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αg*s-Compactness and αg*s- Connectedness in Topological Spaces

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Abstract

In this paper, we introduce the new concepts αg*s-compactness and αg*s-connectedness in topological spaces and obtained some of their properties using αg*s-closed sets.

Keywords: αg*s-closed sets, αg*s-continuous maps, αg*s-connectedness and αg*s-compactness.

1. Introduction

Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line. The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers [1- 6] have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness and connectedness have been introduced and investigated.

P. G Patil and T. D. Rayanagoudar [7] introduced and studied the properties of αg*s-closed sets in topological spaces. The aim of this paper is to introduce the concept of αg*s-compactness and αg*s-connectedness in topological spaces and is to give some characterizations of αg*s-compactness and αg*s-connectedness. Further it is proved that αg*s-connectedness is under αg*s-irresolute surjections.

2. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) (or simply X, Y and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ), cl(A), Int(A), αcl(A) and Ac denote the closure of A, interior of A, α-closure of A and the compliment of A in X respectively.

The following definitions are useful in the sequel.

Definition 2.1: Let (X, τ) be a topological space. Then,

1. A subset A of (X, τ) is called αg*s-closed set [7] if αcl(A) ⊆ U whenever A ⊆ U and U is gs-open in (X, τ).
2. A topqadxological space (X, τ) is said to be GO-compact [1] (resp. αGO-compact [2]) if every g-open (resp. αg-open) cover of (X, τ) has a finite subcover.
3. A topological space (X, τ) is said to be GPR-compact [4] (resp. ω-compact [6]) if every GPR-open (resp. ω-open) cover of (X, τ) has a finite subcover.

Definition 2.2: A function f : (X, τ) → (Y, σ) is called

1. αg*s-continuous [7] (resp. αg*s-irresolute [7]) if the inverse image of every closed (resp. αg*s-closed) set in (Y, σ) is αg*s-closed in (X, τ).
2. strongly αg*s-continuous [8] (resp. perfectly αg*s- continuous [8]) if the inverse image of every αg*s-closed (resp. αg*s-closed) set in (Y, σ) is closed (resp. clopen) in (X, τ).

Definition 2.3: A topological space (X, τ) is said to be $_{\alpha gs}T^*_{1/2}$-space [7] if every αg*a-closed set is closed.

3. αg*s-Compactness in Topological Spaces

In this section, we introduce the concept of αg*s-compactness and obtained some of their properties.

Definition 3.1: A collection $\{A_i: i\in I\}$ of αg*s-open sets in a topological space X is called αg*s-open cover of a subset A in X if $A \subseteq \bigcup_{i\in I} A_i$.

Definition 3.2: A topological space X is called αg*s-compact if every αg*s-open cover of X has a finite subcover.

Definition 3.3: A subset A of a topological space X is called αg*s-compact relative to X if for every collection $\{A_i: i\in I\}$ of αg*s-open subsets of X such that $A \subseteq \bigcup_{i\in I} A_i$ there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{i\in I_0} A_i$.

Definition 3.4: A subset A of a topological space X is called αg*s-compact if A is αg*s-compact of the subspace of X.

Theorem 3.5: A αg*s-closed subset of αg*s-compact space is αg*s-compact relative to X.

Proof: Let A be a αg*s-closed subset of a topological space X. Then A^c is αg*s-open in X. Let $S = \{A_i: i\in I\}$ be a αg*s-open cover of A by αg*s-open subsets in X. Then $S^* = S \cup A^c$ is a αg*s-open cover of X. That is $X = [\cup \{A_i: i\in I\}] \cup A^c$. By hypothesis X is αg*s-compact and hence S* is reducible to a finite subcover of X say $X = A_{i1} \cup A_{i2} \cup ... \cup A_{in} \cup A^c$, $A_{ik} \in S^*$. But A and A^c are disjoint. Hence $A \subseteq A_{i1} \cup A_{i2} \cup .....\cup A_{in} \in S$. Thus a αg*s-open cover S of A contains a finite subcover. Hence A is αg*s-compact relative to X.

Theorem 3.6: Every αg*s-compact space is compact.

Proof: Let X be a αg*s-compact space. Let $\{A_i : i\in I\}$ be an open cover of X. Then $\{A_i: i\in I\}$ is a αg*s-open cover of X as every open set is αg*s-open set. Since X is αg*s-compact, the αg*s-open cover $\{A_i: i\in I\}$ of X has a finite subcover say $\{A_i: i=1...n\}$ for X. Hence X is compact.

Theorem 3.7: If X is compact and ${}_{\alpha gs}T^*_{1/2}$-space, then X is αg*s-compact.

Theorem 3.8: Every αGO-compact space is αg*s-compact.

Proof: Let X be a αGO-compact space. Let $\{A_i: i\in I\}$ be a αg*s-open cover of X by αg*s-open set in X. Since every αg*s-open set is αg-open, so that $\{A_i: i \in I\}$ is αg-open cover of X. Since X is αGO-compact, the αg-open cover $\{A_i: i\in I\}$ of X has a finite subcover say $\{A_i: i=1...n\}$ of X. Hence X is αg*s-compact.

Theorem 3.9: Let f: X → Y be surjective, αg*s-continuous function. If X is αg*s-compact, then Y is compact.

Proof: Let $\{A_i: i\in I\}$ be an open cover of Y. Since f is αg*s-continuous function, then $\{f^{-1}(A_i): i\in I\}$ is αg*s-open cover of X has a finite subcover say $\{f^{-1}(A_i): i=1.....n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = \bigcup_{i=1}^{n} A_i$. Since f is surjective, that is $Y = \bigcup_{i=1}^{n} A_i$. Thus $\{A_1, A_2, \ldots A_n\}$ is a finite subcover of $\{A_i: i\in I\}$ for Y. Hence Y is compact.

Theorem 3.10: If a function f: X → Y is αg*s-irresolute and a subset B of X is αg*s-compact relative to X, then the image f(B) is αg*s-compact relative to Y.

Proof: Let $\{A_i: i\in I\}$ be any collection of αg*s-open sets in Y such that $f(B) = \bigcup_{i\in I} A_i$. Then $B \subseteq \bigcup_{i\in I} f^{-1}(A_i)$, where $\{f^{-1}(A_i: i \in I\}$ is αg*s-open set in X. Since B is αg*s-compact relative to X, there exists finite subcollection $\{A_1, A_2, ..., A_n\}$ such that $B \subseteq \bigcup_{i\in I_0} f^{-1}(A_i)$ Therefore $f(B) \subseteq \bigcup_{i\in I_0} A_i$. Hence f(B) is αg*s-compact relative to Y.

Theorem 3.11: If a function f: X → Y is strongly αg*s-continuous from a compact space X onto a topological space Y, then Y is αg*s-compact.

Proof: Let $\{A_i: i\in I\}$ be a αg*s-open cover of Y. Since f is strongly αg*s-continuous, $\{f^{-1}(A_i) : i\in I\}$ is an open cover of X. Again since X is compact space, the open cover $\{f^{-1}(A_i): i\in I\}$ of X has a finite subcover say $\{f^{-1}(A_i): i = 1 ... n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = \bigcup_{i=1}^{n} A_i$ so that $Y = \bigcup_{i=1}^{n} A_i$. That is $\{A_1, A_2 \ldots A_n\}$ is a finite subcover of $\{A_i : i\in I\}$ for Y. Hence Y is αg*s-compact.

Theorem 3.12: A function f: X → Y is perfectly αg*s-continuous from a compact space X onto a topological space Y, then Y is αg*s-compact.

Proof: Since every perfectly αg*s-continuous function is a strongly αg*s-continuous and by Theorem 3.11, Y is αg*s-compact.

Theorem 3.13: A topological space X is αg*s-compact if and only if every family of αg*s-closed sets of X having finite intersection property has a non-empty intersection.

Proof: Suppose X is αg*s-compact. Let {A_i: i∈I} be a family of αg*s-closed sets with finite intersection property. To prove that $\bigcap_{i\in I} A_i \neq \phi$.

Suppose $\bigcap_{i\in I} A_i = \phi$. Then $X - \bigcup_{i\in I} A_i = X$. Which implies $\bigcup_{i\in I}(X - A_i) = X$..

Thus the cover {X – A_i; i∈I} is a αg*s-open cover of X. Since X is αg*s-compact, the αg*s-open cover {X – A_i; i∈I} has a finite subcover say {X – A_i; i=1…...n}. This implies X = $\bigcup_{i=1}^{n}(X - A_i)$ which implies that X = X – $\bigcap_{i=1}^{n} A_i$ which implies X – X = X - $\left[X - \bigcap_{i=1}^{n} A_i\right]$ implies that $\phi = \bigcap_{i=1}^{n} A_i$. This contradicts the assumption. Hence $\bigcap_{i=1}^{n} A_i \neq \phi$.

Conversely, suppose every family of αg*s-closed sets of X with finite intersection property has a non-empty intersection. To prove that X is αg*s-compact. Suppose X is not a αg*s-compact. Then there exists a αg*s-open cover of X say {G_i: i∈I} having no finite sub-cover. This implies for any finite sub family {G_i: i =1...n} of {G_i: i∈I} we have $\bigcup_{i=1}^{n} G_i \neq X$ which implies that $X - \bigcup_{i=1}^{n} G_i \neq$ X – X, which implies $\bigcap_{i=1}^{n}(X - G_i) \neq \phi$. Then the family {X – G_i: i∈I} of αg*s-closed sets has a finite intersection property. Also by assumption $\bigcap_{i=1}^{n}(X - G_i) \neq \phi$ which implies X – $\bigcup_{i=1}^{n} G_i \neq \phi$. So that $\bigcup_{i=1}^{n} G_i \neq X$. This implies {$G_i$: i ∈ I} is not a cover of X. This contradicts the fact that {G_i: i∈I} is a cover for X. Thus a αg*s-open cove {G_i: i∈I} has a finite subcover {G_i: i =1…..n}. Hence X is αg*s-compact.

Theorem 3.14: The image of a αg*s-compact space under a strongly αg*s-continuous function is αg*s-compact.

Theorem 3.15: The image of a αg*s-compact space under a αg*s-irresolute function is αg*s-compact.

4. Countably αg*s-Compactness in Topological Spaces

In this section, we study the concept of countably αg*s-compactness in topological spaces and investigated some of their properties.

Definition 4.1: A topological space X is said to be countably αg*s-compact if every countable αg*s-open cover of X has a finite subcover.

Theorem 4.2: If X is a countably αg*s-compact space, then f is countably compact.

Theorem 4.3: If X is countably compact and $_{\alpha gs}T^*_{1/2}$-space, then X is countably αg*s-compact.

Theorem 4.4: Every αg*s-compact space is countably αg*s-compact.

Proof: Let X be a αg*s-compact space. Let $\{A_i: i\in I\}$ be a countable αg*s-open cover of X containing αg*s-open sets. Then $\{A_i: i\in I\}$ is a αg*s-open cover $\{A_i: i\in I\}$ of X which has a finite subcover say $\{A_i : i=1...n\}$. Hence X is countably αg*s-compact.

Theorem 4.5: If f: X → Y is αg*s-continuous function from a countably αg*s-compact space X onto a topological space Y, then Y is countably compact.

Proof: Let $\{A_i : i\in I\}$ be a countable open cover of Y. Since f is αg*s-continuous, then $\{f^{-1}(A_i): i\in I\}$ is countable αg*s-open cover of X. Again since X is countably αg*s-compact, the countable αg*s-open cover $\{f^{-1}(A_i): i\in I\}$ of X has a finite subcover say $\{f^{-1}(A_i): i=1...n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ implies $f(X) = \bigcup_{i=1}^{n} A_i$. Thus $Y = \bigcup_{i=1}^{n} A_i$. Therefore $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i: i\in I\}$ for Y. Hence Y is countably compact.

Theorem 4.6: Let f: X → Y be αg*s-continuous function from a countably αg*s-compact space X onto a topological space Y. If Y is $_{\alpha gs}T^*_{1/2}$-space, then Y is countably αg^*s-compact.

Proof: Let $\{A_i: i\in I\}$ be a countable αg^*s-open cover of Y by αg^*s-open sets in Y. Since Y is ${}_{\alpha gs}T^*_{1/2}$-space, $\{A_i : i\in I\}$ is a countable open cover of Y. Then $\{f^{-1}(A_i) : i\in I\}$ is a countable αg^*s-open cover of X as f is αg^*s-continuous. Again since X is countably αg^*s-compact, the countable αg^*s-open cover $\{f^{-1}(A_i): i\in I\}$ of X has a finite subcover say $\{f^{-1}(A_i): i=1....n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies that $f(X) = \bigcup_{i=1}^{n} A_i$. That is $Y = \bigcup_{i=1}^{n} A_i$. Thus $\{A_1, A_2, ...A_n\}$ is a finite subcover of a countable αg^*s-open cover $\{A_i : i\in I\}$ for Y. Hence Y is countably αg^*s-compact.

Theorem 4.7: Let f: X $\rightarrow$Y be strongly αg^*s-continuous function from a countably compact space X onto a topological space Y, then Y is countably αg^*s-compact.

Proof: Let $\{A_i: i\in I\}$ be a countable αg^*s-open cover of Y by αg^*s-open sets in Y. Then $\{f^{-1}(A_i): i\in I\}$ is a countable open cover of X as f is strongly αg^*s-continuous function. Since X is countably compact, the countable open cover $\{f^{-1}(A_i): i\in I\}$ of X has a finite subcover say $\{f^{-1}(A_i): i=1, ..., n\}$. Therefore $X = \bigcup_{i=1}^{n} f^{-1}(A_i)$ which implies that $f(X) = \bigcup_{i=1}^{n} A_i$. Then $Y = \bigcup_{i=1}^{n} A_i$. Thus $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i: i\in I\}$ for Y. Hence Y is countably αg^*s-compact.

Theorem 4.8: If a function f: X $\rightarrow$ Y is perfectly αg^*s-continuous from a countably compact space X onto a topological space Y, then Y is countably αg^*s-compact.

Theorem 4.9: The image of a countably αg^*s-compact space under αg^*s-irresolute function is countably αg^*s-compact.

Theorem 4.10: A space X is countably αg^*s-compact if and only if every countable family of αg^*s-closed sets of X having finite intersection property (f. i. p.) has a non empty intersection.

Definition 4.11: A topological space X is said to be αg^*s-Lindelöf if every αg^*s-open cover of X has a countable subcover.

Theorem 4.12: Every αg^*s-Lindelöf space is Lindelöf.

Theorem 4.13: If X is Lindelöf and ${}_{\alpha gs}T^*_{1/2}$-space, then X is αg*s-Lindelöf space.

Theorem 4.14: Every αg*s-compact space is αg*s-Lindelöf space.

Proof: Let X be a αg*s-compact space. Let $\{A_i: i\in I\}$ be αg*s-open cover of X. Then $\{A_i: i\in I\}$ has a finite subcover say $\{A_i: i=1...n\}$ as X is αg*s-compact. Since every finite subcover is always a countable subcover. And therefore $\{A_i: i=1.....n\}$ is a countable subcover of $\{A_i: i\in I\}$ for X. Hence X is αg*s-Lindelöf space.

Theorem 4.15: If a function f: (X, τ) → (Y, σ) is αg*s-continuous from a αg*s-Lindelöf space X onto a topological space, then Y is Lindelöf space.

Theorem 4.16: The image of αg*s-Lindelöf space under αg*s-irresolute function is αg*s-Lindelöf.

Proof: Let f: (X, τ) → (Y, σ) be a αg*s-irresolute function form a αg*s-Lindelöf space X onto a topological space Y. Let $\{A_i: i\in I\}$ be a αg*s-open cover of Y. Then $\{f^{-1}(A_i): i\in I\}$ is αg*s-open cover of X as f is αg*s-irresolute. Since X is αg*s-Lindelöf, the αg*s-open cover $\{f^{-1}(A_i): i\in I\}$ of X has a countable subcover say $\{f^{-1}(A_{i_n}): n\in N\}$. Therefore $X = \bigcup_{n\in N} f^{-1}(A_{i_n})$ which implies $f(X) = Y = \bigcup_{n\in N} A_{i_n}$, that is $\{A_{i_n}: n\in N\}$ is a countable subfamily of $\{A_i: i\in I\}$ for Y. Hence Y is Lindelöf space.

Theorem 4.17: If a function f: (X, τ) → (Y, σ) is strongly αg*s-continuous function from a Lindelöf space X onto a topological space Y, then Y is αg*s-Lindelöf.

Proof: Let $\{A_i: i\in I\}$ be an αg*s-open cover of Y. Since f is strongly αg*s-continuous, $\{f^{-1}(A_i): i\in I\}$ is open cover of X. Again since X is Lindelöf, the open cover $\{f^{-1}(A_i): i\in I\}$ of X has a countable subcover say $\{f^{-1}(A_{i_n}): n\in N\}$. Therefore $X = \bigcup_{n\in N} f^{-1}(A_{i_n})$ which implies $f(X) = Y = \bigcup_{n\in N} A_{i_n}$. So $\{A_{i_n}: n\in N\}$ is a countable subcover of $\{A_i: i\in I\}$ for Y. Hence Y is αg*s-Lindelöf.

Theorem 4.18: If X is αg*s-Lindelöf and countably αg*s-compact space, then X is αg*s-compact.

Proof: Suppose X is countably αg*s-compact and αg*s-Lindelöf space. Let $\{A_i: i\in I\}$ be an αg*s-open cover of X. Since X is αg*s-Lindelöf, $\{A_i: i\in I\}$ has a countable subcover say $\{A_{i_n}: n\in N\}$. Therefore $\{A_{i_n}: n\in N\}$ is a countable subcover of X and

{ A_{i_n} : n∈N} is a subfamily of {A_i : i∈I} and so { A_{i_n} : n∈N} is a countably αg*s-open cover of X. Again since X is countably αg*s-compact, { A_{i_n} : n∈N} has a finite subcover say { A_{i_n} : n∈N} ⊆ {A_i : i∈I}. Therefore { A_{i_n} : n∈N} is a finite subcover of {A_i : i∈I} for X. Hence X is αg*s-compact space.

Theorem 4.19: The image of a αg*s-Lindelöf space under a strongly αg*s-continuous function is αg*s-Lindelöf.

5. αg*s-Connectedness in Topological Spaces

Definition 5.1: A topological space X is said to be αg*s -connected if X can not be expressed as a disjoint union of two non-empty αg*s-open sets.

A subset of X is αg*s -connected if it is αg*s-connected as a subspace.

Remark 5.2 Every αg*s-connected space is connected but the converse need not be true in general, which follows from the following example.

Example 5.3 Let X = {a, b, c} and let τ = {X, φ}. Then (X, τ) is connected but not an αg*s-connected space because X = {a} ∪ {b, c}, where {a} and {b, c} are αg*s-open sets in X..

Theorem 5.4: For a topological space X the following are equivalent.

(i) X is αg*s-connected.

(ii) X and φ are the only subsets of X which are both αg*s -open and αg*s - closed.

(iii) Each αg*s-continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i) ⇒ (ii) : Let G be any αg*s-open and αg*s-closed subset of X. Then G^c is both αg*s -open and αg*s-closed. Since X is disjoint union of the αg*s-open sets G and G^c implies from the hypothesis of (i) that either G = φ or G = X.

(ii) ⇒ (i) : Suppose that X = A ∪ B where A and B are disjoint non-empty αg*s -open subsets of X. Then A is both αg*s-open and αg*s-closed. By assumption A = φ or X.

Therefore X is αg*s-connected.

(ii) ⇒ (iii) : Let f : X → Y be a αg*s-continuous map. Then X is covered by αg*s - open and αg*s-closed covering $\{f^{-1}(Y): y \in (Y)\}$. By assumption $f^{-1}(y) = \varphi$ or X for each y ∈ Y. If $f^{-1}(y) = \varphi$ for all y ∈ Y, then f fails to be a map. Then there exists only one point y ∈ Y such that $f^{-1}(y) = \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) ⇒ (ii): Let G be both αg*s-open and αg*s-closed in X. Suppose G = φ. Let f : X → Y be a αg*s-continuous map defined by f(G) = y and $f(G^c) = \{w\}$ for some distinct points y and w in Y .

By assumption, f is constant. Therefore we have G = X

Theorem 5.5: If f : X → Y is a αg*s –continuous, onto and X is αg*s -connected, then Y is connected.

Proof: Suppose that Y is not connected. Let Y = A ∪ B where A and B are disjoint non-empty open set in Y. Since f is αg*s-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty αg*s-open sets in X. This contradicts the fact that X is αg*s-connected. Hence Y is connected.

Theorem 5.6: If f : X → Y is a αg*s-irresolute surjection and X is αg*s-connected, then Y is αg*s-connected.

Theorem 5.7: Suppose that X is a $_{\alpha gs}T^*_{1/2}$-space, then X is connected if and only if it is αg*s -connected.

Proof: Suppose that X is connected. Then X cannot be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a αg*s-connected space. Let A and B be any two αg*s-open subsets of X such that X = A∪B, where A ∩ B = φ and A ⊂ X, B ⊂ X. Since X is $_{\alpha gs}T^*_{1/2}$-space and A, B are αg*s-open sets, A, B are open subsets of X, which contradicts that X is connected. Therefore X is αg*s-connected.

Conversely, every open set is αg*s -open. Therefore every αg*s-connected space is connected.

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