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The Portfolio Standard Risk Model Based on Rank Dependent Expected Utility Model with Quadratic Utility Function

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Abstract

This study aims to develop a portfolio standard risk model by using rank dependent expected utility (RDEU) with the utility function is quadratic function and the probability weighting function is a concave peace-wise linear function. The research results show that portfolio standard risk based on RDEU is a function of risk and expected return of Markowitz portfolio model.

Keywords: Rank dependent expected utility, quadratic utility function, portfolio standard risk

1. INTRODUCTION

Over the past decades, researchers have given a considerable attention to the development of portfolio selection theory, namely Mean-Variance (M-V) theory, expected utility (EU) theory, and growth optimal portfolio (GOP) theory. The M-V theory was first developed by Markowitz [1], in which expected return and portfolio risk are determined by the mean and variance of asset in financial market, e.g. stock. The theory postulates that a portfolio value is said to be optimal if the weighting of each asset in the portfolio is determined in such a way that the portfolio mean is maximum, while the portfolio risk is minimum. The EU theory is a portfolio model which offers a rational, general framework for making decision, in which portfolio is selected by maximizing expected utility function preferred by investors under uncertainty. This theory was first proposed by Von-Neumann and Morgenstern [2]. The GOP theory is a portfolio model which is based on the maximum of geometric mean. The model was first proposed by Williams [3], then developed by other researchers, among others Raju and Selvaraju [4].

The EU theory has given a significant contribution to the development of economics and finance which concerns with risks and uncertainty [5]. Despite this fact, however, only very few researchers in economics and finance have employed the theory owing to the fact that the theory is too theoretical compared to the other two mentioned above [6]. For this reason, Yaari [7] developed a theory of parallel risks by modifying the axiom of independence that is postulated by Von-Neumann and Morgenstern [8], where attitudes towards risks are characterized by a distortion applied to cumulative distribution function (instead of asset utility function), and is called the dual expected utility (DEU) theory.

RDEU expressed by equation (1) was first proposed by Quiggin [6], where the DEU model is a special case of the RDEU model. It is called a special case because the utility function $U(x)$ in the DEU model is a linear function [8], whereas the utility function of $U(x)$ in (1) or (2) is non-linear [9], for example: quadratic utility function, exponential utility function, and logarithmic utility function.

Empirically speaking, the application of RDEU in selecting optimal portfolio has been investigated by a number of researchers, for example Cenci and Filippini [8] and Gao et al. [9]. These researchers used linear utility function in which investor utility changes constantly. Hence, the method of applying RDEU in their portfolio selection model is called as a linear approach.

The current study aims to develop a model of portfolio standard risk by using RDEU with the non-linear utility function, in particular the quadratic utility function. To this end, we firstly formulated an RDEU model in which the probability weighting function is assumed as a continuous function. Next, we used the obtained RDEU model to determine the standard risk of portfolio.

This paper consist of five parts. Part 1 is an introduction which describes several models of portfolio, and research purposes. Part 2 reviews the RDEU theory. Part 3 explains the development of RDEU model with quadratic utility function. Part 4 presents the application of RDEU model to determine a model of portfolio standard of risk. Part 5 gives the conclusions.

2. REVIEW OF THE RDEU THEORY

The RDEU model is a generalization of the EU model. The definition of RDEU, according to Segal [10], is as follows. Suppose that *X* is a random variable of assets in a financial market that is defined on probability space $(\Omega, \mathcal{F}, P)$ with the values or outcomes of $x_1 \leq x_2 \leq, \dots, \leq x_n$ with the probability measure of each is $p_1, p_2, \dots, p_n$ then for $p = (p_1, p_2, \dots, p_i),\ i = 1, 2, \dots, n$

$$RDEU(X) = U(x_1)g(p_1) + \sum_{i=2}^{n} U(x_i)h_i(p) \tag{1}$$

in which $h_i(p) = g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right),\ (i = 1, 2, \dots, n)$ and the probability weighting funtion g: [0,1] → [0,1] is an monotonically inceasing function in such that *g*(0)=0 and g(1)=1. Next, the function U: $[0, \infty) \rightarrow (-\infty, +\infty)$ is the utility function of investor's preference for selecting optimal portfolio for which *U*(0)=0. Afterwards, [9, 11], shows that the utility function *U*(*x*) is a also monotonically increasing, and concave.

RDEU in equation (1) can be expressed with the Lebesque-Stieltjes integral, as follows

$$RDEU(X) = \int_{\Omega} U(x) dg(F_X(x)) \tag{2}$$

where $F_X(x) = \Pr(X \leq x), x\epsilon(-\infty, +\infty)$ is a cumulative distribution function of random variable *X*. If $F_X(x)$ is differentiable, and hence has an associated a density function $f = F_X'$. If the utility function *U*(*x*) and the probability weighting function *g*(*x*) are continuous [9, 12], then equation (2) can be expressed with the Riemann-Stieltjes integral or the Riemann integral, as follows

$$RDEU(X) = \int_{-\infty}^{+\infty} U(x)\, dg(F_X(x))$$

$$= \int_{-\infty}^{+\infty} U(x) g'(F_X(x)) f(x) dx \tag{3}$$

where $g'(F_x) = \frac{dg}{dF_x}$.

RDEU is also developed from the EU theory. Therefore, several axioms in the UE theory are met by the RDEU theory. The axioms are as follows [9]. If $\succcurlyeq$ is the preference relation of investor (decision maker) for selecting an asset in the financial market, the following axioms apply.

A1). If X_1 and X_2 with the cumulative distribution function F_{X_1} and F_{X_2} in such a way that $F_{X_1} = F_{X_2}$, then $X_1 \sim X_2$.

A2). The preference relation $\succcurlyeq$ is a weak order preference, meaning that preference $\succcurlyeq$ fulfills the following properties: complete, transitive, and reflexive.

A3). If X_1 is first stochastic dominance (FSD) X_2 or $F_{X_1} \leq F_{X_2}$, then $X_1 \succcurlyeq X_2$

A4). The preference relation $\succcurlyeq$ is continuous

Furthermore, the comonotonic axiom of the DEU theory by Yaari [7] is also fulfilled by the theory of RDEU, thus

A5). If $X_1, X_2,$ and X_3 are three comonotonic random variables, and $X_1 \succcurlyeq X_2$ then for $\lambda \varepsilon [0,1]$, $\lambda X_1 + (1-\lambda)X_2 \succcurlyeq \lambda X_2 + (1-\lambda)X_3$

As mentioned in the introduction section1, investor (decision maker) attitudes towards the risk of investment choice is characterized by a distortion applied to cumulative distribution function. Therefore, axiom A1 states that if the outcome of random variable X_1 and the outcome of random variable X_2 serve the same the cumulative distribution function, decision maker (investor) can choose either X_1 or X_2 since both can have the same benefits in the investment choice. Next, axiom A3 states that if a decision maker believes that $F_{X_1} \leq F_{X_2}$ will happen, then the decision maker will prefer X_1 to X_2.

3. RDEU WITH QUADRATIC UTILITY FUNCTION

We propose a RDEU model with the utility function $U(x)$ is *a* quadratic function ([13], 14], [15]), namely

$$U(x) = x - ax^2, \; 0 < x < \frac{1}{2a}, \;\; 0<a<1 \tag{4}$$

and the probability weighting function $g(x)$ is a concave peace-wise linear function, as follows

$$g(x) = \begin{cases} bx, & \text{if } 0 \leq x \leq Pr(X \leq v) \\ cx + d, & \text{if } Pr(X \leq v) < x \leq 1 \end{cases}, \; b > c > 0 \tag{5}$$

where X is a random variable defined on probability space $(\Omega, \mathcal{F}, P)$

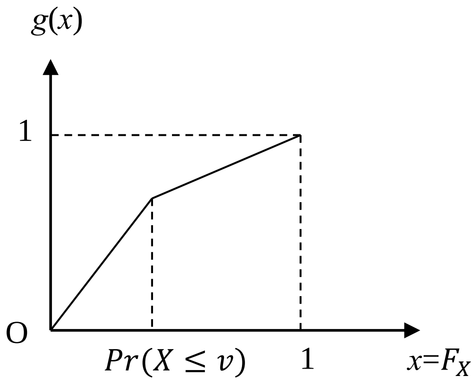

Figure 1: Graph of Function, $g(x)$

The graphical representation of function $g(x)$ is shown by Figure 1. $F_X(v)=Pr(X \leq v)$ is the value of the cumulative distribution function of the random variable X at $v\epsilon(-\infty,+\infty)$.

The values of b, c and d in equation (5) can be eliminated by using the properties of concavity, monotonicity, and also the continuity of function $g(x)$, so, we have (see also [8])

$$1 \leq b \leq \frac{1}{\Pr(X\leq v)}, c = \frac{1-b\Pr(X\leq v)}{1-\Pr(X\leq v)},\ d = (b-c)\Pr(X \leq v),\ b > c \tag{6}$$

The function g is the continuous function on every subinterval $[0,\ Pr(X \leq v\ )]$ and $[Pr(X \leq V,\ 1]$ of $[0,1]$. This function has a left and right derivative at the end of the subinterval, but has no derivative at point $x = Pr(X \leq v)$ [16]. Therefore, the first derivative of function $g(x)$ is

$$g'(F_x) = \begin{cases} b, & \text{jika } 0 \leq F_X \leq Pr(X \leq v) \\ c, & \text{jika } Pr(X \leq v) < F_X \leq 1 \end{cases},\ b > c > 0 \tag{7}$$

If we assume that the random variable X is normally distributed with $E(X) = m,$ where $m \in (-\infty, +\infty)$, and the deviation standard of the random variable X is σ, then (7) becomes

$$g'(F_X(x)) = \begin{cases} b, & \text{if } x \in (-\infty, m] \\ c, & \text{if } x \in [m, +\infty) \end{cases},\ b > c > 0$$

and (3) becomes

$$RDEU(X) = \int_{-\infty}^{+\infty} U(x)\, g'\big(F_x(x)\big) f(x) dx$$

$$= b\int_{-\infty}^{m} U(x)\, g'\big(F_x(x)\big) f(x)dx$$

$$+ c\int_{m}^{+\infty} U(x)\, g'\big(F_x(x)\big) f(x)dx \qquad (8)$$

and $Pr(X \leq m) = \frac{1}{2}$. By using (6), we obtain $1 \leq b \leq 2$, and $c = 2 - b$. Because $b > c > 0$, then $1 < b < 2$ and $0 < c < 1$. Thus, the values of b and c in equations (5), (7), and (8) satisfies the inequality $0 < c < 1 < b < 2$. Furthermore, the random variable X has a density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right),\ x \in (-\infty, +\infty)$.

If the value of a in equation (4) close to 0, then $x \to \infty$ and the value of the utility function $U(x)$ is greater. The RDEU approximation value of the random variable X is

$$RDEU(X) = -\frac{a\sigma^2(c+b)}{2} + \frac{(c-b)(1-2am)\sigma}{\sqrt{2\pi}} + \frac{(c+b)(m-am^2)}{2}$$

$$= -a\sigma^2 + \frac{(2-2b)(1-2am)\sigma}{\sqrt{2\pi}} + (m - am^2) \qquad (9)$$

where $m < \frac{1}{2a}$, $0 < a < 1$, and $1 < b < 2$ (the proof can be seen on the appendix). Therefore, the RDEU is a non-linear function of the standard of deviation σ and the expectation of the random variable X, which is $E(X) = m$.

4. THE PORTFOLIO STANDARD RISK MODEL BASED ON THE RDEU MODEL

Suppose that W is a portfolio of investors in a financial market, where $w = (w_1, w_2, \dots, w_n)$ $w_i(i = 1, 2, \dots, n)$ is the weight of asset *i*, for example stock in the portfolio *W*. The weight w_i satisfy the equation $\sum_{i=1}^{n} w_i = 1,\ w_i \geq 0$.

Furthermore, suppose that the return of stock asset *i* at time $t \in [0, T]$ in portfolio W is R_{it} or simplified with $R_i(i = 1, 2, \dots, n)$. The portfolio return of W is $R_p = \sum_{i=1}^{n} w_i R_i$. The expected return of portfolio $E(R_p)$ [17] is

$$E(R_p) = \sum_{j=1}^{n} w_j E(R_j) \qquad (10)$$

and the variance of portfolio σ_p^2 is

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \qquad (11)$$

Therefore, we obtain from (9) for the random variable of portfolio return $R_p = X$ that the approximation value of standar risk of portfolio W using RDEU [8], is

$$\varphi(R_p) = U(E(R_p)) - RDEU(R_p)$$

$$= m - am^2 + a\sigma_p^2 - \frac{(2-2b)(1-2am)\sigma_p}{\sqrt{2\pi}} - (m - am^2)$$

$$= a\sigma_p^2 + \frac{(2b-2)(1-2am)\sigma_p}{\sqrt{2\pi}} \qquad (12)$$

where $m < \frac{1}{2a}, 0 < a < 1,\ 1 < b < 2$, and the formula of σ_p^2 has been stated in equation (11). Thus, if expected return of portfolio W in equation (12) is $E(R_p) = m$, then the approximation value of portfolio standard risk is as follows

$$\varphi(R_p) = a\sigma_p^2 + \frac{(2b-2)(1-2aE(R_p))\sigma_p}{\sqrt{2\pi}} \qquad (13)$$

Where $E(R_p) < \frac{1}{2a}, 0 < a < 1,\ 1 < b < 2$, and the formula of $E(R_p)$ has been stated in equation (10). The lower the a and b values in equation (13), the lower the value of portfolio standard risk as well.

5. CONCLUSION

This study aims to develop a model of portfolio standard risk by using the RDEU model. The study is a development of past research conducted by Cenci and Filippini [8] which employed the identity utility function, and the probability weighting function is a peace-wise linear function. In the current study we use the quadratic utility function that is continuous, concave, increasing, and bounded. To develop the model, we assume that the probability weighting function is a concave peace-wise linear function, and has a first derivative on each subinterval of it's domain [0,1], and that the random variable of portfolio return is normally distributed.

Result of developing the model shows that the portfolio standar risk based on RDEU is a function of risk σ_p and expected return $E(R_p)$ as well as parameter a $(0 < a < 1)$ and b $(1 < b < 2)$ where σ_p and $E(R_p)$ are function of the weight of the assets i in the portfolio W ie w_i $(i = 1, 2, \dots, n)$. The smaller the value of a and b, the smaller the value of portfolio standard risk $\varphi(R_p)$.

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APPENDIX

Supose that X is a random variable defined on probability space $(\Omega, \mathcal{F}, P)$. If X is normally distributed with $E(X) = m,\ m \in (-\infty, +\infty)$, and has a density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right), x \in (-\infty, +\infty)$. We replace the utility function equation (4) and also the probability density function of random variable X in the integral equation (8), then we have

$$RDEU(X) = \frac{b}{\sigma\sqrt{2\pi}} \int_{-\infty}^{m} (x - ax^2)\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx$$

$$+ \frac{c}{\sigma\sqrt{2\pi}} \int_{m}^{+\infty} (x - ax^2)\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx$$

We substitute $u = \frac{x-m}{\sigma}$. If x=m, u=0, $x \to -\infty, u \to -\infty$, then we have

$$RDEU(X) = \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{0} [(u\sigma + m) - a(u\sigma + m)^2]\exp\left(-\frac{u^2}{2}\right) du$$

$$+ \frac{c}{\sqrt{2\pi}} \int_{0}^{+\infty} [(u\sigma + m) - a(u\sigma + m)^2]\exp\left(-\frac{u^2}{2}\right) du$$

$$= \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{0} [-a\sigma^2 u^2 + (\sigma - 2am\sigma)u + (m - am^2)]exp\left(-\frac{u^2}{2}\right) dx$$

$$+ \frac{c}{\sqrt{2\pi}} \int_{0}^{+\infty} [a\sigma^2 u^2 + (\sigma - 2am\sigma)u + (m - am^2)]exp\left(-\frac{u^2}{2}\right) dx$$

$$= -\frac{ab\sigma^2}{\sqrt{2\pi}} \int_{0}^{+\infty} u^2 exp\left(-\frac{u^2}{2}\right) dx + \frac{b(\sigma-2am\sigma)}{\sqrt{2\pi}} \int_{0}^{+\infty} uexp\left(-\frac{u^2}{2}\right) dx$$

$$+ \frac{b(m-am^2)}{\sqrt{2\pi}} \int_{0}^{+\infty} exp\left(-\frac{u^2}{2}\right) dx + \frac{cb\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{0} u^2 exp\left(-\frac{u^2}{2}\right) dx$$

$$+ \frac{c(\sigma-2am\sigma)}{\sqrt{2\pi}} \int_{-\infty}^{0} uexp\left(-\frac{u^2}{2}\right) dx + \frac{c(m-am^2)]}{\sqrt{2\pi}} \int_{-\infty}^{0} exp\left(-\frac{u^2}{2}\right) dx.$$

$$= -\frac{ab\sigma^2}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{2\pi}\right) + \frac{b(\sigma-2am\sigma)}{\sqrt{2\pi}}.(-1) + \frac{b(m-am^2)}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{2\pi}\right)$$

$$+ \frac{ac\sigma^2}{\sqrt{2\pi}}\left(-\frac{1}{2}\sqrt{2\pi}\right) + \frac{c(\sigma-2am\sigma)}{\sqrt{2\pi}}.(1) + \frac{c(m-am^2)}{\sqrt{2\pi}}\left(\frac{1}{2}\sqrt{2\pi}\right)$$

$$= -\frac{a\sigma^2(c+b)}{2} + \frac{(c-b)(1-2am)\sigma}{\sqrt{2\pi}} + \frac{(c+b)(m-am^2)}{2}$$