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Hypergroup associated with hyperset

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Abstract

The objective of this paper is the study of hypergroup associated with a hyperset. In this paper we show that from an S-hyperset $(Q, \circ')$ we can derive a hypergroup $(Q, \circ)$, called the associate hypergroup of the S-hyperset $(Q, \circ')$ and we study some properties of this hypergroup $(Q, \circ)$.

AMS subject classification:
Keywords: hyperaction, hyperset, co-subhyperset, hypergroup, cyclic hypergroup.

1. Introduction

Hyperstructure theory was surfaced up in 1934 when F. Marty [10] defined hypergroups. Since then many researchers have studied in this field and published several papers on it. Sen, Ameri and Chowdhury [11] introduce the notion of hyperset and studied some algebraic structures on it. In this paper we derive a hypergroup from a hyperset and study some properties of hyperset in the light of associate hypergroup. We recall some definitions and theorems on hyperset and hypergroup from [4], [5], [11], [12] which are needed to develop this paper.

Throughout the paper S denotes a monoid, Q denotes a non-empty set and $P(Q)$ denotes the set of all subsets of Q.

2. Hyperset and Hypergroup

Definition 2.1. [11] A (left) hyperaction of S on Q is a mapping $\circ' : S \times Q \mapsto P(Q)$ (usually denoted by $\circ'(x, q) \mapsto x \circ' q$) for all $x \in S$ and $q \in Q$.

Let $A \in P(Q)$ and $x \in S$. We define $x \circ' A \in P(Q)$ by

$$x \circ' A = \begin{cases} \bigcup_{a \in A}(x \circ' a), & \text{if } A \neq \emptyset \\ \emptyset, & \text{if } A = \emptyset \end{cases}$$

If there exists a hyperaction $\circ'$ of S on Q such that

(i) $(xy) \circ' q = x \circ' (y \circ' q)$

(ii) $q \in 1 \circ' q$, for all $x, y \in S$ and $q \in Q$, then we say that $(Q, \circ')$ is an S-hyperset of type1.

If there exists a hyperaction $\circ'$ of S on Q such that

(i) $(xy) \circ' q = x \circ' (y \circ' q)$

(ii) $1 \circ' q = \{q\}$, for all $x, y \in S$ and $q \in Q$, then we say that $(Q, \circ')$ is an S-hyperset of type2.

An S-hyperset of type2 is always an S-hyperset of type1. Unless otherwise stated a hyperset means an S-hyperset of type2.

Definition 2.2. [4] A hyperoperation on a nonempty set H is a mapping $\circ : H \times H \longrightarrow P(H)$ (usually denoted by $\circ(x, y) \longrightarrow x \circ y$) for all $x, y \in H$.
Let $A, B \in P(H)$ and $x \in H$. We define $A \circ B, x \circ A, A \circ x \in P(H)$ by

$$A \circ B = \begin{cases} \cup\{a \circ b : a \in A \text{ and } b \in B\}, & \text{if } A \neq \emptyset \text{ and } B \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

$$x \circ A = \begin{cases} \cup\{x \circ a : a \in A\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

and

$$A \circ x = \begin{cases} \cup\{a \circ x : a \in A\}, & \text{if } A \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

Then $x \circ \phi = \phi \circ x = \phi$.
A set H together with a hyperoperation $\circ$ is called a hypergroupoid and is denoted by $(H, \circ)$.

Definition 2.3. [4] A hypergroupoid $(H, \circ)$ is called a semihypergroup if

$$a \circ (b \circ c) = (a \circ b) \circ c \text{ for all } a, b, c \in H.$$

Definition 2.4. [4] A semihypergroup $(H, \circ)$ is called a hypergroup if

$$H \circ a = a \circ H = H \text{ for all } a \in H.$$

Definition 2.5. [4] A hypergroup $(H, \circ)$ is called a commutative hypergroup if

$$a \circ b = b \circ a \text{ for all } a, b \in H.$$

Note 2.6. Let $(H, \circ)$ be a hypergroup. Let $a, b \in H$ such that $a \circ b = \emptyset$. Then $H \circ (a \circ b) = \emptyset$. But $H \circ (a \circ b) = \cup\{h \circ (a \circ b) : h \in H\} = \cup\{(h \circ a) \circ b : h \in H\} = (H \circ a) \circ b = H \circ b = H$. Therefore $H = \emptyset$, which is a contradiction. Therefore $a \circ b \neq \emptyset$ for all $a, b \in H$.

There are semihypergroups $(H, \circ)$ where $a \circ b$ may be empty for some $a, b \in H$.

Example 2.7. Let $H = \{2, 3, 4\}$. Let $a, b \in H$. Define a hyperoperation $\circ$ on H by

$$a \circ b = \{p \in H : p \neq 1 \text{ and } p \text{ divides both of } a \text{ and } b\}.$$

$\circ$	2	3	4
2	$\{2\}$	ϕ	$\{2\}$
3	ϕ	$\{3\}$	ϕ
4	$\{2\}$	ϕ	$\{2, 4\}$

Then $(H, \circ)$ is a semihypergroup, where $a \circ b$ may be empty for some $a, b \in H$.

Throughout this paper we follow the convention of the book [4] i.e. we assume that $a \circ b \neq \emptyset$ for all $a, b \in H$.

Theorem 2.8. [4] A semihypergroup $(H, \circ)$ is a hypergroup if and only if

$$a, b \in H \text{ there exist } x, y \in H \text{ such that } b \in a \circ x \text{ and } b \in y \circ a.$$

We give an example of a semihypergroup which is not a hypergroup

Example 2.9. Let $H = \{1, 2, 3, 4\}$. Let $a, b \in H$. Define a hyperoperation $\circ$ on H by

$$a \circ b = \{p \in H : p \text{ divides both of } a \text{ and } b\}.$$

$\circ$	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
3	$\{1\}$	$\{1\}$	$\{1, 3\}$	$\{1\}$
4	$\{1\}$	$\{1, 2\}$	$\{1\}$	$\{1, 2, 4\}$

Then $(H, \circ)$ is a semihypergroup. Choose $a = 2$ and $b = 3$. There exist no $x, y \in H$ such that $b \in a \circ x$ and $b \in y \circ a$. Therefore $(H, \circ)$ is not a hypergroup.

Definition 2.10. [4] Let $(H, \circ)$ be a semihypergroup and K be a nonempty subset of H. Then $(K, \circ)$ is a subsemihypergroup of H if K is itself a semihypergroup under the hyperoperation $\circ$ restricted to K. It is clear that a nonempty subset K of H is a subsemihypergroup if and only if $K \circ K \subseteq K$.

Theorem 2.11. [4] Let $(H, \circ)$ be a semihypergroup. A nonempty subset K of $(H, \circ)$ is a subsemihypergroup if and only if

$$a, b \in K \Longrightarrow a \circ b \subseteq K \text{for all } a, b \in H.$$

Theorem 2.12. [4] A subsemihypergroup $(K, \circ)$ of a hypergroup $(H, \circ)$ is said to be subhypergroup if $(K, \circ)$ is itself a hypergroup. It is clear that a subsemihypergroup $(K, \circ)$ is a subhypergroup if and only if

$$a \circ K = K \circ a = K \text{ for all } a \in K.$$

It can be shown that the Intersection (if it is nonempty) of subsemihypergroups of a semihypergroup is a subsemihypergroup.

Theorem 2.13. [4] Nonempty intersection of two subsemihypergroups of a semihypergroup is a subhypergroup.

3. Associate hypergroup

In this section we show that from an S-hyperset $(Q, \circ')$ we can derive a hypergroup $(Q, \circ)$, called the associate hypergroup of the S-hyperset $(Q, \circ')$. Then study some properties of $(Q, \circ')$ in the light of the associate hypergroup $(Q, \circ)$.

Let $(Q, \circ')$ be an S-hyperset. To develop the section we need the following Lemma:

Lemma 3.1. Let A and B be two subsets of an S-hyperset $(Q, \circ')$ and $x \in S$. Then

(i) $A \subseteq B \Longrightarrow x \circ' A \subseteq x \circ' B$

(ii) $x \circ' (A \cup B) = (x \circ' A) \cup (x \circ' B)$ and

(iii) $x \circ' (A \cap B) \subseteq (x \circ' A) \cap (x \circ' B)$.

Proof.

(i) $x \circ' A = \cup\{x \circ' a : a \in A\} \subseteq \cup\{x \circ' a : a \in B\} = x \circ' B$.

(ii) $A \subseteq A \cup B$ and $B \subseteq A \cup B \Longrightarrow x \circ' A \subseteq x \circ' (A \cup B)$ and $x \circ' B \subseteq x \circ' (A \cup B)$ $\Longrightarrow (x \circ' A) \cup (x \circ' B) \subseteq x \circ' (A \cup B)$.
Conversely $p \in x \circ' (A \cup B) \Longrightarrow p \in x \circ' q$ for some $q \in A \cup B \Longrightarrow p \in x \circ' q$, where $q \in A$ or $q \in B \Longrightarrow p \in x \circ' A$ or $p \in x \circ' B \Longrightarrow p \in (x \circ' A) \cup (x \circ' B)$ $\Longrightarrow x \circ' (A \cup B) \subseteq (x \circ' A) \cup (x \circ' B)$. Therefore $x \circ' (A \cup B) = (x \circ' A) \cup (x \circ' B)$ for all $x \in S$.

(iii) We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $x \circ' (A \cap B) \subseteq x \circ' A$ and $x \circ' (A \cap B) \subseteq x \circ' B$. Therefore $x \circ' (A \cap B) \subseteq (x \circ' A) \cap (x \circ' B)$. ∎

Theorem 3.2. Let $(Q, \circ')$ be an S-hyperset. Define a hyperoperation $\circ$ on Q by

$$p \circ q = S \circ' p \cup S \circ' q \text{ for all } p, q \in Q.$$

Then $(Q, \circ)$ is a hypergroup.

Proof. Since $p \in S \circ' p$ and $q \in S \circ' q$, we have $p, q \in p \circ q$. Therefore $p \circ q \neq \emptyset$. Let $p, q, r \in Q$. Then

$$\begin{aligned}
(p \circ q) \circ r &= \cup\{u \circ r : u \in p \circ q\} \\
&= \cup\{S \circ' u \cup S \circ' r : u \in p \circ q\} \\
&= (\cup\{S \circ' u : u \in S \circ' p \cup S \circ' q\}) \cup S \circ' r \\
&= (S \circ' (S \circ' p \cup S \circ' q)) \cup S \circ' r \\
&= ((S \circ' (S \circ' p)) \cup (S \circ' (S \circ' q))) \cup S \circ' r \text{(using Lemma 3.1)} \\
&= ((SS) \circ' p \cup (SS) \circ' q) \cup S \circ' r \\
&= S \circ' p \cup S \circ' q \cup S \circ' r \\
&= p \circ (q \circ r).
\end{aligned}$$

It follows that $(Q, \circ)$ is a semihypergroup. Let $p \in Q$. Then

$$\begin{aligned}
p \circ Q &= \cup\{p \circ q : q \in Q\} \\
&= \cup\{S \circ' p \cup S \circ' q : q \in Q\} \\
&= S \circ' p \cup (\cup\{S \circ' q : q \in Q\}) \\
&= \cup\{S \circ' q : q \in Q\} \\
&= S \circ' Q \\
&= Q.
\end{aligned}$$

Similarly we can show that $Q \circ p = Q$ for all $p \in Q$. It follows that $(Q, \circ)$ is a hypergroup.

Since $p \circ q = q \circ p$ for all $p, q \in Q$, therefore $(Q, \circ)$ is a commutative hypergroup. It is obvious that $p^2 = p \circ p = S \circ' p$ for all $p \in Q$. Then $p \circ q = S \circ' p \cup S \circ' q = p \circ p \cup q \circ q = p^2 \cup q^2$ for all $p, q \in Q$. Since $p \circ q \circ r = S \circ' p \cup S \circ' q \cup S \circ' r$, we have $p^3 = p \circ p \circ p = S \circ' p \cup S \circ' p \cup S \circ' p = S \circ' p = p^2$. Therefore $p^n = p^2$ for all positive integer $n (\geq 2)$. ∎

Definition 3.3. Let $(Q, \circ')$ be an S-hyperset. Define a hyperoperation $\circ$ on Q by

$$p \circ q = S \circ' p \cup S \circ' q \text{ for all } p, q \in Q.$$

By the Theorem 3.2, $(Q, \circ)$ is a hypergroup The hypergroup $(Q, \circ)$ is called the associate hypergroup of the hyperset $(Q, \circ')$.

Definition 3.4. [11] Let $(Q, \circ')$ be an S-hyperset. A nonempty subset P of Q is said to be a subhyperset if

$$x \circ' P \subseteq P \text{ for all } x \in S.$$

Lemma 3.5. [3] Let P be a subset of an S-hyperset $(Q, \circ')$. Then $P \subseteq S \circ' P$.

Theorem 3.6. [3] Let P be a nonempty subset of an S-hyperset $(Q, \circ')$. Then the following statements are equivalent:

(i) P is a subhyperset of Q.

(ii) $S \circ' P \subseteq P$.

(iii) $S \circ' P = P$.

Theorem 3.7. [11] Let $(Q, \circ')$ be an S-hyperset. Then

(i) Q is itself a subhyperset.

(ii) For each $q \in Q$, $S \circ' q$ is the smallest subhyperset containing q.

(iii) Union and Intersection of two subhypersets are subhyper sets.

Definition 3.8. [3] An S-hyperset $(Q, \circ')$ is said to be connected if for all $p, q \in Q$ there exists $x \in S$ such that $p \in x \circ' q$.

Definition 3.9. [3] An S-hyperset $(Q, \circ')$ is said to be simple if it has no proper subhyperset.

Theorem 3.10. [3] Let $(Q, \circ')$ be an S-hyperset. Then the following statements are equivalent:

(i) $(Q, \circ')$ is connected,

(ii) $S \circ' q = Q$ for all $q \in Q$,

(iii) $(Q, \circ')$ is simple.

Theorem 3.11. Let $(Q, \circ')$ be an S-hyperset and $(Q, \circ)$ be the associate hypergroup. Let P be a nonempty subset of Q. Then $(P, \circ')$ is a subhyperset of $(Q, \circ')$ if and only if $(P, \circ)$ is a subhypergroup of $(Q, \circ)$.

Proof. Let $(P, \circ')$ be a subhyperset of $(Q, \circ')$ and $p, q \in P$. Then $p \circ q = S \circ' p \cup S \circ' q$. By the Theorem 3.6, $p \circ q \subseteq P \cup P = P \Longrightarrow (P, \circ)$ is a subsemihypergroup of $(Q, \circ)$.

Let $p \in P$. Then

$$\begin{aligned} p \circ P &= \cup\{p \circ q : q \in P\} \\ &= \cup\{S \circ' p \cup S \circ' q : q \in P\} \\ &= S \circ' p \cup (\cup\{S \circ' q : q \in P\}) \\ &= \cup\{S \circ' q : q \in Q\} \\ &= S \circ' P \\ &= P \end{aligned}$$

(Using the Theorem 3.6). $(Q, \circ)$ being commutative hypergroup , $p \circ P = P \circ p = P$ for all $p \in P \Longrightarrow (P, \circ)$ is a subhypergroup of $(Q, \circ)$.
Conversely assume that $(P, \circ)$ is a subhypergroup of $(Q, \circ)$. Then $S \circ' P = \cup\{S \circ' p : p \in P\} = \cup\{p \circ p : p \in P\} \subseteq P \circ P = P \Longrightarrow (P, \circ')$ is a subhyperset of $(Q, \circ')$. ■

Lemma 3.12. Let $(A, \circ)$ and $(B, \circ)$ be two subhypergroups of the associate hypergroup $(Q, \circ)$ of the hyperset $(Q, \circ')$. Then $A \circ B = A \cup B$.

Proof. Given that $(A, \circ)$ and $(B, \circ)$ are subhypergroups of the hypergroup $(Q, \circ)$. By the Theorem 3.11, $(A, \circ')$ and $(B, \circ')$ are subhypersets of the S-hyperset $(Q, \circ')$. Now

$$\begin{aligned} A \circ B &= \cup\{p \circ q : p \in A,\ q \in B\} \\ &= \cup\{S \circ' p \cup S \circ' q : p \in A,\ q \in B\} \\ &= S \circ' A \cup S \circ' B \\ &= A \cup B \end{aligned}$$

(Using the Theorem 3.6). ■

Definition 3.13. A commutative hypergroup $(H, \circ)$ is said to be inner irreducible if

$H = H_1 \circ H_2 \Longrightarrow H_1 \cap H_2 \neq \emptyset$, for any pair of subhypergroups $(H_1, \circ)$ and $(H_1, \circ)$.

Definition 3.14. **[3]** A subhyperset $(P, \circ')$ of an S-hyperset $(Q, \circ')$ is said to be co-subhyperset of Q if $Q \setminus P$ is also a subhyperset of Q. From definition it is clear that if P be a co-subhyperset then $Q \setminus P$ is also a co-subhyperset.

Theorem 3.15. The associate hypergroup $(Q, \circ)$ of an hyperset $(Q, \circ')$ is inner irreducible if and only if the hyperset $(Q, \circ')$ has no co-subhyperset.

Proof. Let the hypergroup $(Q, \circ)$ be inner irreducible. Let $(P, \circ')$ be a co-subhyperset of $(Q, \circ')$. Then $(P, \circ')$ and $(Q \setminus P, \circ')$ are subhypersets of $(Q, \circ')$. By the Theorem 3.11, $(P, \circ)$ and $(Q \setminus P, \circ)$ are subhypergroups of $(Q, \circ)$.

Now $Q = P \cup Q \setminus P = P \circ (Q \setminus P)$ (using Lemma 3.12).
The associate hypergroup $(Q, \circ)$ is inner irreducible and $Q = P \circ Q \setminus P$. Then by

Definition 3.13, $P \cap Q \setminus P \neq \emptyset$, which is a contradiction. It follows that the hyperset $(Q, \circ')$ has no co-subhyperset. Conversely assume that the hyperset $(Q, \circ')$ has no co-subhyperset.

Let $(A, \circ)$ and $(B, \circ)$ be two subhypergroups of the associate hypergroup $(Q, \circ)$ such that $Q = A \circ B$. By the Theorem 3.11, $(A, \circ')$ and $(B, \circ')$ are subhypersets of $(Q, \circ')$. By the Lemma 3.12, $Q = A \cup B$. We show that $A \cap B \neq \emptyset$.

Let $A \cap B = \emptyset$. Since $Q = A \cup B$, we have $B = Q \setminus A$ and $A = Q \setminus B$. This implies that both of $(A, \circ')$ and $(B, \circ')$ are co-subhypersets of $(Q, \circ')$, which contradicts the hypothesis. Therefore $A \cap B \neq \emptyset$. It follows that the associate hypergroup $(Q, \circ)$ is inner irreducible. ■

Definition 3.16. **[2]** Let $(H, \circ)$ be a hypergroup. If there is an element $h \in H$ such that $H = \cup\{h^n : n \geq 1\}$ then we say that $(H, \circ)$ is a cyclic hypergroup generated by h. The element h is called a generating element of the cyclic hypergroup $(H, \circ)$.
If there is a positive integer n such that $H = h^n$ then the cyclic hypergroup $(H, \circ)$ is called n-single power cyclic and h is called n-generating element of the hypergroup.

In the following theorem we show a correspondence between the connectedness of the S-hyperset $(Q, \circ')$ and the cyclicity of the associate hypergroup $(Q, \circ)$.

Theorem 3.17. A hyperset $(Q, \circ')$ is connected if and only if the associate hypergroup $(Q, \circ)$ is 2-single power cyclic and each $p \in Q$ is a 2-generating element of the hypergroup $(Q, \circ)$.

Proof. Let the S-hyperset $(Q, \circ')$ be connected. Then by the Theorem 3.10, $S \circ' p = Q$ for all $p \in Q$. Therefore $Q = p \circ p = p^2$ for all $p \in Q$. This shows that the hypergroup $(Q, \circ)$ is 2-single power cyclic and each $p \in Q$ is a 2-generating element of the hypergroup $(Q, \circ)$.

Conversely assume that the associate hypergroup $(Q, \circ)$ is 2-single power cyclic and each $p \in Q$ is a 2-generating element of the hypergroup $(Q, \circ)$. Then $Q = p^2 = p \circ p$ for all $p \in Q$. Therefore $Q = S \circ' p$ for all $p \in Q$. Then by the Theorem 3.10, the S-hyperset $(Q, \circ')$ is connected. ■

Theorem 3.18. **[3]** Let $(Q, \circ')$ be an S-hyperset. We define a relation τ on Q by

$$q \tau p \iff \text{either } q = p \text{ or there exists } x \in S \text{ such that } p \in x \circ' q \text{ for all } p, q \in Q.$$

Then the relation $\tau^\star = \cup\{(\tau \cup \tau^{-1})^n : n \geq 1\}$ is an equivalence relation on Q containing τ and satisfying the property $p \in x \circ' q \Longrightarrow q\tau^\star p$ for all $p, q \in Q$ and $x \in S$.

Theorem 3.19. **[3]** Let $(Q, \circ')$ be an S-hyperset and P be a co-subhyperset of Q. Then $p \in P,\ p\tau^\star q \Longrightarrow q \in P$.

Theorem 3.20. **[3]** Let $(Q, \circ')$ be an S-hyperset. Then for each $q \in Q$ the equivalence class $q\tau^\star$ is an indecomposable subbhyperset of Q.

Definition 3.21. Let $(Q, \circ')$ be an S-hyperset. Let $x \in S$ and $q, p \in Q$ be such that $p \in x \circ' q$. If there is an element $y \in S$ such that $q \in y \circ' p$ then we say that the S-hyperset $(Q, \circ')$ is retrievable.

Theorem 3.22. An S-hyperset $(Q, \circ')$ is retrievable if and only if $\tau = \tau^\star$.

Proof. By the Definition 3.21, we say that an S-hyperset $(Q, \circ')$ is retrievable if and only $\tau = \tau^{-1}$.

Let $\tau = \tau^{-1}$. Then τ is an equivalence relation and hence $\tau = \tau^\star$. Let $\tau = \tau^\star$. Then $\tau^{-1} \subseteq \tau \cup \tau^{-1} \subseteq \tau^\star = \tau$. Again $(p, q) \in \tau \Longrightarrow (q, p) \in \tau^{-1} \subseteq \tau \Longrightarrow (q, p) \in \tau \Longrightarrow (p, q) \in \tau^{-1} \Longrightarrow \tau \subseteq \tau^{-1}$. It follows that $\tau = \tau^{-1}$. Thus we get $\tau = \tau^{-1}$ if and only if $\tau = \tau^\star$. ■

Definition 3.23. Let $(Q, \circ')$ be an S-hyperset and T be a nonempty subset of Q. We define a subset $\sigma_Q(T)$ of Q as follows:

$$\sigma_Q(T) = \{p \in Q : \text{ there exists } x \in S \text{ such that } x \circ' p \cap T \neq \emptyset\}$$

Theorem 3.24. Let $(Q, \circ')$ be an S-hyperset. Then the following statements are equivalent:

(i) $(Q, \circ')$ is retrievable.

(ii) $\sigma_Q(P) = P$ for all subhypersets $(P, \circ')$ of $(Q, \circ')$.

(iii) Every subhyperset of $(Q, \circ')$ is a co-subhyperset.

Proof. $(i) \Longrightarrow (ii)$.
Let $(Q, \circ')$ be retrievable and $(P, \circ')$ be a subhyperset of $(Q, \circ')$. Let $p \in P$. Then by Definition 3.4, $x \circ' p \subseteq P$ for all $x \in S \Longrightarrow x \circ' p \cap P \neq \emptyset$ for all $x \in S$
$\Longrightarrow p \in \sigma_Q(P) \Longrightarrow P \subseteq \sigma_Q(P)$.
Again $p \in \sigma_Q(P) \Longrightarrow x \circ' p \cap P \neq \emptyset$ for some $x \in S$
$\Longrightarrow$ there exists $q \in Q$ such that $q \in x \circ' p$ and $q \in P$
$\Longrightarrow$ there is $y \in S$ such that $p \in y \circ' q$ and $q \in P$(since $(Q, \circ')$ is retrievable)
$\Longrightarrow p \in S \circ' P \subseteq P$(since $(P, \circ')$ is a subhyperset of $(Q, \circ')$)
$\Longrightarrow \sigma_Q(P) \subseteq P$.
It follws that $\sigma_Q(P) = P$.
$(ii) \Longrightarrow (iii)$.
Let $\sigma_Q(P) = P$ for all subhypersets $(P, \circ')$ of $(Q, \circ')$. We show that $(Q \setminus P, \circ')$ is a subhyperset of $(Q, \circ')$. Let $p \in Q \setminus P$. Then $p \notin P \Longrightarrow p \notin \sigma_Q(P) \Longrightarrow x \circ' p \cap P = \emptyset$ for all $x \in S \Longrightarrow x \circ' p \subseteq Q \setminus P$ for all $x \in S \Longrightarrow (Q \setminus P, \circ')$ is a subhyperset of$(Q, \circ') \Longrightarrow (P, \circ')$ is a co-subhyperset of $(Q, \circ')$.
$(iii) \Longrightarrow (i)$.
Let every subhyperset of $(Q, \circ')$ is a co-subhyperset. Let $p, q \in Q$ and $x \in S$ be such that $p \in x \circ' q$. Then $(x \circ' q) \cap (S \circ' p) \neq \emptyset$. We show that there exists $y \in S$ such that

$q \in y \circ' p$. If possible let $q \notin y \circ' p$ for all $y \in S \Longrightarrow q \notin S \circ' p \Longrightarrow q \in Q \setminus (S \circ' p)$. By the Theorem 3.7, $S \circ' p$ is a subhyperset and by hypothesis $S \circ' p$ is a co-subhyperset. Then $\Longrightarrow x \circ' q \subset Q \setminus (S \circ' p) \Longrightarrow (x \circ' q) \cap (S \circ' p) = \emptyset$, a contradiction. This follows that the hyperset $(Q, \circ')$ is retrievable. ∎

Theorem 3.25. An S-hyperset $(Q, \circ')$ is retrievable. Then every inner irreducible subhypergroup of the associate hypergroup $(Q, \circ)$ is 2-single power cyclic.

Proof. Let $(Q, \circ')$ be retrievable and $(P, \circ)$ be a inner irreducible subhypergroup of the associate hypergroup $(Q, \circ)$. By the Theorem 3.11, $(P, \circ')$ is a subhyperset of $(Q, \circ')$. By the Theorem 3.15, the subhyperset $(P, \circ')$ has no co-subhyperset. Now $p \in P \Longrightarrow (S \circ' p, \circ')$ is a subhyperset of $(P, \circ')$ but $(P \setminus S \circ' p, \circ')$ is not a subhyperset of $(P, \circ')$. We assert that $P = S \circ' p$. Otherwise $P \setminus S \circ' p \neq \emptyset \Longrightarrow$ there exists $q \in P \setminus S \circ' p \Longrightarrow$ there must exists $x \in S$ such that $x \circ' q \cap S \circ' p \neq \emptyset \Longrightarrow$ there exists $r \in x \circ' q$ and $r \in S \circ' p$. $(Q, \circ')$ is retrievable and $r \in x \circ' q \Longrightarrow$ there exists $z \in S$ such that $q \in z \circ' r$. Now $q \in z \circ' r$ and $r \in S \circ' p \Longrightarrow q \in S \circ' p \Longrightarrow q \in S \circ' p$, a contradiction. Therefore $P = S \circ' p = p \circ p = p^2 \Longrightarrow$ the subhypergroup $(P, \circ')$ is 2-single power cyclic. ∎

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