in F\}$. Thus we have

$$\Sigma_0 = \phi,\ \Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i},\ \Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$$

and

$$\Sigma_1 = \{\text{all c.p filters}\}.$$

From 1.4 Definition, if $a \leq b$,then $\Sigma_a \subseteq \Sigma_b$. But $\Sigma_a \subseteq \Sigma_b$ need not imply $a \leq b$.

Definition 2.7. [6] The spectrum of a locale is defined as follows.

$$Sp(L) = (\{\text{all c.p filters}\}, \{\Sigma_a : a \in L\}).$$

Then Sp(L) is a topological space with the topology $\Omega(Sp(L)) = \{\Sigma_a : a \in L\}$.

Definition 2.8. [6] A locale L is said to be spatial if it is isomorphic to $\Omega(X)$ of some topological space X

As in classical topology, the point free topology have seperation axioms. Subfit and fit correspond to T_1 axiom of classical topology.

Definition 2.9. [6] A locale is said to be sub fit if for $a, b \in L$, $a \nleq b$, then $\exists c$, such that $a \vee c = 1$ and $b \vee c \neq 1$.

The rather below relation [6] In a locale L, for $a, b \in L$, we say that a is rather below b, denoted by $a \prec b$, if there exist $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$.

Definition 2.10. [6] A locale L is said to be regular if $a = \bigvee\{x : x \prec a\}$ for every $a \in L$.

Definition 2.11. [2] A locale L is said to have S_2' property if for any $a, b \in L$, if $a \vee b = 1$ with $a \neq 1$ and $b \neq 1$, then there exist u, v with $u \wedge v = 0$,$v \nleq a$, $u \nleq b$.

Definition 2.12. [2] A locale L is called normal if it satisfies the condition:
If $a \vee b = 1$, then there exist u, v such that $a \vee v = 1$, $u \vee b = 1$, $u \wedge v = 0$.

Definition 2.13. [6] A cover of a locale L is a subset $A \subseteq L$ such that $\bigvee A = 1$.

A sub cover of a cover A is a subset $B \subseteq A$ such that $\bigvee B = 1$. A locale is said to be compact if each cover has a finite sub cover.

Definition 2.14. [6] An element $p \neq 1$ in a lattice L is said to be meet irreducible if for any $a, b \in L$, $a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$.

3. Ideals $[a, \Sigma_b]$ for $a, b \in L$

Let L be a locale and $O(L)$ denote the collection of all order preserving maps on L. That is

$$O(L) = \{f ; f : L \to L \quad \text{is order preserving}\}.$$

Define $f \leq' g$ if and only if $f(a) \leq g(a)$ $\forall a \in L$. Since $f(a) \leq f(a)$ $\forall a \in L$, the relation $\leq'$ is reflexive. Let $f \leq' g$ and $g \leq' f$. Then $f(a) \leq g(a)$ and $g(a) \leq f(a)$ $\forall a \in L$. Thus $f(a) = g(a)$, $\forall a \in L$. Hence the relation $\leq'$ is anti symmetric. Let $f \leq' g$ and $g \leq' h$. Then $f(a) \leq g(a)$ and $g(a) \leq h(a)$ $\forall a \in L$. Since the relation $\leq$ is transitive, we have $f(a) \leq h(a)$ $\forall a \in L$. Hence the relation $\leq'$ is transitive. Then $\leq'$ is a partial order on O(L).

If $\vee$ and $\wedge$ denote the respective join and meet corresponding to $\leq'$, defining $f \vee g : L \to L$ by $(f \vee g)(a) = f(a) \vee g(a)$ and $f \wedge g : L \to L$ by $(f \wedge g)(a) = f(a) \wedge g(a)$, we obtain that $O(L)$ is a locale with bottom **0** and top **1**, where
0,**1** : L $\to L$ are defined by **0**(a)=0 and **1**(a)=1 $\forall a \in L$.

Definition 3.1. For each $a, b \in L$, define $[a, \Sigma_b] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_b\}$. Some simple properties of $[a, \Sigma_b]$ have been verified in the following lemmas.

Lemma 3.2.

a) $[a, \Sigma_b]$ is an ideal.

b) If Σ_b is meet irreducible element of $\Omega(Sp(L))$, then $[a, \Sigma_b]$ is prime ideal.

Proof.

a) Since $\Sigma_{\mathbf{0}(a)} = \Sigma_0 = \phi \subset \Sigma_b$, we get $\mathbf{0} \in [a, \Sigma_b]$. Let $f, g \in [a, \Sigma_b]$. Then $\Sigma_{f(a)} \subset \Sigma_b$ and $\Sigma_{g(a)} \subset \Sigma_b$. Hence $\Sigma_{f(a)} \cup \Sigma_{g(a)} \subset \Sigma_b$. That is, $\Sigma_{f(a) \vee g(a)} \subset \Sigma_b$ which implies $\Sigma_{(f \vee g(a))} \subset \Sigma_b$. Thus $f \vee g \in [a, \Sigma_b]$. Hence $[a, \Sigma_b]$ is a sub-join semi lattice.

Now let $f \in [a, \Sigma_b]$ and $g \leq f$ in $O(L)$. Since $g \leq f$, we have $g(a) \leq f(a)$. Hence $\Sigma_{g(a)} \subset \Sigma_{f(a)}$. Thus $\Sigma_{g(a)} \subset \Sigma_{f(a)} \subset \Sigma_b$. That is $\Sigma_{g(a)} \subset \Sigma_b$ which implies $g \in [a, \Sigma_b]$. Hence $[a, \Sigma_b]$ is an ideal.

b) Assume Σ_b is a meet irreducible element of $\Omega(Sp(L))$. From (a), $[a, \Sigma_b]$ is an ideal. Let $f \wedge g \in [a, \Sigma_b]$. Then $\Sigma_{(f \wedge g)(a)} \subseteq \Sigma_b$. That is $\Sigma_{f(a)} \cap \Sigma_{g(a)} \subseteq \Sigma_b$. Since Σ_b is meet irreducible, either $\Sigma_{f(a)} \subseteq \Sigma_b$ or $\Sigma_{g(a)} \subseteq \Sigma_b$. Hence $f \in [a, \Sigma_b]$ or $g \in [a, \Sigma_b]$. Thus $[a, \Sigma_b]$ is a prime ideal. ■

In [7] Johnstone has defined lattice without bottom element or Top element as pseudo lattice. Using the same terminology we can define pseudo subframe as follows.

Definition 3.3. Pseudo subframe M of a frame L is a subset M of L which is closed under all joins and non empty finite meets so that $\mathbf{1}_L \notin M$.

Proposition 3.4. $[a, \Sigma_b]$ is a pseudo subframe of $O(L)$.

Proof. Let I be a non empty indexed set and let $f_i \in [a, \Sigma_b] \quad \forall i \in I$. Then $\Sigma_{f_i(a)} \subset \Sigma_b, \quad \forall i \in I$. Therefore $\bigcup \Sigma_{f_i(a)} \subset \Sigma_b$ and $\bigcap \Sigma_{f_i(a)} \subset \Sigma_b$. Thus $\Sigma_{\bigvee f_i(a)} \subset \Sigma_b$ and $\Sigma_{\bigwedge f_i(a)} \subset \Sigma_b$. That is $\bigvee f_i \in [a, \Sigma_b]$ and $\bigwedge f_i \in [a, \Sigma_b]$. Hence $[a, \Sigma_b]$ is a complete lattice. Also $[a, \Sigma_b]$ satisfies infinite distributive law as $O(L)$ satisfies the same. But $\mathbf{1} \notin [a, \Sigma_b]$ if $b \neq 1$. Hence $[a, \Sigma_b]$ is a pseudo subframe of $O(L)$ and $[a, \Sigma_1]$ is the locale $O(L)$. ■

Lemma 3.5. If $a_1 \leq a_2$,then $[a_1, \Sigma_b] \supseteq [a_2, \Sigma_b]$.

Proof. Suppose $a_1 \leq a_2$.Then $f(a_1) \leq f(a_2) \quad \forall f \in O(L)$. $f \in [a_2, \Sigma_b]$ implies $\Sigma_{f(a_2)} \subset \Sigma_b$. Then $\Sigma_{f(a_1)} \subset \Sigma_{f(a_2)} \subset \Sigma_b$. Thus $f \in [a_1, \Sigma_b]$. Therefore $[a_1, \Sigma_b] \supseteq [a_2, \Sigma_b]$. ■

Lemma 3.6. If $b_1 \leq b_2$, then $[a, \Sigma_{b_1}] \subseteq [a, \Sigma_{b_2}]$.

Proof. Let $b_1 \leq b_2$.Then $\Sigma_{b_1} \subset \Sigma_{b_2}$. $f \in [a, \Sigma_{b_1}]$ implies $\Sigma_{f(a)} \subset \Sigma_{b_1} \subset \Sigma_{b_2}$. Hence $f \in [a, \Sigma_{b_2}]$. Therefore $[a, \Sigma_{b_1}] \subset [a, \Sigma_{b_2}]$. ■

Lemma 3.7. If $a \leq b$, then $O(L) = \bigcup_{f \in O(L)} [a, \Sigma_{f(b)}]$.

Proof. If $a \leq b$, $f(a) \leq f(b)$ for all $f \in O(L)$. Let $h \in O(L)$. Then $h(a) \leq h(b)$. Thus $\Sigma_{h(a)} \subseteq \Sigma_{h(b)}$ so that $h \in [a, \Sigma_{h(b)}]$. Therefore $O(L) \subseteq \bigcup_{f \in O(L)} [a, \Sigma_{f(b)}]$. Also $\bigcup_{f \in O(L)} [a, \Sigma_{f(b)}] \subseteq O(L)$. Hence $O(L) = \bigcup_{f \in O(L)} [a, \Sigma_{f(b)}]$. ■

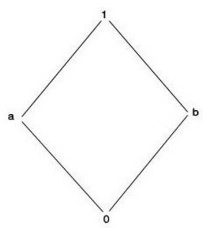
Remark 3.8. f_b denotes the constant function on L with the value b. That is $f_b(x) = b \ \forall x \in L$.

Lemma 3.9. For any $a, b, c \in L$, $[a, \Sigma_b] = [a, \Sigma_c]$ if and only if $\Sigma_b = \Sigma_c$.

Proof. If $\Sigma_b = \Sigma_c$, then clearly $[a, \Sigma_b] = [a, \Sigma_c]$.

Conversely let $[a, \Sigma_b] = [a, \Sigma_c]$. Since $\Sigma_{f_b(a)} \subseteq \Sigma_b$, $f_b \in [a, \Sigma_b] = [a, \Sigma_c]$. Then $\Sigma_{f_b(a)} \subseteq \Sigma_c$. That is $\Sigma_b \subseteq \Sigma_c$. In a similar manner $\Sigma_c \subseteq \Sigma_b$. Hence $\Sigma_b = \Sigma_c$. ■

Examples 3.10. 1). Let the locale L be given as follows.



Then $\Sigma_0 = \phi$, $\Sigma_a = \{F_1\}$, $\Sigma_b = \{F_2\}$, $\Sigma_1 = \{F_1, F_2\}$, where completely prime filters F_1 and F_2 are given by $F_1 = \{a, 1\}$, $F_2 = \{b, 1\}$. Then $O(L) = \{f_1, f_2, \ldots, f_{33}\}$, where the order preserving maps $f_i : L \to L$ is given by the following formulas.

$f_1(x) = \mathbf{0}(x) = 0 \quad \forall x \in L$

$f_2(0) = 0$	$f_3(0) = 0$	$f_4(0) = 0$	$f_5(0) = 0$	$f_6(0) = 0$	$f_7(0) = 0$
$f_2(a) = 0$	$f_3(a) = 0$	$f_4(a) = 0$	$f_5(a) = 0$	$f_6(a) = 0$	$f_7(a) = 0$
$f_2(b) = 0$	$f_3(b) = 0$	$f_4(b) = 0$	$f_5(b) = a$	$f_6(b) = a$	$f_7(b) = b$
$f_2(1) = a$	$f_3(1) = b$	$f_4(1) = 1$	$f_5(1) = a$	$f_6(1) = 1$	$f_7(1) = b$

$f_8(0) = 0$	$f_9(0) = 0$	$f_{10}(0) = 0$	$f_{11}(0) = 0$	$f_{12}(0) = 0$	$f_{13}(0) = 0$
$f_8(a) = 0$	$f_9(a) = a$	$f_{10}(a) = a$	$f_{11}(a) = a$	$f_{12}(a) = a$	$f_{13}(a) = b$
$f_8(b) = b$	$f_9(b) = 0$	$f_{10}(b) = a$	$f_{11}(b) = b$	$f_{12}(b) = 1$	$f_{13}(b) = 0$
$f_8(1) = 1$	$f_9(1) = a$	$f_{10}(1) = a$	$f_{11}(1) = 1$	$f_{12}(1) = 1$	$f_{13}(1) = b$

$f_{14}(0) = 0$	$f_{15}(0) = 0$	$f_{16}(0) = 0$	$f_{17}(0) = 0$	$f_{18}(0) = 0$	$f_{19}(0) = 0$
$f_{14}(a) = b$	$f_{15}(a) = b$	$f_{16}(a) = b$	$f_{17}(a) = b$	$f_{18}(a) = b$	$f_{19}(a) = a$
$f_{14}(b) = 0$	$f_{15}(b) = a$	$f_{16}(b) = b$	$f_{17}(b) = b$	$f_{18}(b) = 1$	$f_{19}(b) = 0$
$f_{14}(1) = 1$	$f_{15}(1) = 1$	$f_{16}(1) = b$	$f_{17}(1) = 1$	$f_{18}(1) = 1$	$f_{19}(1) = 1$

$f_{20}(0) = 0$	$f_{21}(0) = 0$	$f_{22}(0) = 0$	$f_{23}(0) = a$	$f_{24}(0) = a$	$f_{25}(0) = a$

$f_{20}(a) = a$ $f_{21}(a) = 1$ $f_{22}(a) = 1$ $f_{23}(a) = a$ $f_{24}(a) = a$ $f_{25}(a) = a$
$f_{20}(b) = a$ $f_{21}(b) = 0$ $f_{22}(b) = 1$ $f_{23}(b) = a$ $f_{24}(b) = a$ $f_{25}(b) = 1$
$f_{20}(1) = 1$ $f_{21}(1) = 1$ $f_{22}(1) = 1$ $f_{23}(1) = a$ $f_{24}(1) = 1$ $f_{25}(1) = 1$

$f_{26}(0) = a$ $f_{27}(0) = b$ $f_{28}(0) = b$ $f_{29}(0) = b$ $f_{30}(0) = b$
$f_{26}(a) = 1$ $f_{27}(a) = b$ $f_{28}(a) = b$ $f_{29}(a) = b$ $f_{30}(a) = 1$
$f_{26}(b) = 1$ $f_{27}(b) = b$ $f_{28}(b) = b$ $f_{29}(b) = 1$ $f_{30}(b) = 1$
$f_{26}(1) = 1$ $f_{27}(1) = b$ $f_{28}(1) = 1$ $f_{29}(1) = 1$ $f_{30}(1) = 1$

$f_{31}(0) = 1$ $f_{32}(0) = a$ $f_{33}(0) = b$
$f_{31}(a) = 1$ $f_{32}(a) = 1$ $f_{33}(a) = 1$
$f_{31}(b) = 1$ $f_{32}(b) = a$ $f_{33}(b) = b$
$f_{31}(1) = 1$ $f_{32}(1) = 1$ $f_{33}(1) = 1$
Then
$[a, \Sigma_0] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_0\}$ =$\{f \in O(L) : \Sigma_{f(a)} = \phi\}$
=$\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$
$[a, \Sigma_a] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{19}, f_{20}, f_{23}, f_{24}, f_{25}\}$
$[a, \Sigma_b] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{27}, f_{28}, f_{29}\}$
$[a, \Sigma_1] = [a, \Sigma_{a\vee b}] = O(L)$
2). For non spatial example,consider L= the Boolean algebra of all regularly open subsets of the real line R. Then for any regularly open subsets U,V we have
$[U, \Sigma_V] = \{f \in O(L); \Sigma_{f(U)} \subseteq \Sigma_V\} = O(L)$, since $Sp(L) = \phi$.

3.1. The locale $J_a, a \in L$

Fix some $a \in L$,and let J_a=$\{[a, \Sigma_b] : b \in L\}$.

Define $[a, \Sigma_b] \wedge [a, \Sigma_c] = [a, \Sigma_{b\wedge c}]$ and $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_{b\vee c}]$. Then $(J_a, \vee)$ and $(J_a, \wedge)$ are commutative monoids in which every element is idempotent. Also

$$[a, \Sigma_b] \vee ([a, \Sigma_b] \wedge [a, \Sigma_c]) = [a, \Sigma_b] \vee [a, \Sigma_{b\wedge c}] = [a, \Sigma_{b\vee(b\wedge c)}] = [a, \Sigma_b]$$

and

$$[a, \Sigma_b] \wedge ([a, \Sigma_b] \vee [a, \Sigma_c]) = [a, \Sigma_b] \wedge [a, \Sigma_{b\vee c}] = [a, \Sigma_{b\wedge(b\vee c)}] = [a, \Sigma_b].$$

Thus absorption laws are satisfied and hence J_a is a lattice. Since $\bigvee a_i$, for $a_i \in L$ exist, J_a is a complete lattice. Again

$$[a, \Sigma_b] \wedge \bigvee [a, \Sigma_{c_i}] = [a, \Sigma_b] \wedge [a, \Sigma_{\vee c_i}]$$

$$= [a, \Sigma_{b\wedge \bigvee c_i}] = [a, \Sigma_{\bigvee b\wedge c_i}] = \bigvee [a, \Sigma_{b\wedge c_i}]$$

$$= \bigvee [a, \Sigma_b] \wedge [a, \Sigma_{c_i}].$$

Hence J_a satisfies infinite distributive law. Thus J_a is a locale of pseudo subframes of $O(L)$ with top element $[a, \Sigma_1]$ and bottom element $[a, \Sigma_0]$.

From example 2.9(2), we get the locale $J_U =$ The one point locale O, for all $U \in L$. From example 2.9(1), we get the locales $J_0 = \{[0, \Sigma_b] : b \in L\}$, $J_a = \{[a, \Sigma_b] : b \in L\}$,
$J_b = \{[b, \Sigma_a] : a \in L\}$ and $J_1 = \{[1, \Sigma_b] : b \in L\}$.

Proposition 3.11. The locale J_a is compact if and only if Sp(L) is compact.

Proof. Assume that J_a is compact.We have to show that Sp(L) is compact. Let $\Sigma_1 = \bigvee_{i\in I} \Sigma_{b_i}$.Then $\Sigma_1 = \Sigma_{\vee b_i}$. Thus $\bigvee[a, \Sigma_{b_i}] = [a, \Sigma_{\vee b_i}] = [a, \Sigma_1]$, using 2.8 Lemma. Hence $\{[a, \Sigma_{b_i}]; i \in I\}$ is a cover for J_a. Since J_a is compact, we have $[a, \Sigma_{b_1}] \vee [a, \Sigma_{b_2}] \vee \ldots\ldots. \vee [a, \Sigma_{b_n}] = [a, \Sigma_1]$ for some $b_1, b_2, b_3, \ldots..b_n \in L$. That is $[a, \Sigma_{b_1\vee b_2\vee b_3\vee \ldots..b_n}] = [a, \Sigma_1]$. Then $\Sigma_{b_1\vee b_2\vee b_3\vee \ldots..b_n} = \Sigma_1$, using 2.8 lemma. Hence Sp(L)is compact.

Conversely assume Sp(L) is compact. Let $\{[a, \Sigma_{b_i}]; i \in I\}$ be a cover of J_a. That is

$$\bigvee[a, \Sigma_{b_i}] = [a, \Sigma_1] \quad \text{or} \quad [a, \Sigma_{\vee b_i}] = [a, \Sigma_1].$$

This gives $\Sigma_{\vee b_i} = \Sigma_1$. Since Sp(L) is compact,we have $\Sigma_1 = \Sigma_{b_1\vee b_2\vee \ldots\vee b_n}$. Hence $[a, \Sigma_{b_i\vee b_2\vee \ldots\ldots\vee b_n}] = [a, \Sigma_1]$.Thus J_a is compact.

3.2 Isomorphism of J_a with a quotient locale of L

Define a relation $\sim_a$ on L by $b\sim_a c$ if $[a, \Sigma_b] = [a, \Sigma_c]$. Since $[a, \Sigma_b] = [a, \Sigma_b]$, we have $b \sim_a b$.Thus the relation $\sim_a$ is reflexive. The relation $\sim_a$ is symmetric from the definition. Also $[a, \Sigma_b] = [a, \Sigma_c]$ and $[a, \Sigma_c] = [a, \Sigma_d]$ implies $[a, \Sigma_b] = [a, \Sigma_d]$.So the relation $\sim_a$ is transitive. Thus relation $\sim_a$ is an equivalence relation. Let (b,c)$\in\sim_a$.We claim that $(b \wedge d, c \wedge d) \in\sim_a$ and $(b \vee \bigvee S, c \vee \bigvee S) \in\sim_a$.

$[a, \Sigma_{b\wedge d}] = [a, \Sigma_b] \wedge [a, \Sigma_d] = [a, \Sigma_c] \wedge [a, \Sigma_d] = [a, \Sigma_{c\wedge d}]$. Hence $(b \wedge d, c \wedge d) \in\sim_a$. Now $[a, \Sigma_{b\vee\bigvee S}] = [a, \Sigma_b] \vee [a, \Sigma_{\bigvee S}] = [a, \Sigma_c] \vee [a, \Sigma_{\bigvee S}] = [a, \Sigma_{c\vee\bigvee S}]$. Hence $(b \vee \bigvee S, c \vee \bigvee S) \in\sim_a$. Thus $\sim_a$ is a congruence on L. Then by [1], $L/\sim_a$ is a locale with respect to the partial order $[a] \leq [b]$ iff $a \leq b$.

Note that if L is spatial, then $L/\sim_a = L$ for all $a \in L$.

Define $\psi_a : L/\sim_a \to J_a$ by $\psi_a([b]) = [a, \Sigma_b]$. Then $\psi_a([b] \wedge [c]) = \psi_a([b \wedge c]) = [a, \Sigma_{b\wedge c}] = [a, \Sigma_b] \wedge [a, \Sigma_c] = \psi_a([b]) \wedge \psi_a([c])$ and $\psi_a(\bigvee[b_i]) = \psi_a([\vee b_i]) = [a, \Sigma_{\vee b_i}] = \bigvee[a, \Sigma_{b_i}] = \bigvee \psi_a([b_i])$. Hence ψ_a is a frame homomorphism. Also ψ_a is one-one and onto. Thus ψ_a is an isomorphism in the category **Frm**. Since the isomorphism is a self dual property, ψ_a is an isomorphism in the category **Loc**.

3.3 Congruence on O(L)

Define a relation R_a on O(L) by $f R_a g$ if $\Sigma_{f(a)} = \Sigma_{g(a)}$. Then R_a is an equivalence relation. Suppose $f R_a g$. Then

$\Sigma_{(f\wedge h)(a)} = \Sigma_{f(a)\wedge h(a)} = \Sigma_{f(a)} \bigcap \Sigma_{h(a)} = \Sigma_{g(a)} \bigcap \Sigma_{h(a)} = \Sigma_{g(a)\wedge h(a)} = \Sigma_{(g\wedge h)(a)}$. Thus $f \wedge h \;\; R_a \;\; g \wedge h$. Also

$\Sigma_{(f\vee\bigvee f_i)(a)} = \Sigma_{f(a)} \vee \Sigma_{\bigvee f_i(a)} = \Sigma_{g(a)} \vee \Sigma_{\bigvee f_i(a)} = \Sigma_{g(a)\vee\bigvee f_i(a)} = \Sigma_{(g\vee\bigvee f_i)(a)}$.
Hence $f \vee \bigvee f_i \;\; R_a \;\; g \vee \bigvee f_i$. Thus R_a is a congruence on O(L).

Then by [1], $O(L)/R_a$ is a locale with respect to the partial order $[f] \leq [g]$ in $O(L)/R_a$ if and only if $f \leq g$ in O(L).

Example 3.12. In example 2.9(1), we have $O(L)/R_a = \{[f_1], [f_9], [f_{13}], [f_{21}]\}$, where

$$[f_1] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$$

$$[f_9] = \{f_9, f_{10}, f_{11}, f_{12}, f_{19}, f_{20}, f_{23}, f_{24}, f_{25}\}$$

$$[f_{13}] = \{f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{27}, f_{28}\}$$

$$[f_{21}] = \{f_{21}, f_{22}, f_{26}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}\}$$

Lemma 3.13. If $(f, g) \in R_a$, then $(f(a), g(a)) \in \sim_a$.

Proof. Let $(f, g) \in R_a$. Then $\Sigma_{f(a)} = \Sigma_{g(a)}$. By lemma 2.8 $[a, \Sigma_{f(a)}] = [a, \Sigma_{g(a)}]$. Hence $(f(a), g(a)) \in \sim_a$. ■

Proposition 3.14. The quotient locale $L/\sim_a$ of L is isomorphic to the quotient locale $O(L)/R_a$ of $O(L)$.

Proof. Define the map $\sigma : O(L)/R_a \to L/\sim_a$ by $\sigma([f]) = [f(a)]$. Then $\sigma([f]) = \sigma([g])$ implies $[f(a)] = [g(a)]$. Hence $[a, \Sigma_{f(a)}] = [a, \Sigma_{g(a)}]$. Then by lemma 2.8 $\Sigma_{f(a)} = \Sigma_{g(a)}$. Hence $[f] = [g]$. Thus the map σ is one one. Also for each $[b] \in L/\sim_a$, $\sigma([f_b]) = [f_b(a)] = [b]$. Thus σ is onto. Now

$$\sigma(\bigvee[f_i]) = \sigma([\bigvee f_i]) = [(\bigvee f_i)(a)] = [\bigvee(f_i(a))] = \bigvee[f_i(a)] = \bigvee \sigma([f_i])$$

and

$$\sigma([f]\wedge[g]) = \sigma([f\wedge g]) = [(f\wedge g)(a)] = [f(a)\wedge g(a)] = [f(a)]\wedge[g(a)] = \sigma(f)\wedge\sigma(g).$$

Hence σ is an isomorphism in **Frm**. Since isomorphism is a self dual property, quotient locale $L/\sim_a$ of L is isomorphic to the quotient locale $O(L)/R_a$ of $O(L)$. ■

3.7 Embedding Theorem for locale L
The locale L can be embedded as a dense sublocale of $O(L)$.

Proof. Define $F : L/\sim_a \to L$ by $F([b]) = b$ and $G : O(L) \to O(L)/R_a$ by $G(f) = [f]$. Then F is one one, G is onto. Consider $F \circ \sigma \circ G : O(L) \to L$ where $\sigma : O(L)/R_a \to L/\sim_a$ is the isomorphism in 3.6. Then for each $b \in L$ we have $f_b \in O(L)$ such that $(F \circ \sigma \circ G)(f_b) = b$. Thus the map $F \circ \sigma \circ G$ is onto. Also since F, G, σ are frame homomorphisms, $F \circ \sigma \circ G$ is a frame homomorphism. Since $F \circ \sigma \circ G$ is an onto frame homomorphism, its adjoint δ is a one one localic map from L to O(L). Since $\mathbf{0} \in \delta(L)$, $\delta(L)$ is dense in O(L). Thus L can be embedded as a dense sublocale of O(L).

3.8 Localic map from Sp(L) to O(L)/R_a

Define $\phi_a : O(L)/R_a \to Sp(L)$ by $\phi_a([f]) = \Sigma_{f(a)}$. Then

$$\phi_a([f] \wedge [g]) = \Sigma_{f \wedge g(a)} = \Sigma_{f(a)} \cap \Sigma_{g(a)} = \phi_a([f]) \cap \phi_a([g])$$

and

$$\phi_a(\bigvee [f]_i) = \phi_a([\vee f_i]) = \Sigma_{\vee f_i(a)} = \bigcup \Sigma_{f_i(a)}.$$

Hence ϕ_a is a frame homomorphism and its adjoint ϕ_a^* is a localic map from Sp(L) to O(L)/R_a.

Lemma 3.15. If $f \in [a, \Sigma_b]$, then [f]$\in [a, \Sigma_b]$.

Proof. Let $f \in [a, \Sigma_b]$ and let $g \in [f]$. Then we have $\Sigma_{f(a)} = \Sigma_{g(a)}$. Since $f \in [a, \Sigma_b]$, $\Sigma_{f(a)} \subseteq \Sigma_b$, which implies $\Sigma_{g(a)} \subseteq \Sigma_b$. Hence $[f] \in [a, \Sigma_b]$. ■

Proposition 3.16. The locale J_a is subfit if and only if for every $[a, \Sigma_b]$ in J_a with $[a, \Sigma_b] \nleq [a, \Sigma_c]$, there exists $\Sigma_d \in Sp(L)$ such that $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$ and $[\mathbf{1}] \nsubseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$.

Proof. Suppose locale J_a is a subfit. Let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \nleq [a, \Sigma_c]$. Since J_a is a subfit, there exist $[a, \Sigma_d]$ such that $[a, \Sigma_b] \vee [a, \Sigma_d] = [a, \Sigma_1]$ and $[a, \Sigma_c] \vee [a, \Sigma_d] \neq [a, \Sigma_1]$. Since $[a, \Sigma_b] \vee [a, \Sigma_d] = [a, \Sigma_1]$, we have $\mathbf{1} \in [a, \Sigma_{b \vee d}]$. Then by lemma 3.9, $[\mathbf{1}] \in [a, \Sigma_{b \vee d}]$. Thus $\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_{b \vee d}$. Hence $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$. Also, if $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$, then $\mathbf{1} \in [a, \Sigma_{c \vee d}]$,a contradiction. Hence $[\mathbf{1}] \nsubseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$.

Conversely, suppose $[a, \Sigma_b] \nleq [a, \Sigma_c]$. Then by hypothesis, $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$. That is $\phi_a([\mathbf{1}]) \in \Sigma_{b \vee d}$. Thus $\mathbf{1} \in [a, \Sigma_{b \vee d}]$ and hence $[a, \Sigma_{b \vee d}] = [a, \Sigma_1]$. If $[a, \Sigma_{c \vee d}] = [a, \Sigma_1]$, then $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$, a contradiction. Hence J_a is subfit. ■

Proposition 3.17. The locale J_a has S_2' property if and only if for every $[a, \Sigma_b] \neq [a, \Sigma_1]$,
$[a, \Sigma_c] \neq [a, \Sigma_1]$ in J_a with $[a, \Sigma_{b \vee c}] = [a, \Sigma_1]$,there exist $\Sigma_d, \Sigma_e \in \Omega(Sp(L))$ such that $\Sigma_e \nsubseteq \Sigma_b$,
$\Sigma_d \nsubseteq \Sigma_c$ and $\phi_a([f]) \subseteq \Sigma_d \wedge \Sigma_e$ implies that $f \in [\mathbf{0}]$.

Proof. Suppose J_a has S_2' property. Let $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. Then there exist $[a, \Sigma_d], [a, \Sigma_e] \in J_a$ such that $[a, \Sigma_d] \wedge [a, \Sigma_e] = [a, \Sigma_0]$, $[a, \Sigma_e] \nleq [a, \Sigma_b]$ and $[a, \Sigma_d] \nleq [a, \Sigma_c]$. Then $\Sigma_d, \Sigma_e \in \Omega(Sp(L))$. If $\Sigma_e \subseteq \Sigma_b$,then $[a, \Sigma_e] \subseteq [a, \Sigma_b]$, a contradiction. Hence $\Sigma_e \nsubseteq \Sigma_b$.Similarly we can prove that $\Sigma_d \nsubseteq \Sigma_c$. Let $\phi_a([f]) \subseteq$

$\Sigma_d \wedge \Sigma_e$. Then $\Sigma_{f(a)} \subseteq \Sigma_{d\wedge e}$. Hence $f \in [a, \Sigma_{d\wedge e}] = [a, \Sigma_0]$. Thus $f \in [\mathbf{0}]$. Hence $\phi_a([f]) \subseteq \Sigma_d \wedge \Sigma_e$ implies $f \in [\mathbf{0}]$.

Conversely, let $[a, \Sigma_b] \neq [a, \Sigma_1], [a, \Sigma_c] \neq [a, \Sigma_1]$ with $[a, \Sigma_{b\vee c}] = [a, \Sigma_1]$. Then by assumption, there exist $\Sigma_d, \Sigma_e \in \Omega(Sp(L))$ with $\Sigma_e \nsubseteq \Sigma_b$, $\Sigma_d \nsubseteq \Sigma_c$ and $\phi_a([f]) \subseteq \Sigma_d \wedge \Sigma_e$ implying that $f \in [\mathbf{0}]$. Also $f \in [a, \Sigma_{d\wedge e}]$, implies $\Sigma_{f(a)} \subseteq \Sigma_{d\wedge e}$. Hence

$$\phi_a([f]) \subseteq \Sigma_d \wedge \Sigma_e \Rightarrow f \in [\mathbf{0}] \Rightarrow f \in [a, \Sigma_0].$$

Hence $[a, \Sigma_d] \wedge [a, \Sigma_e] = [a, \Sigma_0]$. Also since $\Sigma_e \nsubseteq \Sigma_b$, $\Sigma_d \nsubseteq \Sigma_c$, $[a, \Sigma_e] \nleq [a, \Sigma_b]$ and $[a, \Sigma_d] \nleq [a, \Sigma_c]$. Thus J_a has S_2' property. ■

Lemma 3.18. If $b \prec c$ in L, then $[a, \Sigma_b] \prec [a, \Sigma_c]$ in J_a.

Proof. Suppose $b \prec c$ in L. Then there exist $d \in L$ such that $b \wedge d = 0$ and $c \vee d = 1$. We have $[a, \Sigma_b] \wedge [a, \Sigma_d] = [a, \Sigma_{b\wedge d}] = [a, \Sigma_0]$ and $[a, \Sigma_c] \vee [a, \Sigma_d] = [a, \Sigma_{c\vee d}] = [a, \Sigma_1]$. Hence $[a, \Sigma_b] \prec [a, \Sigma_c]$ in J_a. ■

Proposition 3.19. If L is a regular locale, then J_a is a regular locale.
Proof of the preposition follows from the lemma 3.12. ■

Proposition 3.20. Locale J_a is normal if and only if for every $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$, there exist $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$ such that $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$,
$[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$ and $\phi_a([f]) \in \Sigma_{u\wedge v} \Rightarrow f \in [\mathbf{0}]$.

Proof. Suppose J_a is normal. Let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. Since J_a is normal, there exist $[a, \Sigma_u], [a, \Sigma_v] \in J_a$ such that

$$[a, \Sigma_b] \vee [a, \Sigma_u] = [a, \Sigma_1], [a, \Sigma_c] \vee [a, \Sigma_v] = [a, \Sigma_1]$$

and

$$[a, \Sigma_u] \wedge [a, \Sigma_v] = [a, \Sigma_0].$$

Then

$$\Sigma_u, \Sigma_v \in \Omega(Sp(L)).$$

Also

$$\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_b \cup \Sigma_u.$$

Hence $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$. Similarly $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$. Now $\phi_a([f]) \in \Sigma_{u\wedge v}$ implies $\Sigma_{f(a)} \subseteq \Sigma_{u\wedge v}$. Hence $f \in [a, \Sigma_{u\wedge v}] = [a, \Sigma_0]$, which implies $\Sigma_{f(a)} \subseteq \Sigma_0$. Hence $f \in [\mathbf{0}]$.

Conversely, let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. By assumption, there exist $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$ such that $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$, $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$ and $\phi_a([f]) \in \Sigma_{u\wedge v} \Rightarrow f \in [\mathbf{0}]$. $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$. Since $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$, $[a, \Sigma_u], [a, \Sigma_v] \in J_a$. Also $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$ implies $\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_{u\vee b}$. Thus $\mathbf{1} \in [a, \Sigma_{u\vee b}]$. Hence $[a, \Sigma_{u\vee b}] = [a, \Sigma_1]$. Similarly we can prove that $[a, \Sigma_{v\vee c}] =$

$[a, \Sigma_1]$. Also $f \in [a, \Sigma_{u\wedge v}]$ implies $\Sigma_{f(a)} = \phi_a([f]) \subseteq \Sigma_{u\wedge v}$. Then by assumption $f \in [a, \Sigma_0]$. Hence $[a, \Sigma_{u\wedge v}] = [a, \Sigma_0]$. ■

Proposition 3.21. The locale J_a is Boolean if and only if for each $b \in L, \Sigma_b$ is a clopen subset of $\Omega(Sp(L))$.

Proof. Suppose that the locale J_a is Boolean. Let $\Sigma_b \in \Omega(Sp(L))$. Then $[a, \Sigma_b] \in J_a$. Since J_a is Boolean, there exist $[a, \Sigma_c] \in J_a$ such that $[a, \Sigma_b] \wedge [a, \Sigma_c] = [a, \Sigma_0]$ and $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. $[a, \Sigma_{b\wedge c}] = [a, \Sigma_0]$ implies $\Sigma_{b\wedge c} = \Sigma_0$ by lemma 2.8. Hence $\Sigma_b \cap \Sigma_c = \phi$. $[a, \Sigma_{b\vee c}] = [a, \Sigma_1]$ implies that $\Sigma_{b\vee c} = \Sigma_1$. Hence $\Sigma_b \cup \Sigma_c = \Sigma_1$. Thus $\Sigma_c \in \Omega(Sp(L))$ is the compliment of Σ_b. Hence Σ_b is both closed and open.

Conversely assume that each Σ_b is clopen. Let $[a, \Sigma_b] \in J_a$.Then $\Sigma_b \in \Omega(Sp(L))$. Since Σ_b is clopen, we have $\Sigma_c = (\Sigma_b)^c \in \Omega(Sp(L))$. Then $[a, \Sigma_c] \in J_a$. Now

$$[a, \Sigma_b] \wedge [a, \Sigma_c] = [a, \Sigma_{b\wedge c}] = [a, \Sigma_b \cap \Sigma_c] = [a, \phi] = [a, \Sigma_0]$$

and

$$[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_{b\vee c}] = [a, \Sigma_b \cup \Sigma_c] = [a, \Sigma_1].$$

Hence the locale J_a is Boolean. ■

3.16 Coproduct of the locales $J_a, a \in L$

In [6], if $L_i, i \in I$ are locales, then the cartesian product $\prod L_i$ together with component wise ordering is a locale. Since each $J_a, a \in L$ is a locale, $J = \prod J_a$ is a locale together with the map $p_a : J_a \to J, a \in L$, defined by $p_a([a, \Sigma_b]) = \prod[b, \Sigma_x]$ where $[b, \Sigma_x] = [b, \Sigma_1]$ for all $b \neq a$ and $[a, \Sigma_x] = [a, \Sigma_b]$. Then $(p_a : J_a \to J)_{a\in L}$ is the coproduct of locales J_a.

Notation Any element of the coproduct locale J is denoted by $\prod[a, \Sigma_{x_a}]$, where $[a, \Sigma_{x_a}] \in J_a$.

Proposition 3.22. The locale J is subfit if and only if each J_a is a subfit.

Proof. Suppose that each J_a is subfit. Let $A = \prod[a, \Sigma_{x_a}]$, $B = \prod[a, \Sigma_{y_a}] \in J$ such that $A \nleq B$. Then there exist $d \in L$ such that $[d, \Sigma_{x_d}] \nleq [d, \Sigma_{y_d}]$. Since J_d is subfit, there exist $[d, \Sigma_z]$ such that $[d, \Sigma_{x_d}] \vee [d, \Sigma_z] = [d, \Sigma_1]$ and $[d, \Sigma_{y_d}] \vee [d, \Sigma_z] \neq [a, \Sigma_1]$. Take $C \in J$ as $C = \prod[a, \Sigma_{z_a}]$ where $[a, \Sigma_{z_a}] = [a, \Sigma_1]$ for $a \neq d$ and $[d, \Sigma_{z_d}] = [d, \Sigma_z]$. Then we have $A \vee C = 1_J$ and $B \vee C \neq 1_J$. Hence J is a subfit locale.

Conversely assume that J is a subfit locale. Let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \nleq [a, \Sigma_c]$. Then let $A = \prod[b, \Sigma_{x_b}]$, $B = \prod[b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_1]$, $[b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and $[a, \Sigma_{x_a}] = [a, \Sigma_b]$ and $[a, \Sigma_{y_a}] = [a, \Sigma_c]$. Then A,B $\in J$ is such that $A \nleq B$. Since J is a subfit locale there exist $C = \prod[b, \Sigma_{z_b}] \in J$ such that $A \vee C = 1_J$ and $B \vee C \neq 1_J$. Then we must have $[a, \Sigma_b] \vee [a, \Sigma_{z_a}] = [a, \Sigma_1]$ and $[a, \Sigma_c] \vee [a, \Sigma_{z_a}] \neq [a, \Sigma_1]$. Thus J_a is a subfit. ■

Proposition 3.23. If the locale J has S_2' property, then each $J_a, a \in L$ has S_2' property.

Proof. Suppose that the locale J has S_2' property. Let $[a, \Sigma_b] \neq [a, \Sigma_1], [a, \Sigma_c] \neq [a, \Sigma_1] \in J_a$ with $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. Let $A = \prod[b, \Sigma_{x_b}], B = \prod[b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and $[a, \Sigma_{x_a}] = [a, \Sigma_b]$ and $[a, \Sigma_{y_a}] = [a, \Sigma_c]$. Then we have $A, B \neq 1_J \in J$ with $A \vee B = 1_J$. Since J has S_2' property, there exist $U = \prod[b, \Sigma_{u_b}], V = \prod[b, \Sigma_{v_b}] \in J$ such that $U \wedge V = 0_J, U \nleq B, V \nleq A$. Thus $[a, \Sigma_{v_a}] \nleq [a, \Sigma_b], [a, \Sigma_{u_a}] \nleq [a, \Sigma_c]$ and $[a, \Sigma_{v_a}] \wedge [a, \Sigma_{u_a}] = [a, \Sigma_0]$. Hence J_a has S_2' property. ■

Proposition 3.24. The locale J_a is normal if and only if J is normal.

Proof. Suppose that each J_a is normal. Let $A = \prod[b, \Sigma_{x_b}], B = \prod[b, \Sigma_{y_b}] \in J$ such that $A \vee B = 1_J$. Then $[a, \Sigma_{x_a}], [a, \Sigma_{y_a}] \in J_a$ with $[a, \Sigma_{x_a}] \vee [a, \Sigma_{y_a}] = [a, \Sigma_1]$ for all $a \in L$. Since J_a is normal, there exist $[a, \Sigma_{u_a}], [a, \Sigma_{v_a}] \in J_a$ such that

$$[a, \Sigma_{x_a}] \vee [a, \Sigma_{v_a}] = [a, \Sigma_1], [a, \Sigma_{y_a}] \vee [a, \Sigma_{u_a}] = [a, \Sigma_1], [a, \Sigma_{v_a}] \wedge [a, \Sigma_{u_a}] = [a, \Sigma_0].$$

Let $U = \prod[b, \Sigma_{u_b}], V = \prod[b, \Sigma_{v_b}]$. Then $U, V \in J$ such that $A \vee V = 1_J, B \vee U = 1_J, U \wedge V = 0_J$. Hence J is normal.

Conversely assume that J is normal. Let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \vee [a, \Sigma_c] = [a, \Sigma_1]$. Consider $A = \prod[b, \Sigma_{x_b}], B = \prod[b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and

$$[a, \Sigma_{x_a}] = [a, \Sigma_b], [a, \Sigma_{y_a}] = [a, \Sigma_c].$$

Then $A, B \in J$ with $A \vee B = 1_J$. Since J is normal, there exist $U = \prod[b, \Sigma_{u_b}], V = \prod[b, \Sigma_{v_b}]$ such that $A \vee V = 1_J, B \vee U = 1_J, U \wedge V = 0_J$. Then $[a, \Sigma_b] \vee [a, \Sigma_{v_a}] = [a, \Sigma_1]$, $[a, \Sigma_c] \vee [a, \Sigma_{u_a}] = [a, \Sigma_1]$, and $[a, \Sigma_{v_a}] \wedge [a, \Sigma_{u_a}] = [a, \Sigma_0]$. Hence J_a is normal. ■

Proposition 3.25. The subset $Y = \{\prod[a, \Sigma_b] : a \in L \;\; [a, \Sigma_b] = [a, \Sigma_1]$ except for finitely many values of $a\}$ of J is compact if each J_a is compact.

Proof. Suppose each J_a is compact. Let $U = \{\prod[x, \Sigma_{y_i}], i \in I\}$ be a cover of Y. Then, $[x, \Sigma_{y_i}] = [x, \Sigma_1]$ except for finitely many values of x say $x_1, x_2, \ldots, x_n$. Since U covers Y ,we have $\vee U = 1_Y$. Thus $\cup[x_j, \Sigma_{y_i}] = [x_j, \Sigma_1]$ for j=1,2,…,n. Since each J_{x_j} is compact there exist a finite sub cover for the cover $\{[x_j, y_i]\}$ j=1,2,…,n. Then it follows U has a finite sub cover. Hence Y is compact. ■

4. Conclusion

New spatial locales $J_a, a \in L$ have been constructed from the locale $O(L)$ of order preserving maps on the locale L. Specific congruences $\sim_a$ on L have been defined using the ideal $[a, \Sigma_b]$ of $O(L)$. It is observed that the quotient locale $L/\sim_a$ is isomorphic to J_a for each $a \in L$. Also the category $B = \{J_a : a \in L\}$ forms the full subcategory of the category **Loc**. The coproduct J constructed using the locales J_a,$a \in L$, satisfies the seperation axioms for locales such as subfit and normal if and only if each J_a is subfit and normal respectively.

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