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Certain Results of Neighborhoods and Partial Sums of Convex Functions based on Ruscheweyh Derivative Operators

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Abstract

In this paper we introduced a new class $C^{\lambda}(\alpha,\beta)$ of convex functions. A subclass $TC^{\lambda}(\alpha,\beta)$ of $C^{\lambda}(\alpha,\beta)$ based with negative coefficients is considered .We obtained some properties related to "coefficient inequality" , "extreme points" , "neighborhood" and "partial sums" of $C^{\lambda}(\alpha,\beta)$ and $TC^{\lambda}(\alpha,\beta)$.

1. INTRODUCTION

Let C denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disk U={z : |z| < 1}. Also, denote T , the subclass of C consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad (1.2)$$

which are univalent and normalized in U . For $f \in C$ of the form (1.1) and $g(z) \in C$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the hadamard product (or convolution) $f * g$ of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (1.3)$$

Definition 1. For -1 ≤α< 1 and $\beta \geq 0$, we consider $C^{\lambda}(\alpha,\beta)$ be the subclass of C consisting of functions of the form (1.1) and satisfying the following

$$Re\left\{\frac{z\,(D^{\lambda}f(z))''}{(D^{\lambda}f(z))'} + (1-\alpha)\right\} > \beta\left|\frac{z\,(D^{\lambda}f(z))''}{(D^{\lambda}f(z))'}\right| \quad (1.4)$$

Where D^{λ} is the Ruscheweyh derivative [6] defined by

$$D^{\lambda}f(z) = f(z)\frac{1}{(1-z)^{\lambda+1}} = z + \sum_{k=2}^{\infty} B_k(\lambda)a_k z^k$$

where

$$B_k(\lambda) = \frac{(\lambda+1)_{k-1}}{(k-1)!} = \frac{(\lambda+1)(\lambda+2)\ldots\ldots\ldots(\lambda+k-1)}{(k-1)!}\lambda \geq 0 \quad (1.5)$$

Also $TC^{\lambda}(\alpha,\beta) = C^{\lambda}(\alpha,\beta) \cap T$ is considered.

In this paper, we study coefficient and extreme points of the subclass $TC^{\lambda}(\alpha,\beta)$.We obtained certain neighborhoods and partial sums results for functions in $TC^{\lambda}(\alpha,\beta)$.

Many authors studied univalent functions based on Ruscheweyh derivative operator as "Al-Amiri" [1] , "Atshan and Joudah" [2] and" Goodman" [3],[4].

2. THE CLASSES $C^{\lambda}(\alpha,\beta)$ AND $TC^{\lambda}(\alpha,\beta)$

In this section we get a necessary and sufficient condition and extreme points for functions *f*(z) in the class $TC^{\lambda}(\alpha,\beta)$

Theorem 2.1. A sufficient condition for a function *f*(z) of the form (1.1) to be in $C^{\lambda}(\alpha,\beta)$ is that

$$\sum_{k=2}^{\infty}\frac{k[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\beta_k(\lambda)|a_k| \leq 1,\ -1 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0. \quad (2.1)$$

Proof. It's a sufficient to show that

$$\beta\left|\frac{z\,(D^{\lambda}f(z))''}{(D^{\lambda}f(z))'}\right| - Re\left\{\frac{z\left(D^{\lambda}f(z)\right)''}{\left(D^{\lambda}f(z)\right)'}\right\} \leq 1-\alpha. \quad (2.2)$$

We have

$$\beta\left|\frac{z\left(D^{\lambda}f(z)\right)''}{\left(D^{\lambda}f(z)\right)'}\right| - Re\left\{\frac{z\left(D^{\lambda}f(z)\right)''}{\left(D^{\lambda}f(z)\right)'}\right\} \leq (1+\beta)\left|\frac{z\left(D^{\lambda}f(z)\right)''}{\left(D^{\lambda}f(z)\right)'}\right|$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}k(k-1)\beta_k(\lambda)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}k\,\beta_k(\lambda)|a_k||z|^{k-1}}$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}k(k-1)\beta_k(\lambda)|a_k|}{1-\sum_{k=2}^{\infty}k\,\beta_k(\lambda)|a_k|}$$

The last inequality is bounded above by 1-α if

$$\sum_{k=2}^{\infty} k[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)|a_k| \leq 1-\alpha ,$$

and the proof is complete .

The above condition is also necessary for $f \in T$, as follows.

Theorem 2.2. A necessary and sufficient condition for f of the form (1.2) to be in $TC^{\lambda}(\alpha,\beta)$ $,-1 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0,$ is that

$$\sum_{k=2}^{\infty} k[(1+\beta)k-(\alpha+\beta)]a_k\beta_k(\lambda) \leq 1-\alpha \tag{2.3}$$

Proof. In view of theorem (2.1), we need only to prove the necessity. If $f \in TC^{\lambda}(\alpha,\beta)$ with z is a real, then

$$(1-\alpha)-\frac{\sum_{k=2}^{\infty}\beta_k(\lambda)\,k\,(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\beta_k(\lambda)\,k\,a_k z^{k-1}} \geq \frac{\beta\,\sum_{k=2}^{\infty}\beta_k(\lambda)\,k\,(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\beta_k(\lambda)\,k\,a_k z^{k-1}}$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty} k[(1+\beta)k-(\alpha+\beta)]a_k\beta_k(\lambda) \leq 1-\alpha$$

Theorem 2.3. The extreme points of $TC^{\lambda}(\alpha,\beta)$ $,-1 \leq \alpha < 1, \beta \geq 0$ are the functions given

$$f_1(z)=z \text{ and } f_k(z)=z-\frac{1-\alpha}{k[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)}\,z^k,$$

$k = 2, 3, \ldots,$ where $\lambda > -1$. Then $f \in TC^{\lambda}(\alpha,\beta)$ if and only if f may be expressed as the form $\sum_{k=1}^{\infty}\mu_k f_k(z)$, where $\mu_k \geq 0$, $\sum_{k=1}^{\infty}\mu_k = 1$.

Proof . Let us express f_1 and f_k as in the above, therefore we can write

$$\sum_{k=1}^{\infty}\mu_k f_k(z) = \mu_1 z+\sum_{k=2}^{\infty}\mu_k f_k(z)$$

$$= \mu_1 z+\sum_{k=2}^{\infty}\mu_k\left(z-\frac{1-\alpha}{k\,[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)}z^k\right)$$

$$= z-\sum_{k=2}^{\infty}\mu_k\,v_k\,z^k \text{ , where}$$

$v_k = \frac{1-\alpha}{k\,[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)}\,\mu_k$. Therefore $f \in TC^{\lambda}(\alpha,\beta)$, since

$\sum_{k=2}^{\infty} v_k \frac{k\,[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)}{1-\alpha} = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 < 1$.

Conversely, assume that $f \in TC^{\lambda}(\alpha,\beta)$, therefore

$$\mu_k = \frac{k\,[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)}{1-\alpha} a_k \,.$$

Then

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

$$= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)\mu_k}{k\,[(1+\beta)k-(\alpha+\beta)]\beta_k(\lambda)} z^k = z - \sum_{k=2}^{\infty} \mu_k\big(z - f_k(z)\big)$$

$$= z - \sum_{k=2}^{\infty} \mu_k\, z + \sum_{k=2}^{\infty} \mu_k f_k(z) = z\Big(1 - \sum_{k=2}^{\infty} \mu_k\Big) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 z + \sum_{k=2}^{\infty} \mu_k f_k(z).$$

The proof is complete.

3. NEIGHBORHOOD RESULTS

This concept is introduced by "Goodman" [3] and studied by "Silverman" [7]. Also, "Thomas Rosy" ," Subramanian and Murugusundaramoorthy" studied a certain results of neighborhood of starlike functions based on ruscheweyh derivatives. In this section we study a neighborhood of functions in the subclass $TC^{\lambda}(\alpha,\beta)$ of convex functions.

Definition 2. For $f \in C$ of the form (1.1) and $\delta \geq 0$, we define η-δ – neighborhood of f by

$$N_{\delta}^{\eta}(f) = \{g \in C : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k^{\eta+2}|a_k - b_k| \leq \delta\} \qquad (3.1)$$

where η is a fixed positive integer . We need the following lemmas to study the η-δ – neighborhood of functions in $TC^{\lambda}(\alpha,\beta)$.

Lemma 3.1. Let $\mu \geq 0$ and $-1 \leq \gamma < 1$. If $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies

$$\sum_{k=2}^{\infty} k^{\mu+2}\,|b^k| \leq \frac{1-\gamma}{1+\beta}\,. \qquad (3.2)$$

Then $g \in C^{\lambda}(\alpha,\beta)$. The result is sharp .

Proof. In view of Theorem (2.1) , we have

$\frac{k\,[(1+\beta)k-(\gamma+\beta)]}{1-\gamma}\beta_k = \frac{k\,[(1+\beta)k-(\gamma+\beta)]}{1-\gamma} \cdot \frac{(\mu+1)(\mu+2)\ldots\ldots(\mu+k-1)}{(k-1)!}$

$$\leq \frac{k^2(1+\beta)(\mu+1)(\mu+2)\ldots\ldots(\mu+k-1)}{(1-\gamma)(k-1)!}.$$

Therefore,

$$\frac{k\,[(1+\beta)k-(\gamma+\beta)]}{1-\gamma}\beta_k \leq \frac{k^{\mu+2}(1+\beta)(\mu+1)(\mu+2)\ldots\ldots(\mu+k-1)}{k^{\mu}(1-\gamma)(k-1)!}$$

Now, we need to prove that

$$H(k,\mu) = \frac{(\mu+1)(\mu+2)\ldots\ldots(\mu+k-1)}{k^{\mu}(k-1)!} \leq 1$$

In view of Lemma (3.1) in [10] , $H(k,\mu)$ is decreasing .Then the result is hold for all k$\geq$ 2.

Lemma3.2.Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T\, -1 \leq \alpha < 1\,, \beta \geq 0\,, \in \geq 0$ .If $\frac{f\,(z)+\in z}{1+\in} \in TC^{\lambda}(\alpha\,,\beta)$, then

$$\sum_{k=2}^{\infty} k^{\eta+2}\, a_k \leq \frac{2^{\eta+1}\,(1-\alpha)(1+\in)}{(2-\alpha+\beta)(1+\lambda)},$$

where either $\eta = 0$ and $\lambda \geq 0$ or $\eta = 1$ and $1 \leq \lambda \leq 2$. The result is sharp with external function

$$f\,(z) = z - \frac{(1-\alpha)(\,1+\in)}{2\,(\,2-\alpha+\beta)(\,1+\lambda)} z^2\,, z \in U$$

Proof. Suppose that $g(z) = \frac{f\,(z)+\in z}{1+\in}$. We have $g(z) = z - \sum_{k=2}^{\infty} \frac{a_k}{1+\in} z^k\,, z \in U$.

In view of Theorem (1.3) , $g(z) = \sum_{k=1}^{\infty} \mu_k g_k(z)$,where $\mu_k \geq 0\,, \sum_{k=1}^{\infty} \mu_k = 1$.

Therefore,

$$g_1(z) = z \qquad , \qquad g_k(z) = z - \frac{(1-\alpha)(1+\in)}{k(k-\alpha+\beta)\beta_k(\lambda)} z^k\,.$$

Hence,

$$g(z) = \mu_1 z + \sum_{k=2}^{\infty} \mu_k \left(z - \frac{(1-\alpha)(1+\in)}{k(k-\alpha+\beta)\beta_k(\lambda)} z^k\right)$$

$= z - \sum_{k=2}^{\infty} \mu_k \frac{(1-\alpha)(1+\in)}{k(k-\alpha+\beta)\beta_k(\lambda)} z^k$ .Therefore , it follows that

$\sum_{k=2}^{\infty} k^{\eta+2}\, a_k \leq sup\; k^{\eta+2} \left(\frac{(1-\alpha)(1+\in)}{k(k-\alpha+\beta)\beta_k(\lambda)}\right)$ .This implies that

$$\sum_{k=2}^{\infty} k^{\eta+2}\, a_k \leq sup\; k^{\eta+1} \left(\frac{(1-\alpha)(1+\in)}{(k-\alpha+\beta)\beta_k(\lambda)}\right).$$

Taking $A(k,\eta\,,\alpha,\epsilon,\lambda) = \frac{k^{\eta+1}\,(1-\alpha)(1+\in)}{(k-\alpha+\beta)\beta_k(\lambda)}$. In view of lemma (3.2) in [11], we get $A(k,\eta\,,\alpha,\epsilon,\lambda)$ is decreasing function of k and the result it's done.

Theorem 3.1.Suppose either $\eta = 0$ and $\eta \geq 0$ or $\eta = 1$ and $1 \leq \eta \leq 2$. Let $-1 \leq \alpha < 1$, and

$$-1 \leq \gamma < \frac{(2-\alpha+\beta)(1+\lambda)-2^{\eta+1}(1-\alpha)(1+\epsilon)(1+\beta)}{(2-\alpha+\beta)(1+\lambda)(1+\beta)}$$

Let $f \in T$ and for all real numbers $0 \leq \epsilon < \delta$, assume $\frac{f(z)+\epsilon z}{1+\epsilon} \in TC^{\lambda}(\alpha,\beta)$. Then the η-$\delta$ neighbourhood of *f*, namely $N_{\delta}^{\eta}(f) \subset C^{\eta}(\alpha,\beta)$, where

$$\delta = \frac{(1-\gamma)(2-\alpha+\beta)(1+\eta)-2^{\eta+1}(1-\alpha)(1+\epsilon)(1+\beta)}{(2-\alpha+\beta)(1+\lambda)(1+\beta)}$$

The result is sharp , with extermal function.

$$f(z) = z - \frac{(1-\alpha)(1+\epsilon)}{2(2-\alpha+\beta)(1+\lambda)} z^2 , z \in U$$

Proof. For a function f of the form (1.2) , let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ *be in* $M_{\delta}^{\eta}(f)$. In view of Lemma (3.2) , we have

$$\sum_{k=2}^{\infty} k^{\eta+2}|b_k| = \sum_{k=2}^{\infty} k^{\eta+2}|a_k - b_k - a_k|$$
$$\leq \sum_{k=2}^{\infty} k^{\eta+2}|a_k - b_k| + \sum_{k=2}^{\infty} k^{\eta+2}|a_k|$$
$$\leq \delta + \frac{2^{\eta+1}(1-\alpha)(1+\epsilon)}{(2-\alpha+\beta)(1+\lambda)}.$$

By Applying Lemma 3.1 , it follows $g \in C^{\eta}(\gamma,\beta)$.

If $\delta + \frac{2^{\eta+1}(1-\alpha)(1+\epsilon)}{(2-\alpha+\beta)(1+\lambda)} \leq \frac{1-\gamma}{1+\beta}$

That is,

$$\delta \leq \frac{(1-\gamma)(2-\alpha+\beta)(1+\lambda)-2^{\eta+1}(1-\alpha)(1+\epsilon)(2+\beta)}{(2-\alpha+\beta)(1+\lambda)(2+\beta)}$$

This completes the proof .

4. PARTIAL SUMS

From earlier papers by "Silverman" [8] and "Silvia" [9] on partial sums of analytic functions , we consider in this section a partial sums of functions in the class C_p^{λ} (α,β) and obtain lower bounds for the ratios or real part of f(z) to f_n(z) and f'(z) to $f_n{}'$(z).

Theorem 4.1. Let $f(z) \in C_p^{\lambda}$ (α,β) be given by (1.1) .Also f_1(z) and f_n(z) defined by

$$f_1(z) = z \quad \text{and} \quad f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k , (n \in N/\{1\}). \qquad (4.1)$$

Suppose that

$$\sum_{k=2}^{\infty} d_k|a_k| \leq 1, \tag{4.2}$$

where $\left(d_k := \frac{k[(1+\beta)k-(\alpha+\beta)]B_k(\lambda)}{1-\alpha}\right)$. Therefore $f \in C^{\lambda}(\alpha,\beta)$. *Furthermore,*

$$\text{Re}\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{d_{n+1}}, z \in U, n \in N \tag{4.3}$$

and

$$\text{Re}\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{d_{n+1}}{1+d_{n+1}}. \tag{4.4}$$

Proof. Since $f_1(z) = z \in C^{\lambda}(\alpha,\beta)$, therefore from Theorem 3.3 and by hypothesis (4.2), we have

$$N(z) \subset C^{\lambda}(\alpha,\beta), \tag{4.5}$$

Which shows that $f \in C^{\lambda}(\alpha,\beta)$.

For the coefficients d_k given by (4.2)

$$d_{k+1} > d_k > 1 . \tag{4.6}$$

Therefore ,

$$\sum_{k=2}^{n}|a_k| + d_{n+1}\sum_{k=n+1}^{\infty}|a_k| \leq \sum_{k=2}^{\infty} d_k|a_k| \leq 1 \tag{4.7}$$

By setting

$$g_1(z) = d_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{d_{n+1}}\right)\right\}$$

$$= d_{n+1}\left\{\frac{f(z)}{f_n(z)} - 1 + \frac{1}{d_{n+1}}\right\}$$

$$= 1 + \frac{d_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{n} a_k z^{k-1}}$$

Applying (4.7), we get that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{n+1}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k| - d_{n+1}\sum_{k=n+1}^{\infty}|a_k|} \leq 1 \tag{4.8}$$

Which readily yields the assertion (4.3) of Theorem 4.1. The function

$$f(z) = z + \frac{z^{n+1}}{d_{n+1}} \tag{4.9}$$

gives sharp result, so that for z = $re^{i\pi/n}$, $\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{d_{n+1}} \to 1 - \frac{1}{d_{n+1}}$ as z→1^-.Similarly, if we take

$$g_2(z) = (1 + d_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{d_{n+1}}{1 + d_{n+1}}\right\}$$

$$= 1 - \frac{(1 + d_{n+1})\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}$$

and making use of (4.7), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1 + d_{n+1})\sum_{k=n+1}^{\infty}|a_k|}{2 - 2\sum_{k=2}^{n}|a_k| - (1 + d_{n+1})\sum_{k=n+1}^{\infty}|a_k|} \leq 1, \quad z \in U. \tag{4.10}$$

Which leads to assertion (4.4) of Theorem 4.1.The bound in (4.4) is sharp for each n∈N with the extremal function *f*(z) given by (4.9).

The proof of Theorem 4.1. is complete.

Theorem 4.2. *If* f(z) of the form (1.1) satisfies the condition (2.1). Then

$$\text{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 1 - \frac{n+1}{d_{n+1}} \tag{4.11}$$

Proof. By setting

$$g(z) = \frac{d_{n+1}}{n+1}\left\{\frac{f'(z)}{f_n'(z)} - \left(1 - \frac{n+1}{d_{n+1}}\right)\right\}$$

$$= \frac{1 + \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}}$$

$$= 1 + \frac{\frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{n} k a_k z^{k-1}}, \text{ Therefore,}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k|a_k|}{2 - 2\sum_{k=2}^{n} k\,|a_k| - \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k|a_k|} \leq 1\,.$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$ if

$$\sum_{k=2}^{n} k\,|a_k| + \frac{d_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k|a_k| \leq 1 \qquad (4.12)$$

The result is sharp for the extremal function

$$f(z) = z + \frac{z^{n+1}}{d_{n+1}}.$$

Theorem 4.3. If $f(z)$ of the form (1.1) satisfies the condition (2.1) then

$$\text{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{d_{n+1}}{n+1+d_{n+1}}.$$

Proof. By setting

$$g(z) = \frac{[(n+1)+d_{n+1}]}{n+1}\left\{\frac{f_n'(z)}{f'(z)} - \frac{d_{n+1}}{[(n+1)+d_{n+1}]}\right\}$$

$$= 1- \frac{(1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1+\sum_{k=2}^{\infty} k a_k z^{k-1}}$$

and making use

$$\sum_{k=2}^{\infty} k|a_k| + (1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k\,|a_k| \leq 1$$

we can deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{(1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k|}{2-2\sum_{k=2}^{\infty} k|a_k| - (1+\frac{d_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k\,|a_k|} \leq 1,$$

Therefore, the result of Theorem 4.3 holds.

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